ON THE OPTIMAL STOCHASTIC CONTROL OF WATER RESOURCES SYSTEMS

A. Szöllösi-Nagy

September 1975 WP-75-111

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ABSTRACT

To get an effective control of large river basin systems, the decision maker wishes to develop optimal operating policies. To establish these policies, the future behaviour of inputs, such as available resources, demand to be satisfied, etc., must be known or rather predicted. Because of the uncertainties inherent in water resources processes, both in quantity and quality, the prediction scheme to be constructed should be able to handle stochastic effects. Moreover, the algorithms should be recursive to avoid cumbersome computations and to be able to be used for real-time forecasting. This is especially important in case of emergency, e.g. flash floods.

A general state space based formulation of water resources systems is given. It is shown that the general model of runoff control systems is able to handle different kinds of uncertainties. Optimal sequential prediction algorithms for linear discrete time stochastic WR system are presented. In the framework of runoff control the case of optimal stochastic dynamic water quality control is discussed and feedback control policies are established.

The algorithms proposed might help the decision maker in working out the optimal operating policies for a large river basin system in the presence of different kinds of uncertainties.

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* to be presented as a semi-tutorial paper at the UNDP/UN Interregional Seminar on River Basin and Interbasin Development, September 1975, Budapest.

INTRODUCTION

It is rather a truism that the Large River Basin and/or Interbasin problems are extremely complex. They involve physical, economical, social, legal, political and several other issues. The collection and dynamic behaviour of the above listed issues are usually called Water Resource(WR) System.

There is no question about the fact that the main problem dealing with WR systems is how to bring them into the 'best' possible states either in a short-term or in the long run. In other words, how to make decisions during either the operational or the planning phase in order to reach maximum utility. The decisions are generally sequential in time, e.g. short-term reservoir policies or long-term investment strategies. The different kinds of water resources decisions are imbedded here into the collective term of runoff control [8]. The purpose of runoff control is to regulate the distribution of water both from quantity and/or quality standpoints. The runoff control problem is essentially a stochastic control problem, mainly due to (1) the randomness of natural and man-made environmental effects and (2) some uncertain economic effects. It is obvious, that the role of predictions in the water resources policy making, or, as we term in selecting an optimal runoff control strategy, is highly important. (Throughout this paper the word prediction is understood in a fairly broad context, it might include economic forecasting as well as hydrologic predictions.)

The purpose of the paper is to propose a systems methodology for solving runoff control problems. As the indicated problems are fairly sophisticated, the tools for their solution are, unfortunately not less sophisticated. Here, we consider optimal sequential prediction/control algorithms using which the runoff control problems can be solved. In the first part, the state space description of WR systems is given. Then the measure of system performance is discussed followed by the determination of an optimal prediction algorithm. The third part deals with the problem of optimal stochastic water quality control, the later being an important issue in the general runoff control. The procedures developed can readily be applied for other runoff control problems too. All in all, the author would like to clarify some runoff control problems from a methodological point of view trying to follow in the meantime Einstein's dictum that "an explanation should be as

simple as possible but no simpler".

Before going into the details here is a remark on the terminology used throughout the paper.

Although these are some rather standard terms, borrowed from econometrics, in water resources management to name random WR systems, here we rather use the terms of teh control science, noting that in many instances we are talking about the same thing and nothing else but the 'name of the game' is different. To prove this and to help the reader's orientation in the labirinths of different terminologies Table 1 contains some technical terms of the control field as well as the corresponding ones of the econometrics. The table is an extension of that given by Mehra [19].

STOCHASTIC DISCRETE TIME WR-SYSTEMS

In the past decades considerable efforts were made to describe the behaviour of WR systems by using the so-called 'black-box' approach. The rainfall/runoff models as well as the input/output econometric models give good examples of this statement. The questions 'What is going on inside the box?' 'How can we relate the internal dynamics to the input/output behaviour?', and 'How can we incorporate our a priori knowledge to the model?' lead to the introduction of a new concept. This is the concept of state. Intuitively speaking, the state is the minimal amount of information about the past history of a system which is required to predict its future behaviour [3].

For deterministic systems it means that the state of a system is a set of quantities $x_1(t)$, $x_2(t)$,...$x_n(t)$ which if known at $t = t_o$ are determined for $t \geq t_o$ by specifying the inputs $u(\cdot)$ to the system for $t \geq t_o$. Subsequently we are going to deal with discrete time systems only where the systems are evoling on the discrete time set $T = \{t : t = t_o, t_o + 1,\ldots,t_f\}$, but are continuous in the state. Here t_o is the initial time, t_f is the final time which may be specified $t_f = N$ or 'free' depending upon the problem. For example, in long-range development problems t_f is usually fixed (planning horizon, N) while in real-time WR control problems it is generally unspecified.

As Åström [3] indicates for stochastic systems we naturally cannot require that the future behaviour be uniquely determined by the actual state x.

TABLE 1.

TERMINOLOGY OF SYSTEM IDENTIFICATION
IN ECONOMETRICS AND CONTROL

Control Theory	Econometrics
Input variable	Exogeneous variable
Output variable	Endogeneous variable
Control function	Decision function
Identification	Specification and estimation of the model
Identifiable model	Justidentified or overidentified model
Unidentifiable model	Underidentified model
Noise	Error
White Gaussian noise	Nonautocorrelated normally distributed error
Colored noise	Autocorrelated error
Measurement noise	Sample uncertainty Measurement error
Process noise	Model uncertainty
Stochastic state variables	Natural uncertainty
Performance index	Objective function
Separation theorem	Certainty Equivalent Principle
Markovian model	Distributed Lag Model
Rational z-transform	Rational lag distribution
Impulse response model	Final form model
Filtering	Exponential smoothing
Prediction	Forecasting
Impulse response function Markov parameters Weighting pattern	Impact, interim and total multipliers

A natural extension of the concept of state to stochastic systems would be to require that the probability distribution of the state variable x at future time be uniquely determined by the actual value of the state. Recalling the properties of a Markov process, we thus require that the system be described as a Markov process.

Consider a general dynamic lumped WR system $\mathcal{S}$, as depicted in Fig. 1. the behaviour of which on the discrete time set T can be described by the stochastic difference equation

$$x(t+1) = \mathcal{F}[x(t),u(t),w(t),t] \qquad (1)$$

where x(t) is the n-vector of the system states, a vector in the n-dimensional state space $X \subset \mathbb{R}^n$, $x(t) \in X$; u(t) is the vector of control variables at time t and is an element of the set of admissible controls $U \subset \mathbb{R}^p$; $u(t) \in U$; w(t) is the s-vector of uncertain disturbances (essentially the process noise), $w(t) \in \mathbb{R}^s$, while the given function $\mathcal{F}$ characterizes the properties of the system $\mathcal{S}$. First of all, a few words about the control u. As it will be discussed later the principal aim of applying any control is to bring our system into a desired, either physical or economical (or both) state. There are, however, certain constraints to be considered when choosing a control strategy. Constraints, usually given either by physical laws or by limited resources (monetary or whatsoever). A control which satisfies the control constraints during the entire time interval $[t_o,t_f]$ is called an admissible control and the set of admissible controls is denoted by U.

In order to evaluate the performance of the system quantitatively a performance measure should be established. Here, the performance of the system $\mathcal{S}$ is characterized by a scalar loss function of the type

$$J = \sum_{t=t_o}^{t_f} Q_t[x(t),u(t)]$$

where $Q_t \geq 0$ is a given cost-functional. Clearly the system performance depends on the states reached and the control efforts taken. Further, we call a control $u^* \in U$ optimal if it minimizes the loss J subject to the behaviour of $\mathcal{S}$, given by (1). However, as the states are random variables

