WORKING PAPER

QUALITATIVE DIFFERENTIAL GAMES: A VIABILITY APPROACH

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FOREWORD

The author defines the playability property of a qualitative differential game, defined by a system of differential equations controlles by two controls. The rules of the game are defined by constraints on the states of each player depending on the state of the other player. This paper characterizes the playability property by a regulation map which associates with any playable state a set of **playable controls**.

In other words, **the players can implement playable solutions to the differential game by playing for each state a static game on the controls of the regulation subset.**

One must extract among theses playable controls the set of **discriminating** and **pure** controls of one of the players. Such controls are defined through an adequate "contingent" Hamilton-Jacobi-Isaacs equation. Sufficient conditions implying the existence of continuous or minimal playable, discriminating and pure feedbacks are provided.

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Contents

Introduction

We consider a two-player differential game whose dynamics are described by

$$\begin{cases} a) & \begin{cases} i) & x'(t) = f(x(t), y(t), u(t)) \\ ii) & u(t) \in U(x(t), y(t)) \end{cases} \\ b) & \begin{cases} i) & y'(t) = g(x(t), y(t), v(t)) \\ ii) & v(t) \in V(x(t), y(t)) \end{cases} \end{cases}$$

The rules of the game are set-valued maps $P : Y \rightsquigarrow X$ and $Q : X \rightsquigarrow Y$, stating the constraints imposed by one player on the other.

The **playability domain** of the game $K \subset X \times Y$ is defined by:

$$K := \{ (x,y) \in X \times Y \mid x \in P(y) \text{ and } y \in Q(x) \}$$

(We consider only the time-independent case for the sake of simplicity).

The **playability property** states that for all initial state $(x_0, y_0) \in K$, there exists a solution to the differential game which is playable in the sense that

$$\forall\, t \geq 0, \quad x(t) \in P(y(t)) \;\; \& \;\; y(t) \in Q(x(t))$$

We shall charaterize it by constructing **decision rules**

$$(x,y,v) \rightsquigarrow \Phi(x,y;v) \;\; \& \;\; (x,y,u) \rightsquigarrow \Psi(x,y;u)$$

which involve the contingent derivatives[1] of the set-valued maps P and Q, with which we build the **regulation map** R mapping each $(x,y) \in K$ to the regulation set

$$R(x,y) = \{ (u,v) \mid u \in \Phi(x,y;v) \text{ and } v \in \Psi(x,y;u) \}$$

[1] We recall that the **contingent cone** $T_K(x)$ to a subset K at $x \in K$ is the closed cone of elements v satisfying

$$\liminf_{h \to 0+} d(x + hv, K)/h = 0$$

The **contingent derivative** of the set-valued map Q from X to Y at a point (x,y) of its graph is the closed positively homogenous set-valued map $DQ(x,y)$ from X to Y defined by

$$\mathrm{Graph}(DQ(x,y)) := T_{\mathrm{Graph}(Q)}(x,y)$$

or, equivalently, by

$$v \in DQ(x,y)(v) \iff \liminf_{h \to 0+, u' \to u} d\left(v, \frac{Q(x+hu') - y}{h}\right) = 0$$

The controls belonging to $R(x,y)$ are called **playable**.

The Playability Theorem states that under technical assumptions, the playability property holds true if and only if

$$\forall\,(x,y)\in K,\;\; R(x,y)\;\neq\;\emptyset$$

and that playable solutions to the game are regulated by the **regulation law**:

$$\forall\,t\geq 0,\quad u(t)\;\in\;\Phi(x(t),y(t);v(t))\quad\&\quad v(t)\;\in\;\Psi(x(t),y(t);u(t))$$

We then introduce the subset

$$A(x,y;v)\;:=\;\{\,u\in U(x,y)\mid (u,v)\in R(x,y)\,\}$$

of **discriminating controls** which allow the first player to associate to any control $v\in V(x,y)$ played by the second player at least a control $u\in U(x,y)$ such that the pair (u,v) is playable and the subset

$$B(x,y)\;:=\;\bigcap_{v\in V(x,y)}A(x,y;v)$$

of **pure controls** which allow the first player to find a control $u\in U(x,y)$ such that (u,v) is playable for all $v\in V(x,y)$.

