A CUTTING PLANE ALGORITHM FOR INTEGER PROGRAMS WITH AN EASY PROOF OF CONVERGENCE

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1. Introduction

Let B be the optimal LP basis for a given problem with m rows and n+m variables.

$$\begin{aligned} \min \quad & c_B y + c_N x \\ \text{s.t.} \quad & By + Nx = b \\ & y, x \geq 0 \quad \text{integer} \quad , \end{aligned}$$

where $B^{-1}b \geq 0$, $c_N \geq c_B B^{-1}N$, and all coefficients are assumed to be integral.

<u>Lemma 1</u> If $c_B B^{-1}b$ is not integral, the constraint

$$c_B y + [c_B B^{-1}N]x \geq [c_B B^{-1}b]$$

is a valid cut for the I.P., where $[t]$ is the lowest integer not less than t.

<u>Proof</u> Since B is optimal

$$c_B y + c_N x \geq c_B B^{-1}b$$

for all feasible y, x, and for any value of c_N satisfying $c_N \geq c_B B^{-1}b$. In particular

$$c_B y + [c_B B^{-1}N]x \geq c_B B^{-1}b \quad .$$

Since c_B, $[c_B B^{-1} N]$ are integral, for all feasible integral values of (y, x)

$$c_B y + [c_B B^{-1} N] x \geq [c_B B^{-1} b]$$

which is therefore a valid cut.

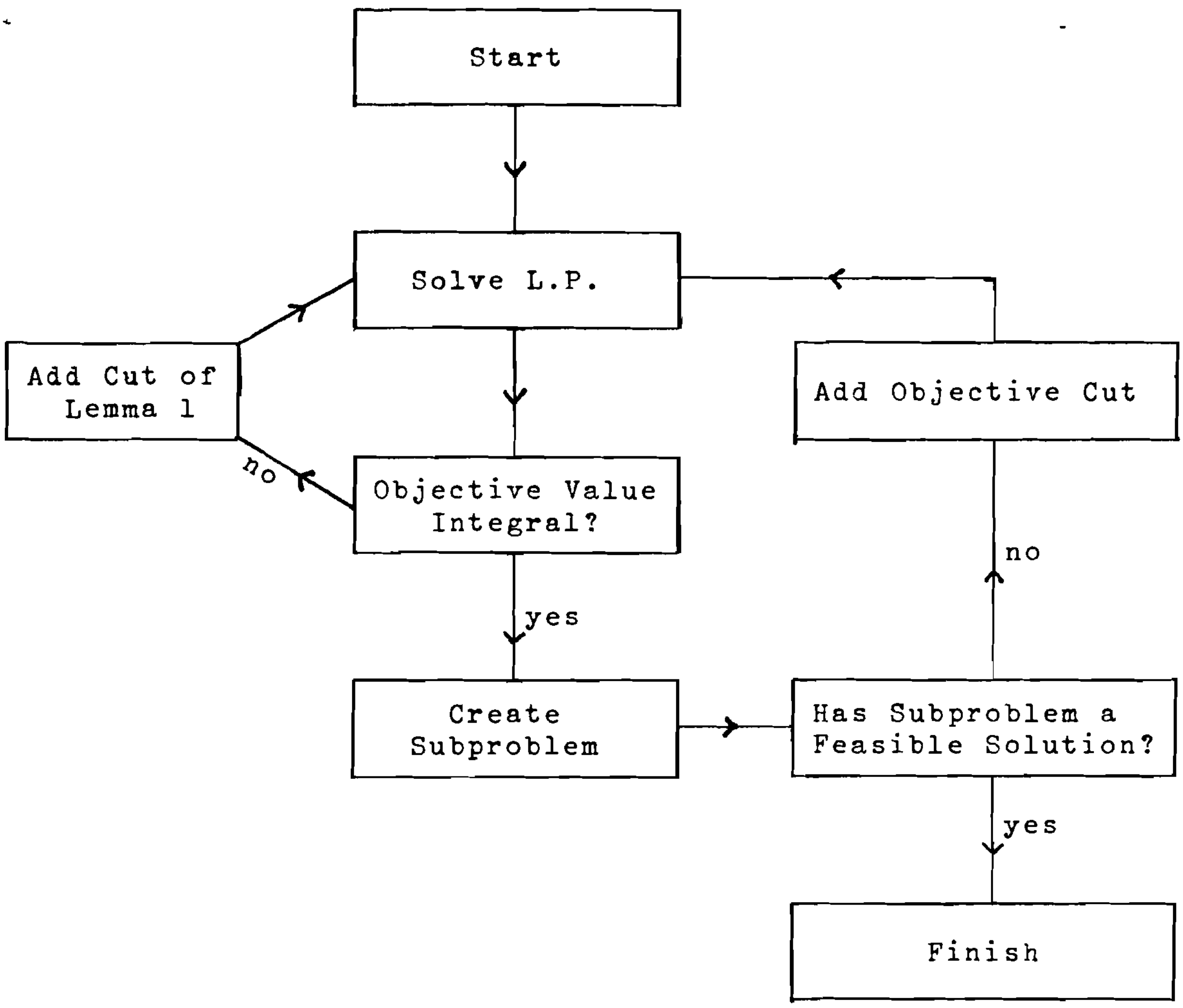
<u>Lemma 2</u> If the cut of Lemma 1 is added to the LP, the optimal objective value increases to at least $[c_B B^{-1} b]$.

<u>Proof</u> Let (y*, x*) be the optimal solution to the new LP,

then $$c_B y^* + c_N x^* \geq c_B y^* + [c_B B^{-1} N] x^* \geq [c_B B^{-1} b]$$

since $c_N \geq [c_B B^{-1} N]$ because c_N is integral and because of the new cut. //

2. The Algorithm



Step 1 Solve the L.P.

Step 2a If the value of the objective is not integral, add the cut of lemma 1 and return to step 1.

Step 2b If the value of the objective is integral, create a subproblem with added constraint

$$c_B y + c_N x = c_B B^{-1} b$$

together with a new objective function (d_B, d_N) chosen only to be independent of the existing m+1 rows. Implement this algorithm on the subproblem. If the subproblem has a feasible solution, it is optimal. If it has no feasible solution, add the cut

$$c_B y + c_N x \geq c_B B^{-1} + 1$$

to the original L.P. and go to step 1.

3. Convergence

Theorem The algorithm of section 2 produces the optimal solution, or shows there is none, after solving only a finite number of L.P. problems, if the L.P. feasible region is bounded.

Proof By induction on n the number of non basic variables.

For n = 0 the algorithm produces the solution or the information that no solution exists after solving at most two L.P.'s.

Assume that the algorithm converges for all programs having up to n non basic variables and now consider a problem having n + 1.

If the subproblem is created, it has only n non basic variables and hence can be solved finitely so that each repetition of step 1 occurs after a finite number of L.P. solutions. Note that the objective value increases by at least 1 every two iterations of step 1. If the L.P. region is bounded, the algorithm must converge finitely. //