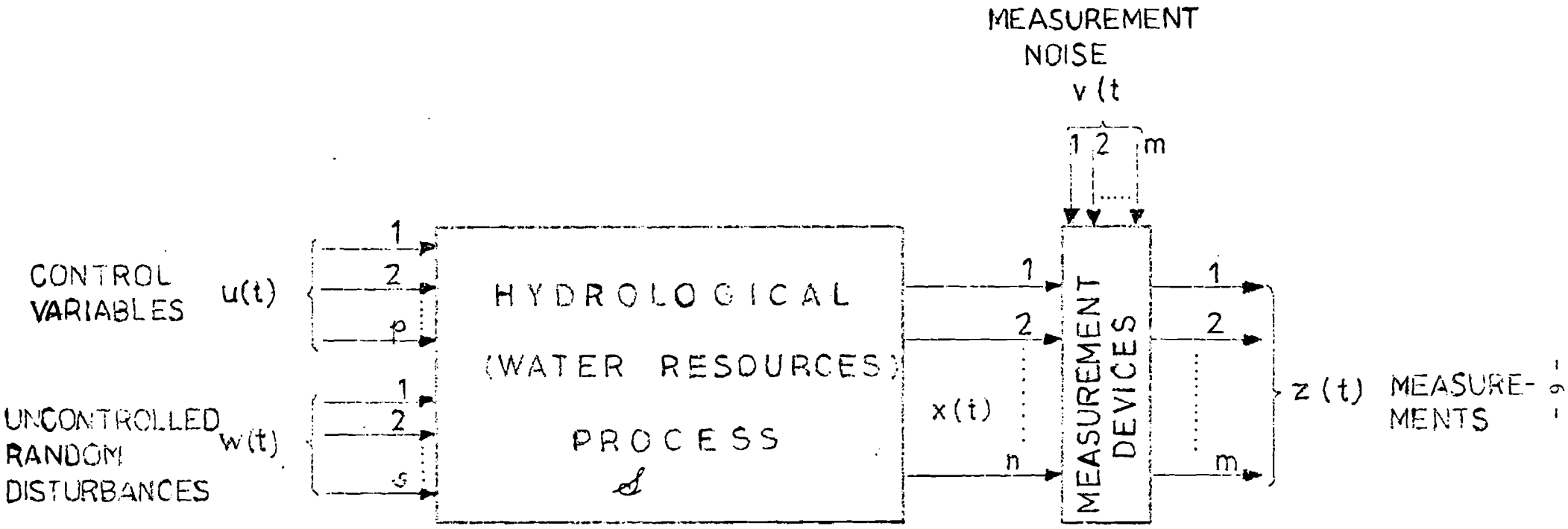

FIGURE 1. HYDROLOGIC PROCESS IDENTIFICATION

the loss J itself is also a random variable, consequently there is no way of defining what is meant by the smallest value of J. Therefore, in the following the expected loss

$$\mathcal{E}\{J\} = \mathcal{E}\left\{ \sum_{t=t_o}^{t_f} Q_t[x(t),u(t)] \right\} \quad , \tag{2}$$

where $\mathcal{E}\{\cdot\}$ denotes the expected value operator, will be chosen as a criterion to be minimized subject to the same constraints as above. To evaluate the system performance one must know exactly the actual state x at time t. In other words, it means that there is no measurement error and sample uncertainty in determining the state. One can expect intuitively that this is rarely the case. There are rather inaccurate 'measurements', z(t), on the state x(t). (Here, the 'measurement' should be understood in a fairly large context, it might mean real measurements, say of rainfall, or the evaluation of certain economic issues such as market effects in water pricing etc.) That is the measurements are given by

$$z(t) = \mathcal{H}[x(t),v(t),t] \tag{3}$$

where $z(t) \in \mathbb{R}^m$ is an m-vector of measurements on the system states (actually the outputs); v(t) is the m-vector of measurement noise and the functional $\mathcal{H}$ is given and characterizes the measurement 'device'. Obviously $m \leq n$ indicating that sometimes not all the state variables are observable. In the case of complete state information z(t) = x(t) while for incomplete state information we have a stochastic measurement vector sequence up to the current time t

$$\mathcal{Z}_t = [z^T(t_o), z^T(t_o + 1), \ldots, z^T(t)]^T$$

consisting of the previous measurement vectors. (The upper T refers to transposition and should not be confused with the discrete time set T.) Clearly, $\mathcal{Z}_t$ is a vector in an m x $(t - t_o + 1)$ dimensional space Z_t, $\mathcal{Z}_t \in Z_t$ and has the 'chain' property

$$\mathcal{Z}_t^T = \left[\mathcal{Z}_{t-1}^T, z^T(t)\right] \quad , \qquad t \in T \tag{4}$$

Now, the stochastic control problem can be formulated as follows:

> Find an admissible control strategy for the WR system described by (1) and (3) such that the criterion (2) is minimal.

As far as the possible control actions are concerned there are two fundamental ways the control can be specified.

(1) Open-loop: $u(t)$, $t = t_o, t_o + 1, \ldots, t_f - 1$, is a fixed time function completely specified before the control starts;

(2) Closed-loop: $u(t)$ is determined as some specific function of $\mathcal{Z}_t$.

In the open-loop case $u(t)$ is a deterministic function. In the closed-loop case $u(t)$, $t = t_o, t_o + 1, \ldots, t_f - 1$ is a stochastic process, as it is a function of a stochastic process $z(\cdot)$. A special case of the closed-loop control occurs when the system states can be observed perfectly, i.e. when $z(t) = x(t)$--no measurement noise. Of course, even in this special case, the control is a stochastic process. In this case, the admissible control strategies are functions which map the state space X into the control space U, $u: X \longrightarrow U$. The closed-loop or feedback optimal control, in case of incomplete state information, is in the form of

$$u^*(t) = \mathcal{L}[\mathcal{Z}_t, t] \quad , \qquad (5)$$

where the functional $\mathcal{L}$ is called the optimal control law or policy. Notice that the optimal control policy specifies how to generate control value at time t from the observed state values up to the time t. In this case the admissible control strategies are functions which map the space of observed states (observed outputs) Z_t into the space of possible control actions U, $u: Z_t \times T \longrightarrow U$. Since the dimension of the space Z_t will increase as t increases, it is much more difficult to determine the control strategy.

As it will be shown later the predictions play a central role in the choice of the proper control strategy. Now, the problem is as follows: the actually observed values of a stochastic process over some interval of

time are given, then the conditional probabilities of all future values should be determined based upon those given values. As a definition, we call a function $\mathscr{P}_t^{\ell}$ ℓ-step-ahead predictor if it maps Z_t into the ℓ-step 'enlarged' state space $X_{+\ell}$

$$\mathscr{P}_t^{\ell} : Z_t \longrightarrow X_{+\ell} \tag{6}$$

The value of this function for a particular value of $\mathcal{Z}_t$ is called an ℓ-step-ahead prediction and is denoted by $\hat{x}(t + \ell \,|\mathcal{Z}_t)$. In the one-step-ahead case the predicted value of the state will be denoted by $\hat{x}(t + 1|\mathcal{Z}_t)$. Obviously, the 'goodness' of prediction must also be evaluated through a given loss function $J(\cdot)$. Now, the prediction problem can be formulated as follows:

> Given the set of measurements $\mathcal{Z}_t$ find and estimate $\hat{x}(t + \ell \,|\mathcal{Z}_t)$ of $x(t + \ell)$, $\ell > 0$, subject to the condition that this estimation (prediction) should minimize the chosen loss function.

Again, as the loss function J itself is a random varibale the minimization should be carried out with respect to the expected loss $\mathscr{E}\{J(\cdot)\}$. The details will be given later.

THE MEASURE OF SYSTEM PERFORMANCE

The selection of the proper performance criterion is a basic issue. Obviously, the better the criterion characterizing the real goals, the more efficient control is achieved. Some examples of setting up performance indices to differnt runoff control problems are given as follows:

> Minimum-Time Problems, where the problem is to transfer a WR system from an arbitrary initial state $x(t_o)$ to a specified target set Ξ, in a minimum time by applying an admissible control $u(t) \in U$, $t \in [t_o, t_f]$. The performance measure to be minimized is

$$\mathscr{E}\{J\} = \mathscr{E}\{t_f - t_o\} \tag{7.a}$$

with t_f the first instant of time when $x(t)$ and Ξ intersect [17] . This is e.g. the case of flash-floods when the flood retention reservoirs should either be emptied or be brought into a prescribed lower level Ξ in a minimum time.

Terminal Control Problems, where the problem is to minimize the deviation of the final state of a WR system from its designed value $d(t_f) \in \mathbb{R}^n$. The performance measure to be minimized might be the following quadratic form

$$\mathcal{E}\{J\} = \mathcal{E}\{\sum_{i=1}^{n} [x_i(t_f) - d_i(t_f)]^2\} \quad .$$

Since positive and negative deviations are equally undesireable, the error is squared. A quadratic performance index is chosen not only because of its easy mathematical handling but because the convergence in the mean square implies convergence in probability [23]. To attach different weights to the different deviations we can insert a real symmetric positive semidefinite nxn matrix Q_o and using matrix notations, the performance measure becomes

$$\mathcal{E}\{J\} = \mathcal{E}\{[x(t_f) - d(t_f)]^T Q_o [x(t_f) - d(t_f)]\}$$

or in a short-hand form

$$\mathcal{E}\{J\} = \mathcal{E}\{\| x(t_f) - d(t_f)\|^2_{Q_o}\} \qquad (7.b)$$

where $\|\cdot\|^2_{Q_o}$ is the squared norm of the deviation vector with respect to Q_o. To illustrate the terminal control problems consider again a storage system consisting of n reservoir and assume that certain irrigation demand $d(t_f)$ should be satisfied by the time t_f. In this example $x(t_f)$ reflects the volume of stored water which can be used for irrigation purposes at t_f, and Q_o consists of the cost associated with the economic losses of the non-sufficient irrigation.