These concepts are particularly relevant for games "against nature" or "disturbances"(see [11,12,26,27] and their references).

Before going further, it may be useful to relate these concepts to more familiar ones through an adequate Hamilton-Jacobi-Isaacs's equation (see[18]).

For that purpose, we characterize the rules P and Q by their indicator functions W_P and W_Q defined respectively by

$$W_P(x,y)\;:=\;\begin{cases}0 & \text{if } x\in P(y)\\ +\infty & \text{if } x\notin P(y)\end{cases}\qquad W_Q(x,y)\;:=\;\begin{cases}0 & \text{if } y\in Q(x)\\ +\infty & \text{if } y\notin Q(x)\end{cases}$$

These functions are only lower semicontinuous, but we can still "differentiate" them by taking their **contingent epiderivatives**[2]. We set

$$H(W_P + W_Q; x, y; u, v) \ := \ D_\uparrow(W_P + W_Q)(x, y)(f(x, y; u), g(x, y; v))$$

We shall prove that

— the game is playable if and only if

$$\inf_{u \in U(x,y), v \in V(x,y)} H(W_P + W_Q; x, y; u, v) \ = \ 0$$

and the regulation map is equal to

$$\begin{cases} R(x, y) = \{(u, v) \in U(x, y) \times V(x, y) \mid \\ H(W_P + W_Q; x, y; u, v) = \inf_{u' \in U(x,y), v' \in V(x,y)} H(W_P + W_Q; x, y; u', v')\} \end{cases}$$

— the first player has a discriminating control if and only if

$$\sup_{v \in V(x,y)} \inf_{u \in U(x,y)} H(W_P + W_Q; x, y; u, v) \ = \ 0$$

and the feedback map A is equal to

$$\begin{cases} A(x, y; v) = \{u \in U(x, y) \mid \\ H(W_P + W_Q; x, y; u, v) = \inf_{u' \in U(x,y)} H(W_P + W_Q; x, y; u', v)\} \end{cases}$$

— the first player has a pure control if and only if

$$\inf_{u \in U(x,y)} \sup_{v \in V(x,y)} H(W_P + W_Q; x, y; u, v) \ = \ 0$$

and the feedback map B is equal to

$$\begin{cases} B(x, y) = \{u \in U(x, y) \mid \sup_{v \in V(x,y)} H(W_P + W_Q; x, y; u, v) \\ = \inf_{u' \in U(x,y)} \sup_{v \in V(x,y)} H(W_P + W_Q; x, y; u', v)\} \end{cases}$$

[2]The contingent derivative $D_\uparrow W(x)$ of a extended function W from X to $\mathbf{R} \cup \{+\infty\}$ at $x \in \mathrm{Dom}(W)$ is defined by

$$\mathcal{E}p D_\uparrow W(x) \ := \ T_{\mathcal{E}p(W)}(x, W(x))$$

or, equivalently, by

$$D_\uparrow W(x)(u) \ = \ \liminf_{h \to 0+, u' \to u} \frac{V(x + hu') - V(x)}{h}$$

We then deal with the main topic of this paper: construct single-valued **playable feedbacks** $(\tilde{u}, \tilde{v})$, such that the differential system

$$\begin{cases} x'(t) &= f(x(t), y(t), \tilde{u}(x(t), y(t)) \\ y'(t) &= g(x(t), y(t), \tilde{v}(x(t), y(t)) \end{cases}$$

has playable solutions for each initial state. By the Playability Theorem, they must be selections of the regulation map R in the sense that

$$\forall\, (x,y) \;\in\; K,\;\; (x,y) \mapsto (\tilde{u}(x,y), \tilde{v}(x,y)) \in R(x,y)$$