Tracking Problems, where the problem is to maintain the system state $x(t)$ as close as possible to the desired state $d(t) \in \mathbb{R}^n$ in the interval $[t_o, t_f]$. The performance measure of these problems is generally in the form of

$$\mathcal{E}\{J\} = \mathcal{E}\{ \sum_{t=t_o}^{t_f} \|x(t) - d(t)\|^2_{Q_1} \} \qquad (7.c)$$

where Q_1 is a real symmetric positive semi-definite nxn weighting matrix. Note that Q_1 may be time-varying rather than constant. Some of the water quality control problems give excellent examples of these sort of tasks. The desired values for the water quality state variables are generally given by standards. For example, let $d_1(t)$ be that for the Biochemical Oxygen Demand (BOD)$\triangleq$ $x_1(t)$ and let $d_2(t)$ be the standard value for the Dissolved Oxygen (DO)$\triangleq$ $x_2(t)$ to be maintained during $[t_o, t_f]$. Moreover, let q_{11} resp. q_{12} be the cost associated with the BOD resp. DO differences. Then the objective function to be minimized is

$$\mathcal{E}\{J\} = \mathcal{E}\{ \sum_{t=t_o}^{t_f} [q_{11}(x_1(t) - d_1(t))^2 + q_{12}(x_2(t)) - d_2(t))^2] \}$$

Obviously, in this case

$$Q_1 = \begin{bmatrix} q_{11} & 0 \\ 0 & q_{12} \end{bmatrix} .$$

(More about the water quality control will be given later)

Minimum-Control-Effort Problems, where the problem is to transfer a WR system from an arbitrary initial state $x(t_o)$ to a specified target set Ξ, with a minimum expenditure of control effort. Obviously, the control to be applied must also be admissible; $u(t) \in U$, $t \in [t_o, t_f]$. The general performance measure of this kind of problem is in the form of

$$\mathcal{E}\{J\} = \mathcal{E}\left\{ \sum_{t=t_o}^{t_f} \|u(t)\|^2_{Q_2} \right\} \qquad (7.d)$$

where Q_2 is a real symmetric positive definite weighting matrix consisting of the cost of 'energy' consumed by applying a particular control policy. Of course, the elements of Q_2 may be functions of time if it is desired to vary the weighting on control effort expenditure during the interval $[t_o, t_f]$. To illustrate this problem is straightforward. One might think of e.g. the long-range dynamic water resources planning problems to be carried out with minimum energy.

Unfortunately, the real-world problems are rarely so simple as the above listed ones. They are more

Complex Problems, where there are multiple, sometimes conflicting, objectives to be reached. For example, a decision maker wants to control a WR system in such a way that the particular final states of the system be close to their desired value and in the meantime the deviations during the operating period also be small by applying as minimal control efforts as possible. In this case, the performance measure

$$\mathcal{E}\{J\} = \mathcal{E}\{\|x(t_f) - d(t_f)\|^2_{Q_o} + \sum_{t=t_o}^{t_f-1} [\,\|x(t) - d(t)\|^2_{Q_1} + \|u(t)\|^2_{Q_2}\,]\} \qquad (7.e)$$

could be used. In the following we assume that the weighting matrices Q_o, Q_1 and Q_2 are being independent of time, noting however, that the results developed are valid for time-dependent matrices too.

By comparing eqs. (7) with (2) the specialities become apparent.

LINEAR STOCHASTIC WR-SYSTEMS

In hydrology and water resources development the linear models are of fundamental importance since most of

the practical problems can successfully be tackled with the assumption of linearity. For linear systems the functionals $\mathcal{F}$ and $\mathcal{H}$ are linear. It is assumed then,that the system $\mathcal{S}$ is governed by the linear stochastic difference equation

$$\mathcal{S}: \begin{cases} x(t+1) = \phi x(t) + \Gamma u(t) + w(t) & (8) \\ z(t) = Hx(t) + v(t) & (9) \end{cases}$$

where again $t \in T$, $x \in X \subset \mathbb{R}^n$, $u \in U \subset \mathbb{R}^p$, $z \in \mathbb{R}^m$ and the uncertainties $\{w(t) : t \in T\}$, $\{v(t) : t \in T\}$ are multivariate Gaussian white noise sequences with zero mean values and the covariances

$$\text{cov } [w(t),w(t)] = \mathcal{E}\{w(t)w^T(\tau)\} = R_1 \delta_{t\tau} \quad (10.a)$$

$$\text{cov } [v(t),v(t)] = \mathcal{E}\{v(t)v^T(\tau)\} = R_2 \delta_{t\tau} \quad (10.b)$$

$$\text{cov } [v(t),w(t)] = \mathcal{E}\{v(t)w^T(\tau)\} = 0 \quad (10.c)$$

where $\delta_{t\tau}$ denotes the Kronecker delta. In (8) Φ is an nxn nonsingular matrix called state transition matrix and Γ is the nxp control gain matrix while in (9) the mxn matrix H is called measurement matrix. In case of time varying systems the matrices Φ, Γ, H, R_1 and R_2 depend on time. Here, for notational simplicity we consider those matrices with constant elements, noting however, that the subsequently developed algorithm are also valid for time varying cases, the only thing we ought to do is just to insert the time as an argument of matrices. As it is indicated by (10.c) we assume that the uncertainties are independent of each other. Anyway, this is an obvious fact. Moreover, it is assumed that w(t) and v(t) are independent of x(t) and the initial state $x(t_o)$ is normal with

$$\mathcal{E}\{x(t_o)\} = \hat{x}(t_o) \quad (11.a)$$

$$\text{cov}[x(t_o),x(t_o)] = \mathcal{E}\{(x(t_o) - \hat{x}(t_o))(x(t_o) - \hat{x}(t_o))^T\} = P(t_o) \quad (11.b)$$

One can argue about the basic assumptions of being the noise process Gaussian white sequences with known covariance matrices. Specially he is right in the second issue because it is hard to say that those values are known in dealing with hydrologic time series. To overcome this difficulty,

an adaptive noise covariance matrix algorithm is introduced in [29]. As concerns the handling of 'colored' noises, by augmenting properly the state vector with the dependent part of the processes, the resulting residual is a white sequence (for details see Porebski [20]).

As far as the performance measure is concerned, it might be selected from eqs.(7) - always carefully considering the objective of the control to be achieved.

OPTIMAL SEQUENTIAL PREDICTION ALGORITHMS FOR LINEAR WR-SYSTEMS

In large river basin management one of the most crucial issues, if not that one, is the lack of reliable predictions. Predictions for the future resources as well as those for the future demand. The need for a reliable prediction scheme is as old as civilization itself and dates back to the good old Egyptian days. Nowadays, to achieve reliable predictions first of all a reliable remote sensing network should be established which then makes the quick collection of information possible, upon which the prediction is based. Here, we do not touch this issue - some details can be found in the WMO Casebook [34] or in [28]. Clearly, for real-time operation of water resource systems, small computers are preferable. Hence, our prediction algorithms must be suited for these small computers. But how? The answer is simple: Using recursive prediction algorithms in which there is no need to store all the past measurements for the purpose of predicting future behaviour of the time series in question. Moreover, these algorithms offer:

(1) The treatment of the information of each measurement in a sequential manner allows for on-line implementations (e.g. by means of data-acquisition by automatic measurement devices connected in real-time mode with a central processor)

(2) Time variable parameters and different types of disturbances can easily be treated. Hence, the suitable prediction scheme should preferably satisfy the following requirements:

- -it should be mathematically tractable
- -it should be easily implemented for small computers
- -it should be generally applicable

-it should yield an 'optimum' prediction
-it should be adaptable to the varying environmental conditions
-it should yield an acceptable convergence.

As it will be shown soon, the state space based prediction models are good candidates for fulfilling the above requirements. We mention in advance that the prediction scheme given below can also be applied for economic forecasting. A somewhat similar approach to economic forecasting is discussed in [11].

It is well-known (see e.g. in Dooge [10]) that a fairly large class of lumped hydrologic systems (e.g. rainfall excess/suface runoff, runoff/runoff transformations of flood-routing etc.) can be described by a single input-single output discrete convolution type of model

$$y(t) = \sum_{\tau=0}^{q} g(\tau)h(t - \tau) \quad , \qquad t,\tau \in T$$

where $h(t)$ is the input of the system (either controllable or not), $g(t)$ is the impulse response of the system having finite memory q and $y(t)$ is the output process. In practice, however, we have only noise computed measurements

$$z(t) = y(t) + v(t)$$

where $v(t)$ is a Gaussian white noise process. Hence, the model is

$$z(t) = \sum_{\tau=0}^{q} g(\tau)h(t - \tau) + v(t) \quad , \qquad t,\tau \in T \tag{12}$$

Note, that although the system was assumed linear, in case of slight non-linearities, the noise process $v(\cdot)$ might be sought as a term including those non-linear disturbances. By defining the vectors

$$\begin{aligned} H(t) &= [h(t),h(t - 1),\ldots,h(t - q)] \quad , \\ x &= [g(o),g(1),\ldots,g(q)]^T \end{aligned} \tag{13}$$

Eq. (12) becomes

$$z(t) = H(t)x + v(t) \tag{13}$$

This equation can be looked upon as a measurement equation for the above defined state vector x, c.f. (9). The missing state equation can also be introduced without much difficulty. As it was assumed that the system is time invariant, its impulse response $g(\cdot)$ does not change with time, i.e. it is assumed to be the same at time $t + 1$ as at time t. Using the above defined state vector, x, this statement can be formulated as

$$x(t + 1) = \Phi x(t) + w(t) \tag{14}$$

where we consider the uncertainties by adding a white Gaussian noise term $w(t)$. Clearly, (14) plays the role of the state equation, c.f. (8), with $\Phi = I$, the identity matrix and $\Gamma = 0$. (Though it is absolutely unnecessary here to indicate Φ, but because later on in the water quality control problem we deal with the $\Phi \neq I$ case we still use the general formulation of (14). For notational simplicity, on the other hand, $H(t)$ will be denoted by H, bearing in mind that in this case it is obviously time variant.) We also assume that the noises have the properties as those of (10).