We shall prove the existence of such continuous single-valued playable feedbacks, as well as more constructive, but discontinuous, playable feedbacks, such as the feedbacks associating the controls of $R(x,y)$ with minimal norm (the playable slow feedbacks, as in [13,36]). More generally, we shall show the existence of possibly set-valued feedbacks associating with any $(x,y) \in K$ the set of controls $(u,v) \in R(x,y)$ which are solutions to a (static) optimization problem of the form:

$$(u,v) \in R(x,y) \;\;\mid\;\; \sigma(x,y;u,v) \leq \inf_{u',v' \in R(x,y)} \sigma(x,y;u',v')$$

or solutions to a noncooperative game of the form:

$$\forall (u',v') \in R(x,y),\;\; a(x,u,v') \leq a(x,u,v) \leq a(x,u',v)$$

In other words, **the players can implement playable solutions to the differential game by playing for each state $(x,y) \in K$ a static game on the controls of the regulation subset $R(x,y)$.**

We also consider the issue of finding **discriminating feedbacks**, which are selections of the set-valued map A. We shall provide for instance sufficient conditions implying that for all continuous feedback $\tilde{v}(x,y) \in V(x,y)$ played by the second player, the first player can find a feedback (continuous or of minimal norm) $\tilde{u}(x,y)$ such that the above differential equation has playable solutions for each initial state.

Finally, we address the question of constructing continuous **pure feedbacks** $\tilde{u}(x,y)$ which have the property of yielding playable solutions of the above differential equation whatever the continuous feedback $\tilde{v}(x,y)$ played by the second player[3].

[3]One can also construct "dynamic feedback controls" which are selections $(\tilde{b}, \tilde{c})$ of the

We use for constructing these feedbacks selection theorems (for instance, Michael's continuous selection theorem, see [29,30,31]), we need to prove the lower semicontinuity of the set-valued maps R, A and B. In the case of the set-valued map B, we need a Lower Semicontinuity Criterion of an infinite intersection of lower semicontinuous maps. We provide such a theorem at the end of this paper, which can be useful for other purposes.

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contingent derivative of the regulation map

$$(\tilde{b}(x,y;u,v),\tilde{c}(x,y:u,v)) \in DR(x,y)(f(x,y;u),g(x,y;v))$$

With these "dynamic feedbacks, players implement the differential system

$$\begin{cases} x'(t) &= f(x(t),y(t),\tilde{u}(x(t),y(t)) \\ y'(t) &= g(x(t),y(t),\tilde{v}(x(t),y(t)) \\ u'(t) &= \tilde{b}(x(t),y(t);u(t),v(t)) \\ v'(t) &= \tilde{c}(x(t),y(t);u(t),v(t)) \end{cases}$$

which yields playable solutions.

In other words, **the players can implement playable solutions to the differential game by playing for each state $(x,y) \in K$ a static game on "velocities" of the controls in the derivative $DR(x,y)(f(x,y;u),g(x,y;v))$ of the regulation subset.**

Minimal selections (b°,c°) provide **heavy trajectories** (see [5]) in the case of control systems

1 Playable Rules

Let us consider only two players, Xavier and Yves. Xavier acts on a state space X and Yves on a state space Y. For doing so, they have access to some knowledge about the global state (x,y) of the system and are allowed to choose controls u in a global state dependent set $U(x,y)$ and v in a global state dependent set $V(x,y)$ respectively.

Their actions on the state of the system are governed by the system of differential inclusions:

$$(1) \qquad \left\{ \begin{array}{ll} a) & \left\{ \begin{array}{lrcl} i) & x'(t) & = & f(x(t),y(t),u(t)) \\ ii) & u(t) & \in & U(x(t),y(t)) \end{array} \right. \\ b) & \left\{ \begin{array}{lrcl} i) & y'(t) & = & g(x(t),y(t),v(t)) \\ ii) & v(t) & \in & V(x(t),y(t)) \end{array} \right. \end{array} \right.$$

We now describe the influences (power relations) that Xavier exerts on Yves and vice-versa through **rules of the game**. They are set-valued maps $P : Y \rightsquigarrow X$ and $Q : X \rightsquigarrow Y$ which are interpreted in the following way. When the state of Yves is y, Xavier's choice is constrained to belong to $P(y)$. In a symmetric way, the set-valued map Q assigns to each state x the set $Q(x)$ of states y that Yves can implement[4].