Now, in the sequential prediction scheme first we have to estimate the state based upon the past and the newest measurements and then to give a prediction for the output process.

Assume that given a prior estimate $\hat{x}(t|t - 1)$ of the system state $x(t)$ at $t \in T$ which is based on previous measurements up to $t - 1$. Then we seek an updated estimate $\hat{x}(t|t)$ which takes into account the new measurement $z(t)$ at $t \in T$. Consider this updated estimation as being the linear combination of the previous state and the new (noisy) measurement

$$\hat{x}(t|t) = \tilde{K}(t)\hat{x}(t|t - 1) + K(t)z(t) \tag{15}$$

where $K(t)$ and $\tilde{K}(t)$ are time varying weighting matrices as yet unsepecified. In fact, we wish to minimize, in a certain sense, the prediction error

$$\tilde{x}(t|t) = \hat{x}(t|t) - x(t) \tag{16}$$

Introducing (13) into (15) and utilizing the 'whiteness' of the noise process, it can readily be seen that (15) is an unbiased estimation only if $\tilde{K}(t) = I - K(t)H$. Hence, the state estimation, $\hat{x}(t|t)$ using the new measurement, $z(t)$ is

$$\hat{x}(t|t) = \hat{x}(t|t-1) + K(t)[z(t) - H\hat{x}(t|t-1)] \tag{17}$$

where $K(t)$ is still unspecified, and the initial condition at $t = t_o$ for the state estimation is given by (11.a) since $\hat{x}(t_o|t_o) = \hat{x}(t_o)$. As a measure of the goodness of the estimation, we use the covariance matrix $P(\cdot)$ of the prediction error defined as

$$P(t|t) = \mathcal{E}\{\tilde{x}(t|t)\tilde{x}^T(t|t)\} \tag{18}$$

the initial condition of which is given by (11.b), since $P(t_o|t_o) = P(t_o)$. It can also easily be seen that the covariance matrix of $\tilde{x}(t|t)$ can be projected from that of $\tilde{x}(t|t-1)$ as

$$P(t|t) = (I - K(t)H)P(t|t-1)(I - K(t)H)^T + K(t)R_2K^T(t) \tag{19}$$

As far as the loss function is concerned we define it similarly to (7.c), i.e.

$$J_t \triangleq J(\tilde{x}(t|t)) = \|\tilde{x}(t)\|_Q^2$$

where Q is any positive semidefinite matrix and for the sake of simplicity let $Q = I$. Having defined the loss function we seek that estimate $\hat{x}(t|t)$ of $x(t)$--in other words, that form of the yet unspecified $K(t)$ - which minimizes the expected loss (as sometimes called Bayesian risk) $B_t = \mathcal{E}\{J_t\}$.

Since B_t is the trace of the error covariance matrix (18), the problem is to minimize the trace norm $\|P(t|t)\|$ of $P(t|t)$, i.e. the length of the estimation error vector. Using the properties of matrix derivatives, it can be seen that the weighting matrix K(t) can be obtained from

$$\frac{\partial}{\partial K(t)}\|P(t|t)\| = 0$$

as

$$K(t) = P(t|t-1)H^T[HP(t|t-1)H^T + R_2]^{-1} \qquad (20)$$

which is used to refer to as the Kalman gain matrix. Now, the next step is the extrapolation of the state variable. Consider the one-step-ahead case, when $\ell = 1$. In the process model (14) $w(\cdot)$ is a white noise sequence so no more information on it is contained in $z(\cdot)$ and thus the best prediction of $w(\cdot)$ that can be made from $z(\cdot)$ is its mean value, i.e. 0, consequently, the one-step-ahead prediction of the state vector, given observations up to $t \in T$ is

$$\hat{x}(t+1|t) = \Phi\hat{x}(t|t) \quad . \qquad (21)$$

The propagation of prediction errors $P(t|t) \longrightarrow P(t+1|t)$ can be determined by computing the predicted error covariance matrix as

$$P(t+1|t) = \mathcal{E}\{\tilde{x}(t+1|t)\tilde{x}^T(t+1|t)\} \quad .$$

Using (21) and (14) and utilizing the fact that the prediction error and model error are independent of each other, we obtain

$$P(t+1|t) = \Phi P(t|t)\Phi^T + R_1 \qquad (22)$$

Using the formulas in the order of (21), (22)--and then with t : = t + 1-- (20)(17) and (19) the celebrated Kalman filter algorithms [14][16] are obtained. The algorithms should be used sequentially, t = 1,2,..., starting with the given initial conditions at time t_o. The complete algorithms,

together with the initial conditions, are summarized in Table 2. We mentioned that in order to obtain the best estimate of the state we actually should compute the conditional distribution of $x(t + 1)$ given $\mathcal{Z}_t$. As the distribution is Gaussian, it is completely given by the mean $\hat{x}$ and the covariance P. Another interesting fact is that the P and K can be precomputed, so there is no need to store all the past observations as the calculation progresses. Due to this fact, the conditional distribution of $x(t + 1)$ given $\mathcal{Z}_t$ is uniquely given by the conditional mean $\hat{x}(t + 1|t) = \mathcal{E}[x(t + 1)|\mathcal{Z}_t]$. If f denotes the conditional density we thus have

$$f[x(t + 1)|\mathcal{Z}_t] = f[x(t + 1)|\hat{x}(t + 1|t)] ,$$

which means that the conditional mean is a sufficient statistic for the conditional distribution of $x(t + 1)$ given $\mathcal{Z}_t$. In other words, it means that the knowledge given of $\hat{x}(t + 1|t)$ is equivalent to the knowledge of $\mathcal{Z}_t$. This arguement can be extended for the $\ell > 1$ case as well. In fact, the collection of algorithms summerized in Table 2. is the prediction functional $\mathcal{P}_t^1$ of (6). It can be seen that the algorithms fulfill the requirements for a suitable prediction scheme laid down previously.

Although it was assumed that the system is truly time variant, it should be stressed, however, that the above formulation can be used for describing slightly time variant systems which are, due to seasonal changes most common in hydrology. The system behaviour can however be considered as being time invariant within a well defined "data window". This data window, of course, is of a moving type. As concerns the length of the moving data window, it is essentially equal to the memory of the system and might be estimated from cross-correlation analysis performed on the input/output processes. The moving data window creates the basis of the sequential prediction. Up to this point we assumed that the noise sequences are Gaussian white ones with known statistics. However, this is far from being true and the noise variance estimation should somehow be included in the algorithms discussed. This can be done by an adaptive algorithm developed by Sage and Husa [24]. For details, the reader should refer to [29]. In that paper examples are also given to illustrate the utility of the proposed prediction scheme using simulated sequences. Finally, we note again that the algorithms can be

TABLE 2. OPTIMAL SEQUENTIAL PREDICTION ALGORITHM

System Model	$x(t+1) = \Phi x(t) + w(t)\ , \quad w(t) \sim N(0, R_1)$
Measurement Model	$z(t) = Hx(t) + v(t)\ , \quad v(t) \sim N(o, R_2)$
Initial Conditions	$\mathcal{E}\{x(t_o)\} = \hat{x}(t_o)$ $\text{cov}[x(t_o), x(t_o)] = P(t_o)$
Other Assumption	$\mathcal{E}\{v(t)w^T(\tau)\} = 0$
State Prediction	$\hat{x}(t+1\|t) = \Phi\hat{x}(t\|t)$
Predicted Error Covariance Matrix	$P(t+1\|t) = \Phi P(t\|t)\Phi^T + R_1$
Predictor Gain Algorithm	$K(t+1) = P(t+1\|t)H^T[HP(t+1\|t)H^T + R_2]^{-1}$
State Estimation Using the New Measurement	$\hat{x}(t+1\|t+1) = \hat{x}(t+1\|t) + K(t+1)[z(t+1) - H\hat{x}(t+1\|t)]$
Error Covariance Matrix Algorithm	$P(t+1\|t+1) = (I - K(t+1)H)P(t+1\|t)(I - K(t+1)H)^T + K(t+1)R_2K^T(t+1)$

used for economical forecasting as well. In this respect the reader should consult Chow [6]. It might also be mentioned that the above algorithms can also be derived in the framework of Bayesian statistics. Schweppe [25] gives an excellent treatment of the BDT applied to dynamic state estimation problems.

OPTIMAL STOCHASTIC WATER QUALITY CONTROL

The water quality control is one of the fundamental categories of the general runoff control. In the following we will discuss a stochastic water quality control model which utilizes the previously discussed optimal stochastic state estimation of the state variables involved and the dynamic programming technique.

It is well-known that the Biochemical Oxygen Demand+) (BOD) and the Dissolved Oxygen (DO) concentration give a fairly good measure for characterizing the quality of a polluted river. If it is assumed that

- -The width and depth of the river are small compared to the length on the section $[r_o, r_f]$ considered and
- -The effects of longitudinal dispersion along the length of the river are small

then, by the mass-balance, the following partial differential equations are obtained:

$$\frac{\partial B(r,t)}{\partial t} + a(r)\,\frac{\partial B(r,t)}{\partial r} = -K_r B(r,t) \qquad (23.a)$$

$$\frac{\partial D(r,t)}{\partial t} + a(r)\,\frac{\partial D(r,t)}{\partial r} = -K_a D(r,t) - K_d B(r,t) + K_a D_s \quad . \qquad (23.b)$$

These are the famous Streeter-Phelps equations [9] [27] where

- r is the distance downstream from the reference point r_o, $r \in [r_o, r_f]$;
- t denotes time;
- $B(r,t)$ is the BOD concentration in [mg/ℓ],
- $D(r,t)$ is the DO concentration in [mg/ℓ],
- K_r is the BOD removal (decay) coefficient [day^{-1}], (say BOD removal by sedimentation),

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+) The BOD is usually defined as the amount of oxygen required by bacteria while stabilizing decomposable organic matter with the help of dissolved oxygen in the water [9].

K_d is the deoxygenation coefficient [day^{-1}],

K_a is the reaeration coefficient [day^{-1}],

D_s is the saturation level of D,

$a(r)$ stream velocity at r.

In fact, the above model describes the self-purification process of the polluted river. (Anyway, the author is aware of the fact that the Streeter-Phelps model has some drawbacks, e.g. in case of industrial wastes. However, the methodology developed below is general enough to handle more sophisticated pollution situations as well and the only thing to do is to add some more balance equations, e.g. for Dissolved Organic Carbon, for the suspended biomass etc.) The above model is of distributed type and can be pretty well applied for space dependent problems, such as e.g. estuary pollution studies. For small rivers, however, a simplified lumped parameter BOD-DO interaction model can be set up by adding a third assumption to those of the distributed model. The assumption is that

-The river can be decoupled into k non-overlapping reaches $\mathcal{R}_i$, $\bigcup_{i=1}^{k} \mathcal{R}_i = [r_o, r_f]$, in such a way that BOD and DO concentrations do not change with respect to $r \in \mathcal{R}_i$ within that particular reach $\mathcal{R}_i$, i.e.

$$\frac{\partial B(r,t)}{\partial r} = 0 \quad , \qquad \frac{\partial D(r,t)}{\partial r} = 0 \quad , \forall r \in \mathcal{R}_i \quad .$$

In other words, a reach is defined as a stretch of the river of some convenient length or of which there is only one treatment facility of any kind. Hence, with this assumption the lumped BOD-DO dynamics is described by the well-known [9] relations

$$\frac{dB(t)}{dt} = - K_r B(t) \tag{24.a}$$

$$\frac{dD(t)}{dt} = - K_a D(t) - K_d B(t) + K_a D_s \tag{24.b}$$

which characterize the pollution situation at some average point $\bar{r} \in \mathcal{R}_i$ in the reach. Until now the effect/addition of effluents have not been taken into account. This can be done by defining the control vector $u(t) = [u_1(t), u_2(t)]^T$, where $u_1(t)$ is for the dumping control of effluents from the sewage treatment plant and $u_2(t)$ is for artificial aeration carried

out along the reach, if there is any. The first control might mean, say, the operation rule for a retention (depletion) reservoir located right after the treatment plant and the second control is the timing schedule for the aeration brushes. For sure, the controls belong to the set of admissible controls $u(t) \in U$. As far as the stochastic effects, such as random disturbances caused by turbulence, model uncertainties, etc., are concerned they were also ignored. Obviously, they can also be easily taken into account by defining the random vector $w(t) = [w_1(t), w_2(t)]^T$ which is assumed again to be a zero mean Gaussian white noise sequence with (10.a). And now, to complete the formulation of the water quality control problem we define the state variables and the performance measure. As we already mentioned in connection with the tracking problems, there are certain water quality standards to be satisfied during the control period. Let those be denoted by $d = [d_1, d_2]^T$ with respect to the BOD and Do concentrations. (They might be time varying but for notational simplicity we assume them to be constant.) Now, we define the state $x(t) = [x_1(t), x_2(t)]^T$ as being composed from the deviation from the desired BOD level $d_1, x_1(t) = B(t) - d_1$, and from $x_2(t) = D_s - D(t)$, which is called oxygen deficit. Clearly, the standard value d_2 corresponding to $x_2(t)$ might be set to zero, since one of the objectives of the water quality control is to maintain high DO values whenever it is possible, e.g. $d_2 = 0$. So, using (24) the complete process model becomes

$$\frac{dx(t)}{dt} = \Phi x(t) + \Gamma u(t) + w(t) \qquad (25)$$

where

$$\Phi = \begin{bmatrix} -K_r & 0 \\ K_d & -K_a \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

since the more the artificial aeration the less the oxygen deficit, and reversely.

Due to the fact, that in practice we have discrete measurement, we hearafter deal with the discrete time model

$$x(t+1) = \Phi x(t) + \Gamma u(t) + w(t) \tag{26}$$

where Φ and Γ are the same matrices as in (25). Eq.(25) is really a linear stochastic difference equation, c.f.(8). The system dynamics is depiced in Fig.2. We mention here, that Φ may be time dependent, but it does not change the picture too much at least from a technical point of view. It can also easily be proved that the system is state controllable. As far as the state measurements are concerned, the situation is that the evaluation of BOD concentration usually needs several days in a laboratory and to determine the optimal real-time control policy instantaneous DO measurements are available only. Again, the noisy measurment at one particular point are assumed to be in the form of (c.f.(9))

$$z(t) = Hx(t) + v(t) \tag{27}$$

where $H = [0,1]^T$ and $v(t) = [0,v_2(t)]^T$ is as (10.b). Surely, the uncertainties have the property of (10.c).

A suitable performance measure by which various control strategies can be compared in order to find the optimal one is in the form of (7.e) or, because of the tricky choice of the state variables, even simpler as

$$\mathcal{E}\{J\} = \mathcal{E}\{\|x(N)\|^2_{Q_o} + \sum_{t=t_o}^{N-1} [\|x(t)\|^2_{Q_1} + \|u(t)\|^2_{Q_2}]\} \tag{28}$$

i.e. the operational time horizon N is fixed and we assume that it progresses with the same 'speed' as the data collection. In other words, we are always optimizing our decisions N step ahead. The elements of the matrices Q_o, Q_1 and Q_2 are the costs related to the treatment efficiency and efforts, and have the same properties as those of (7.e).

And now comes the solution. Up to time t the measurements $\mathcal{Z}_{t-1} = [z^T(t_o), z^T(t_o+1), \ldots, z^T(t-1)]^T$ have been observed and the problem is to determine the control strategy u such that the criterion (28) is minimal. The criterion can be split up into two parts as