Hence, the **playability domain** of the game is the subset $K \subset X \times Y$ defined by:

$$(2) \qquad K \ := \ \{ \ (x,y) \in X \times Y \ \mid \ x \in P(y) \ \text{ and } \ y \in Q(x) \ \}$$

Naturally, we must begin by providing sufficient conditions implying that the playability domain is non empty. Since the playability domain is the subset of fixed-points (x,y) of the set-valued map $(x,y) \rightsquigarrow P(y) \times Q(x)$, we can use one of the many fixed point theorems to answer this type of questions[5].

From now on, we shall assume that the playability domain associated with the rules P and Q is not empty.

Definition 1.1 (Playability Property) *We shall say that the differential game enjoys the* **playability property** *if and only if, for all initial state*

[4]We can easily extend the results below to the time-dependent case using the methods of [2].

[5]For instance, Kakutani's Fixed Point Theorem furnishes such conditions: *Let $L \subset X$ and $M \subset Y$ be compact convex subsets and $P : M \rightsquigarrow L$ and $Q : L \rightsquigarrow M$ be closed maps with nonempty convex images. Then the playability domain is not empty.*

$(x_0, y_0) \in K$, *there exists a solution to the differential game (1) which is* **playable** *in the sense that*

$$\forall\, t \geq 0, \quad x(t) \;\in\; P(y(t)) \;\;\&\;\; y(t) \;\in\; Q(x(t))$$

We need now to define **playable rules**. For that purpose, we associate with the rules P and Q acting on the states **decision rules** Φ and Ψ acting on the controls defined in the following way:

Definition 1.2 *Xavier's* **decision rule** *is the set-valued map* Φ *defined by*

$$(3) \qquad \begin{cases} \Phi(x, y; v) \\ = \{\, u \in U(x,y) \mid f(x,y,u) \in DP(y,x)(g(x,y,v)) \,\} \end{cases}$$

and Yves's **decision rule** *is the set-valued map* Ψ *defined by*

$$(4) \qquad \begin{cases} \Psi(x, y; u) \\ = \{\, v \in V(x,y) \mid g(x,y,v) \in DQ(x,y)(f(x,y,u)) \,\} \end{cases}$$

We associate with them the **regulation map** R *defined by*

$$(5) \qquad R(x,y) \;=\; \{\, (u,v) \mid u \in \Phi(x,y;v) \;\;\&\;\; v \in \Psi(x,y;u) \,\}$$

The subset $R(x,y)$ *is called the* **regulation set** *and its elements* **playable controls**.

In other words, we have associated with each state (x,y) of the playability domain a static game on the controls defined by the decision rules. This new game on controls is playable if the subset $R(x,y)$ is nonempty. This property deserves a definition.

Definition 1.3 *We shall say that* P *and* Q *are* **playable rules** *if their graphs are closed, the playability domain* K *defined by (2) is non empty and if for all pair* $(x,y) \in K$, *the values* $R(x,y)$ *of the regulation map are nonempty.*

We still need a definition of transversality of the rules before stating our first theorem.