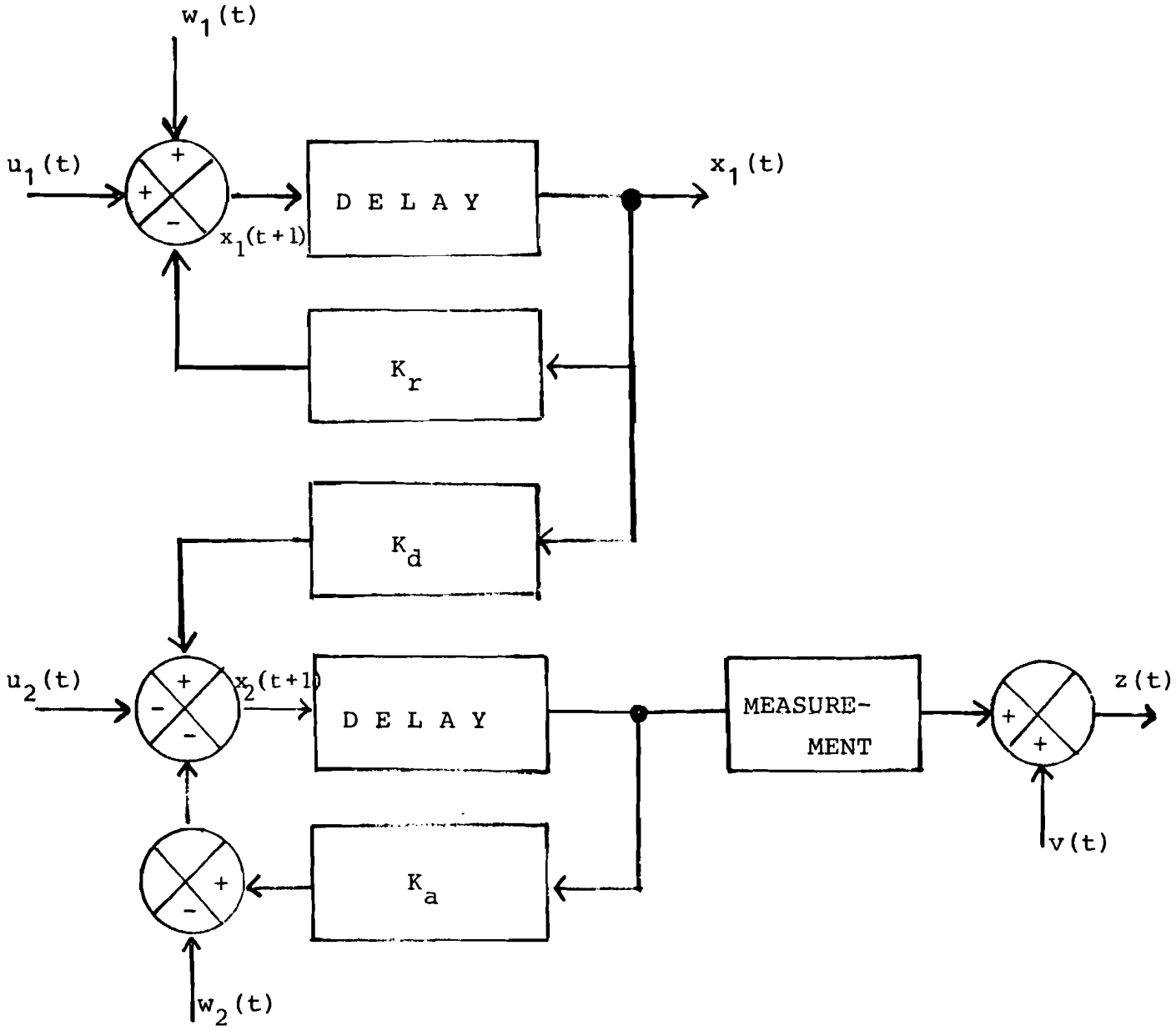

FIGURE 2. THE DYNAMICS OF THE DISCRETE TIME LUMPED STOCHASTIC WATER QUALITY CONTROL SYSTEM

$$\mathcal{E}\left\{\sum_{\tau=t_o}^{t-1}\left[\|x(\tau)\|_{Q_1}^2 + \|u(\tau)\|_{Q_2}^2\right]\right\} +$$

$$+\mathcal{E}\left\{\|x(N)\|_{Q_o}^2 \sum_{\tau=t}^{N-1}\left[\|x(\tau)\|_{Q_1}^2 + \|u(\tau)\|_{Q_2}^2\right]\right\} .$$

Clearly, the second term depends on u(t) only. Assuming that a unique minimum exists, it follows from the Optimization Lemma A.1.(see Appendix) that

$$\min_{u(t)} \mathcal{E}\left\{\|x(N)\|_{Q_o}^2 + \sum_{\tau=t}^{N-1}\left[\|x(\tau)\|_{Q_1}^2 + \|u(\tau)\|_{Q_2}^2\right]\right\} =$$

$$= \mathcal{E}\left\{\min_{u(t)} \mathcal{E}\left\{\|x(N)\|_{Q_o}^2 + \sum_{\tau=t}^{N-1}\left[\|x(\tau)\|_{Q_1}^2 + \|u(\tau)\|_{Q_2}^2\right] | \mathcal{Z}_{t-1}\right\}\right\} , \qquad (29.a)$$

where $\mathcal{E}\{\cdot|\mathcal{Z}_{t-1}\}$ denotes conditional expectation given $\mathcal{Z}_{t-1}$, the first $\mathcal{E}$ of the right hand side denotes expectation with respect to the distribution of $\mathcal{Z}_{t-1}$, and the minimum is taken with respect to all admissible strategies which express u(t) as a function of $\mathcal{Z}_{t-1}$. Repeating the given above for (29.a) for t = t + 1,t + 2,...,N - 1 under the assumption that all the unique minima with respect to u(t),u(t + 1),...,u(N - 1) exist, we obtain

$$\min_{u(t),\ldots,u(N-1)} \mathcal{E}\left\{\|x(N)\|_{Q_o}^2 + \sum_{\tau=t}^{N-1}\left[\|x(\tau)\|_{Q_1}^2 + \|u(\tau)\|_{Q_2}^2\right]\right\} = \mathcal{E}\left\{\mathcal{V}(\mathcal{Z}_{t-1},t)\right\}$$

where

$$\mathcal{V}(\mathcal{Z}_{t-1},t) = \min_{u\in U} \mathcal{E}\left\{\|x(N)\|_{Q_o}^2 + \sum_{\tau=t}^{N-1}\left[\|x(\tau)\|_{Q_1}^2 + \|u(\tau)\|_{Q_2}^2\right] | \mathcal{Z}_{t-1}\right\} \qquad (29.b)$$

i.e. in a detailed form

$$\mathcal{V}(\mathcal{Z}_{t-1},t) = \min_{u(t)} \mathcal{E}\left\{\|x(t)\|_{Q_1}^2 + \|u(t)\|_{Q_2}^2 + \min_{u(t+1)} \mathcal{E}\left\{\|x(t+1)\|_{Q_1}^2 +\right.\right.$$

$$\left.\left. + \|u(t+1)\|_{Q_2}^2 + \min_{u(t+2)} \mathcal{E}\left\{\cdots | \mathcal{Z}_{t-1}\right\}\cdots\right\}\right\} .$$

Using the (29.b) definition of $\mathcal{V}(\cdot)$ for t + 1 we have

$$\mathcal{V}(\mathcal{Z}_t, t+1) = \min_{u(t+1)} \mathcal{E}\left\{ \|x(t+1)\|^2_{Q_1} + \|u(t+1)\|^2_{Q_2} + \min_{u(t+2)} \mathcal{E}\left\{ \cdots \mid \mathcal{Z}_t \right\} \cdots \right\} .$$

$$= \min_{u \in U} \mathcal{E}\left\{ \|x(N)\|^2_{Q_1} + \sum_{\tau=t+1}^{N-1} \left[\|x(\tau)\|^2_{Q_1} + \|u(\tau)\|^2_{Q_2} \right] \mid \mathcal{Z}_t \right\} \tag{29.c}$$

combining Eqs. (29.b) and (29.c) we obtain the following functional equation for $\mathcal{V}$:

$$\mathcal{V}(\mathcal{Z}_{t-1}, t) = \min_{u \in U} \mathcal{E}\left\{ \|x(t)\|^2_{Q_1} + \|u(t)\|^2_{Q_2} + \mathcal{V}(\mathcal{Z}_t, t+1) \mid \mathcal{Z}_{t-1} \right\} \tag{30}$$

which is called Bellman equation and in fact is the result of the Principle of Optimality[+]. The recurrence functional equation (30) creates the basis for the dynamic programming optimization in order to find the optimal control strategy. The equation itself is fairly complicated due to the fact that the dimension of $\mathcal{Z}_t$ increases with t. However, (30) can be simplified by taking into account the system structure. To evaluate (30) the conditional distributions of x(t) and $\mathcal{Z}_t$ given $\mathcal{Z}_{t-1}$ should be determined. It follows from (4) that the first components of $\mathcal{Z}_t$ are identical to those of $\mathcal{Z}_{t-1}$, i.e. to determine the conditional distribution of $\mathcal{Z}_t$ given $\mathcal{Z}_{t-1}$ it is sufficient to know the distribution of z(t) given $\mathcal{Z}_{t-1}$. Due to (27), however, it is determined by the conditional distribution of x(t) given $\mathcal{Z}_{t-1}$, which is uniquely given by the conditional mean $\hat{x}(t|t-1) = \mathcal{E}\{x(t)|\mathcal{Z}_{t-1}\}$, since this is a sufficient statistic (c.f. the arguments given below (22)). In other words, it means that the conditional mean $\hat{x}(t|t-1)$ can be introduced instead of $\mathcal{Z}_{t-1}$ in (30), i.e.