Definition 1.4 *We shall say that the rules* P *and* Q *are* **transversal** *if for all* $(x,y) \in K$, *for all perturbations* $(e,f) \in X \times Y$, *there exists* (u,v) *satisfying*

$$(6) \qquad \begin{cases} i) & u \in DP(y,x)(v) + e \\ ii) & v \in DQ(x,y)(u) + f \end{cases}$$

We shall say that the are **strongly transversal** *if*

$$(7) \quad \begin{cases} \forall (x,y) \in K,\ \exists c > 0, \delta > 0 \ \textit{such that} \ \forall (x',y') \in B_K((x,y),\delta), \\ \forall\ (e,f) \in X \times Y, \ \textit{there exist solutions} \ (u,v) \ \textit{to} \\ \begin{cases} i) & u \in DP(y',x')(v) + e \\ ii) & v \in DQ(x',y')(u) + f \end{cases} \\ \textit{satisfying} \\ \max(\|u\|,\|v\|) \ \le \ \max(\|e\|,\|f\|) \end{cases}$$

We will also assume that the rules are **sleek**[6].

We shall now derive from the Viability Theorem a characterization of the playability property.

Theorem 1.1 (Playability Theorem) *Let us assume that the functions f and g*

$$(8) \qquad \begin{cases} i) & \textit{are continuous} \\ ii) & f(x,y,\cdot) \ \textit{and} \ g(x,y,\cdot) \ \textit{are affine} \\ iii) & \textit{have a linear growth} \end{cases}$$

that the feedback maps U and V

$$(9) \qquad \begin{cases} i) & \textit{are upper semicontinuous with compact convex images} \\ ii) & \textit{have a linear growth} \end{cases}$$

and that the rules P and Q are

$$(10) \qquad \textit{sleek and transversal}$$

Then the rules P and Q enjoy the playability property if and only if they are playable. Furthermore, the controls $u(\cdot)$ and $v(\cdot)$ which provide playable solutions obey the following **regulation law**

$$(11) \ \forall\, t \ge 0, \quad u(t) \ \in \ \Phi(x(t),y(t);v(t)) \ \ \& \ \ v(t) \ \in \ \Psi(x(t),y(t);u(t))$$

Proof — We apply the Viability Theorem (see [17], [3, Theorem 4.2.1., p.180])to the control system

$$\begin{cases} i) & (x'(t),y(t)) \ = \ (f(x(t),y(t),u(t)),g(x(t),y(t),v(t))) \\ ii) & (u(t),v(t)) \ \in \ U(x(t),y(t)) \times V(x(t),y(t)) \end{cases}$$

[6] A subset K is sleek at $x_0 \in K$ if the set-valued map $x \rightsquigarrow T_K(x)$ is lower semicontinuous at x_0. In this case, the contingent cone is convex and coincides with the Clarke tangent cone. K is sleek if it is sleek at every point of K. Convex subsets and C^1-manifolds are sleek. A set-valued map is sleek if it graph is sleek.(see [1])

which satisfy the assumptions of the Viability Theorem. It remains to prove that the playability domain of the differential game is a viability domain of the above control system, i.e., that for any global state $(x,y) \in K$ of the system, there exist controls u and v such that the pair $(f(x,y,u), g(x,y,v))$ belongs to the contingent cone $T_K(x)$.

Since K is the intersection of the graphs of Q and P^{-1}, we need to use a sufficient condition for the contingent cone to an intersection to be equal to the intersection of the contingent cones.

The graphs of Q and P^{-1} are sleek because the rules of the game are supposed to be so. Furthermore,

$$T_{\mathrm{Graph}(P^{-1})}(x,y) - T_{\mathrm{Graph}(Q)}(x,y) \;=\; X \times Y$$

because the maps P and Q are transversal: For any $(e,f) \in X \times Y$, there exists (u,v) such that $(u, v-f)$ belongs to the graph of Q and $(u+e, v)$ to the graph of P^{-1}, i.e., that $(e,f) = (u+e,v) - (u,v-f)$.