+) An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must consitute an optimal policy with regard to the state resulting from the first decision [4].

$$\mathcal{W}(\hat{x}(t|t-1),t) = \mathcal{V}(\mathcal{Z}_{t-1},t)$$

$$= \min_{u(t),\ldots,u(N-1)} \mathcal{E}\left\{\|x(N)\|^2_{Q_o} + \right. \tag{31}$$

$$\left. + \sum_{\tau=t}^{N-1} \left[\|x(\tau)\|^2_{Q_1} + \|u(\tau)\|^2_{Q_2}\right] | \hat{x}(t|t-1)\right\} \quad ,$$

and the Bellman equation becomes

$$\mathcal{W}(\hat{x}(t|t-1),t) = \min_{u\in U} \mathcal{E}\left\{\|x(t)\|^2_{Q_1} + \|u(t)\|^2_{Q_2} + \mathcal{W}(\hat{x}(t+1|t),t+1) | \hat{x}(t|t-1)\right\} \tag{32}$$

Despite its form, this is a considerable simplification because the dimension of $\hat{x}$ is constant and usually much lower than that of $\mathcal{Z}_t$. Clearly, the initial condition for (32) is

$$\mathcal{W}(\hat{x},N) = \mathcal{E}\left\{\|x(N)\|^2_{Q_o} | \hat{x}\right\} \quad . \tag{33}$$

As you may recall, we came to the conclusion in the previous chapter dealing with the prediction algorithms, that the conditional distribution of $x(N)$ given $\mathcal{Z}_{N-1}$ is normal with mean $\hat{x}$ and covariance $P(N)$. Applying a simple relation given in the Appendix (see there as Lemma A.2) we have for (33) that

$$\mathcal{W}(\hat{x},N) = \|\hat{x}\|^2_{Q_o} + \|Q_o P(N)\| \tag{34}$$

which is clearly a solution for the Bellman equation at time $t = N$. As far as the other time instants are concerned, we assume the solution in a similar quadratic form

$$\mathcal{W}(\hat{x},t) = \|\hat{x}\|^2_{S(t)} + s(t) \tag{35}$$

where $S(t)$ and $s(t)$ are as yet unspecified. Eq.(35) is apparently true for $t = N$ and gives (34). By induction, it is assumed that (35) holds for $t + 1$ and then it will be shown that it holds for t. To evaluate (32) the conditional distributions of $x(t)$ and $\hat{x}(t+1|t)$ given $\mathcal{Z}_{t-1}$ should be known. It is known from the prediction study that the conditional distribution of

$x(t)$ given $\mathcal{Z}_{t-1}$ is normal with mean $\hat{x}(t|t-1)$ and covariance matrix $P(t|t-1)$. Therefore, the first term of (32), by the use of Lemma A.2, becomes

$$\mathcal{E}\left\{\|\hat{x}(t)\|^2_{Q_1} | \mathcal{Z}_{t-1}\right\} = \|\hat{x}(t|t-1)\|^2_{Q_1} + \|Q_1 P(t|t-1)\| \quad . \tag{36}$$

Considering eqs.(21) and (17) and the presence of the control function at the same time, we have

$$\hat{x}(t+1|t) = \Phi\hat{x}(t|t-1) + \Gamma u(t) + \Phi K(t)[z(t) - H\hat{x}(t|t-1)] \tag{37.a}$$

Since the sequence $z(t) - H\hat{x}(t|t-1)$, called 'innovations' [13], is again a Gaussian white noise and its conditional distribution given $\mathcal{Z}_{t-1}$ is normal with zero mean and covariance matrix $HP(t|t-1)H^T + R_2$ we have for the statistics of (37)

$$\mathcal{E}\left\{\hat{x}(t+1|t) | \mathcal{Z}_{t-1}\right\} = \Phi\hat{x}(t|t-1) + \Gamma u(t) \tag{37.b}$$

$$\mathrm{cov}[\hat{x}(t+1|t) | \mathcal{Z}_{t-1}] = \Phi K(t)\left[HP(t|t-1)H^T + R_2\right]K^T(t)\Phi^T \tag{37.c}$$

Using the above results the Bellman equation (32) becomes

$$\begin{aligned}\mathcal{W}(\hat{x}(t|t-1),t) = \min_{u\in U} \Big\{ &\|\hat{x}(t|t-1)\|^2_{Q_1} + \|Q_1 P(t|t-1)\| + \|u(t)\|^2_{Q_2} + \\ &+ \|\Phi\hat{x}(t|t-1) + \Gamma u(t)\|^2_{S(t+1)} \\ &+ \|S(t+1)\Phi K(t)[HP(t|t-1)H^T + R_2]K^T(t)\Phi^T\| + s(t+1)\Big\}\end{aligned} \tag{38}$$

And now, if we are looking for an optimal feeback control in the form like (5), or more specifically using the form $u + L\hat{x}$, where L is of course unknown by completing squares in (38) and then elaborating some elementary but tedious algebra, we find that the minimum is obtained for

$$u^*(t) = - L(t)\hat{x}(t|t-1) \tag{39}$$

where the optimal control matrix is

$$L(t) = [Q_2 + \Gamma^T S(t+1)\Gamma]^{-1}\Gamma^T S(t+1)\Phi \tag{40}$$

and the solution is in the form of (35) with the recurrence relation

$$S(t) = \Phi^T S(t+1)\Phi + Q_1 - L^T(t)[Q_2 + \Gamma^T S(t+1)\Gamma]L(t) \tag{41}$$

with the initial condition $S(N) = Q_o$. (A similar expression can be obtained for $s(t+1)$ but it is not used directly to calculate the optimal policy.) It can be proved [2] that (39) gives really an optimum policy. The complete control algorithms are summarized in Table 3. It can be shown [15] that the solution is unique and stable since

- the controlled water quality process specified by Φ and Γ of (26) is state controllable,
- the matrices Q_o, Q_1 and Q_2 of the performance measure (28) are positive definite.

And now, some remarkable properties of the optimal control are briefly summarized. Notice that the optimal control strategy (39) can be separated into two parts, namely into an algorithm computing the conditional mean of the states at time t given the observations up to t - 1 (this is apparently done by the Kalman filter) and into the computation of the control matrix (40). The later depends only on the system dynamics Φ and Γ and the parameters Q_o, Q_1 and Q_2 of the performance measure and is independent of the uncertainties. In other words L(t) can be precalculated and has the same form as the deterministic optimal control solved by dynamic programming. Clearly, the stochastic effects are taken into account by using the stochastic state estimation algorithms. This is a very important and deep result known as Separation Theorem (in control [12]) or Certainty Equivalence Principle (in econometrics [26]). Summarizing, the separation theorem states that, for linear systems with quadratic cost functions and subject to additive Gaussian white noises, the optimum stochastic controller is realized by cascading an optimal estimator (predictor) with a deterministic optimum controller. This is depicted in Figure 3, where the fat arrows mean vector and the system dynamics is shown in detail in Figure 2. It has been mentioned

TABLE 3. OPTIMAL STOCHASTIC CONTROL ALGORITHM

Stochastic Control System Model Incomplete State Information Control Constraints	$\mathcal{S}\begin{cases} x(t+1) = \Phi x(t) + \Gamma u(t) + w(t), & w(t)\sim N(0,R_1) \\ z(t) = Hx(t) + v(t), & v(t)\sim N(0,R_2) \\ u(t) \in U \end{cases}$
Initial Conditions: the same as at the prediction algorithms and	$S(N) = Q_o$
Performance Measure	$\mathcal{E}\{J\} = \mathcal{E}\{\|x(N)\|^2_{Q_o} + \sum_{t=t_o}^{N-1} [\|x(t)\|^2_{Q_1} + \|u(t)\|^2_{Q_2}]\}$
The Problem: Minimize $\mathcal{E}\{J\}$ subject to the control and system dynamics $\mathcal{S}$ constraints.	
Optimal Control Policy	$u^*(t) = -L(t)\hat{x}(t\ t-1)$ $\hat{x}(t\|t-1)$ is obtained from the prediction algorithms, where $L(t) = [Q_2 + \Gamma^T S(t+1)\Gamma]^{-1}\Gamma^T S(t+1)\Phi$ $S(t) = \Phi^T S(t+1)\Phi + Q_1 - L^T(t)[Q_2 + \Gamma^T S(t+1)\Gamma]L(t)$
Minimal Expected Loss	$\min\mathcal{E}\{J\} = \|\hat{x}(t_o)\|_{S(t_o)} + \|S(t_o)P(t_o)\| + \sum_{t=t_o}^{N-1} \|S(t+1)R_1\| + \sum_{t=t_o}^{N-1} \|P(t\|t-1)L^T(t)\Gamma^T S(t+1)\Phi\|$

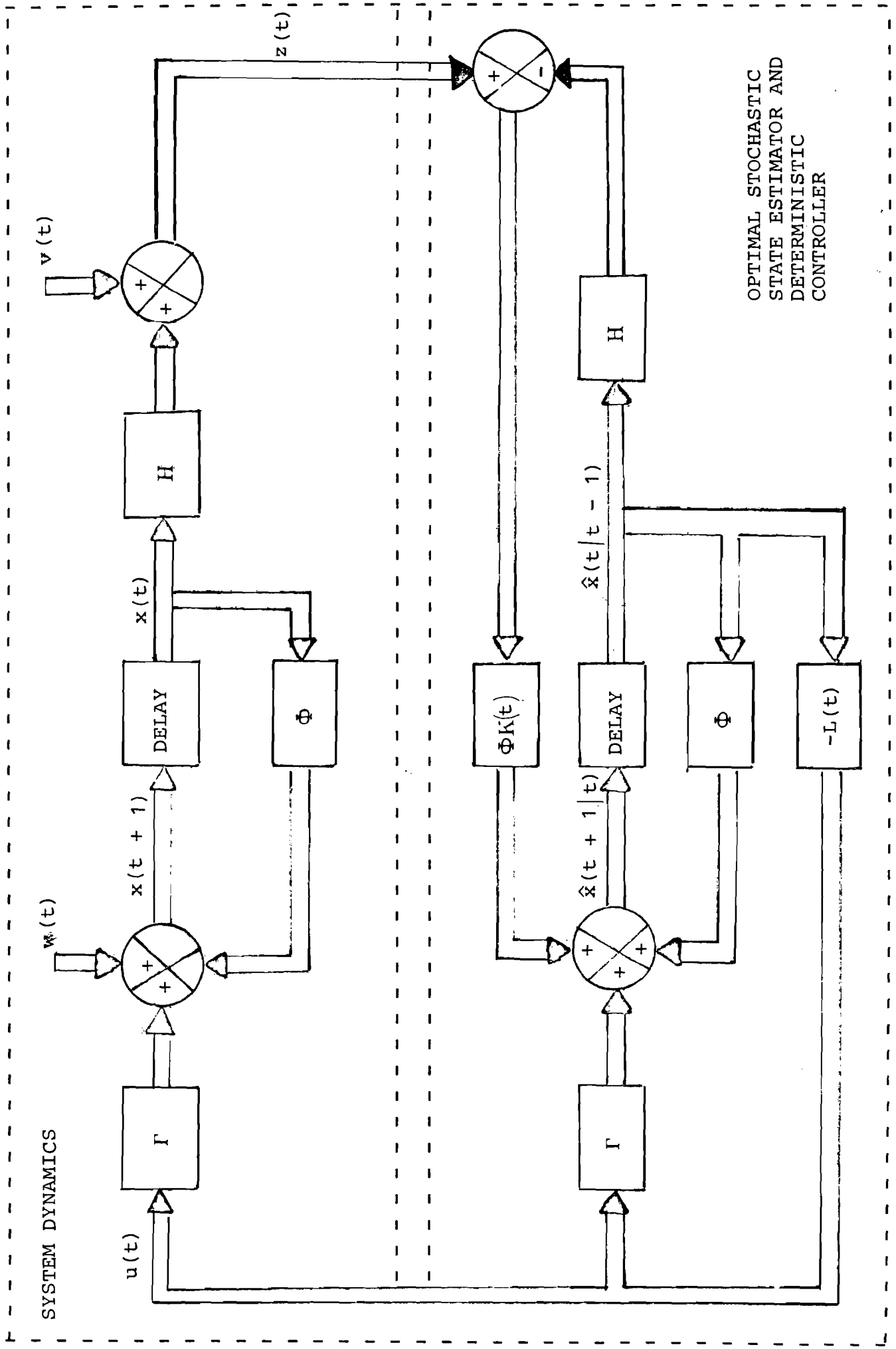

FIGURE 3. BLOCK DIAGRAM OF THE OPTIMAL STOCHASTIC WR-SYSTEMS CONTROL

as Akashi and Nose [1] have shown quite recently, that the separation theorem holds even if all the random variables are non-gaussian and correlated. Well, this was the solution of the lumped stochastic water quality control problem using which the on-line real-time control of polluted rivers becomes feasible. The distributed parameter case, however, is much more difficult both from methodological and practical computational standpoints. An effort to the optimal stochastic distributed water quality control is found in [30], but at this stage that is far from being readily applicable.

As far as other water resource systems are concerned, the above methodology might presumably be used as an extension to the long-range dynamic water resources development models, reported in [7], by considering stochastic effects. The same statement holds for stochastic linear runoff control problems (linear reservoirs etc.) as well.

SUMMARY AND CONCLUSIONS

In this paper the role of predictions in water resources policy making was analyzed. A general state space based formulation of WR systems has been introduced. It was shown that this general model of runoff control systems is able to handle different kinds of uncertainties. The objective of the WR systems control were briefly touched and then the different measures of system performance were discussed.

Optimal sequential prediction algorithms for linear discrete time stochastic WR systems have been discussed in detail and the advantages of the Kalman filtering technique have been taken. In the framework of runoff control the case of optimal stochastic water quality control has been considered. Using the discretized lumped parameter Streeter-Phelps equation the optimal treatment control has been determined by means of stochastic dynamic optimization. It was shown that the stochastic optimization process can be separated into two parts, namely stochastic state prediction and deterministic dynamic programming. In this way the optimal feedback control strategies have been obtained. The procedures discussed offer that

- using time domain formulation, the usual frequency-domain based computations can be avoided on the one hand and the problem becomes mathematically tractable on the other;
- due to the recursiveness of the algorithms the scheme can

- easily be implemented even for small computers and are applicable for real-time on-line forecasting control, always taking into consideration the newest information gathered;
- due to the state space formulation, it is generally applicable for the most general hydrologic time series (water quantity and/or quality), the joint prediction/control of multidimensioanl time series (which might include some economic data) becomes feasible even in the presence of different kinds of uncertainties;
- the algorithms give optimal prediction control in Bayes sense (Bayesian minimum variance estimators);
- the algorithms fulfill the requirement of adaptivity to changing environmental conditions as through a moving data window it allows slight modifications in the model parameters;
- the algorithms are convergent and stable under very general conditions.

Finally, we mention that due to the sometime uncertain future goals of the water resources planning the theory a fuzzy systems (Zadeh [37]) and that of the random cost functions (Rozanov [22]) offer powerful techniques.

Clearly, quite a lot of methodological work should be done in the future to clarify the different sophisticated issues of the runoff control. This paper did not want to be anything other than a humble contribution to those efforts.

APPENDIX

Lemma A.1.

Let $x \in X$ and $z \in Z$ be two scalar stochastic variables defined on a probability space and let the control variable $u \in U$ be admissible. Let the loss function J be a function which maps $X \times Z \times U$ into the real numbers, $J : X \times Z \times U \longrightarrow R$. The expected loss is then $\mathcal{E}\{J(x,z,u)\}$ where $\mathcal{E}$ denotes mathematical expectation with respect to x and z. In case of incomplete state information the admissible control strategies are the functions $u : Z \longrightarrow U$. The control action thus has to be based on information of one of the variables only. Let $\min_{u(z)} \mathcal{E}\{J(x,z,u)\}$ denote the minimum of $\mathcal{E}\{J(x,z,u)\}$ with respect to all admissible control strategies and let $\mathcal{E}\{\cdot \mid z\}$ a conditional mean given z. Assume that the function $f(z,u) = \mathcal{E}\{J(x,z,u) \mid z\}$ has a unique minimum with respect to $u \in U, \forall\, z \in Z$. Let $u^*(z)$ denote the value of u for which the minimum is achieved. Then the optimization Lemma states that

$$\min_{u(z)} \mathcal{E}\{J(x,z,u)\} = \mathcal{E}\{J(x,z,u^*(z))\} = \mathcal{E}_z\{\min_u \mathcal{E}\{J(x,z,u) \mid z\}\} \qquad \text{(I)}$$

where $\mathcal{E}_z$ denotes expectation with respect to the distribution of z[32].

Proof

For all admissible strategies we have

$$f(z,u) \geq f(z,u^*(z)) = \min_u f(z,u)$$

Hence

$$\mathcal{E}\{J(x,z,u)\} = \mathcal{E}_z\{f(z,u)\} \geq \mathcal{E}_z\{f(z,u^*(z))\} = \mathcal{E}\{J(x,z,u^*(z))\}$$

$$= \mathcal{E}_z\{\min_u \mathcal{E}\{J(x,z,u) \mid z\}\}$$

Minimizing the left hand side with respect to all admissible strategies we have

$$\min_{u(z)} \mathcal{E}\{J(x,z,u)\} \geq \mathcal{E}\{J(x,z,u^*(z))\} = \mathcal{E}_z\{\min_u \{J(x,z,u) \mid z\}\} \qquad \text{(II)}$$

Since $u^*(z) \in U$ is also an admissible strategy we have on the other hand

$$\mathcal{E}\{J(x,z,u^*(z))\} \geq \min_{u(z)} \mathcal{E}\{J(x,z,u)\} \qquad \text{(III)}$$

Combining the inequalities (II) and (III) we find (I) and the Lemma is proven.

Lemma A.2

Let x be a Gaussian random vector with mean $\hat{x}$ and covariance P and let S be a nonnegative definite matrix. Then

$$\mathcal{E}\{\|x\|_S^2\} = \|x\|_S^2 + \|SP\|$$

(Note that the trace norm $\|\cdot\|$ should not be confused with the squared vector norm with respect to S, $\|\cdot\|_S^2$.)

The proof is straightforward. Consult [3].

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