Hence, by [4, Corollary 7.6.5., p.441], we deduce that

$$\begin{cases} T_K(x) \\ \qquad = T_{\mathrm{Graph}(P^{-1})}(x,y) \cap T_{\mathrm{Graph}(Q)}(x,y) \\ \qquad = \mathrm{Graph}(DP(y,x))^{-1} \cap \mathrm{Graph}(DQ(x,y)) \end{cases}$$

Therefore, K is a viability domain if and only if the regulation map R has nonempty values, i.e., if and only if the rules of the game are playable. $\square$

The regulation law (11) describes how the players must behave to keep the state of the system playable. A first question arises: do the domains of the set-valued maps

$$\begin{cases} i) & \Phi(x,y) : v \rightsquigarrow \Phi(x,y;v) \\ ii) & \Psi(x,y) : u \rightsquigarrow \Psi(x,y;u) \end{cases}$$

coincide with $U(x,y)$ and $V(x,y)$ respectively?

Proposition 1.1 *We posit the assumptions of Theorem 1.1. Let us assume that for all $(x,y) \in K$,*

$$(12) \qquad \begin{cases} i) & \mathrm{Dom}(\Phi(x,y)) &=& V(x,y) \\ ii) & \mathrm{Dom}(\Psi(x,y)) &=& U(x,y) \end{cases}$$

Then the rules are playable.

Proof — We deduce it from Kakutani's Fixed Point Theorem, since the set $R(x,y)$ is the set of fixed points of the set-valued map

$$(u,v) \rightsquigarrow \Phi(x,y;v) \times \Psi(x,y;u)$$

defined on the convex compact subset $U(x,y) \times V(x,y)$ into itself. This set-valued map has non empty values by assumption, which are moreover convex since the rules P and Q being sleek, the graphs of the contingent derivatives $DP(x,y)$ and $DQ(x,y)$ are convex. They are also closed. This implies that the graph of $(u,v) \rightsquigarrow \Phi(x,y;v) \times \Psi(x,y;u)$ is closed. Hence we can apply Kakutani's Fixed Point Theorem[7]. □

Once the playability of the game is established, how can it be played?

The question arises to know whether Xavier can associate to any control $v \in V(x,y)$ played by Yves at least a control $u \in U(x,y)$ such that the pair (u,v) is playable, or even better, whether he can find a control $u \in U(x,y)$ such that (u,v) is playable for all $v \in V(x,y)$. These ideal situations (for Xavier) deserve a definition.

Definition 1.5 (Discriminating and Pure Controls) *We say that Xavier has* **discriminating controls** *if for any $(x,y) \in K$ and $v \in V(x,y)$, the subset*

$$A(x,y;v) := \{ u \in U(x,y) \mid (u,v) \in R(x,y) \} \tag{13}$$

of discriminating controls is nonempty. It has **pure controls** *if and only if for any $(x,y) \in K$ the subset*

$$B(x,y) := \bigcap_{v \in V(x,y)} A(x,y;v) \tag{14}$$

of pure controls is nonempty.

We have to examine whether the set-valued map B has nonempty values.

Proposition 1.2 *Let us assume that for any $u \in U(x,y)$, there exists u' satisfying*

$$\forall\, v \in \Psi(x,y;u),\ u' \in \Phi(x,y;v)$$

Then the set-valued map B has nonempty values.

[7] We can use instead the Browder-Ky Fan Theorem and replace condition (12) by: a sufficient condition of the form

$$\begin{cases} \forall\, (u,v) \in U(x,y) \times V(x,y), \ \ 0 \in \\ (f(x,y;u), g(x,y;v)) - T(x,y) - A(T_{U(x,y)}(u) \times T_{V(x,y)}(v)) \end{cases}$$

where A is a linear operator from $Z_X \times Z_Y$ to $X \times Y$

Proof — We observe that $B(x,y)$ is the subset of fixed points of the "square product" $\Phi(x,y) \square \Psi(x,y)$ defined by

$$(\Phi(x,y) \square \Psi(x,y))(u) := \bigcap_{v \in \Psi(x,y;u)} \Phi(x,y;v) \tag{15}$$

(see [6]). This set-valued map has nonempty values by assumption, which are obviously convex and compact. Furthermore, its graph is closed. Since $U(x,y)$ is convex and compact, Kakutani's Fixed Point Theorem implies the existence of a fixed point. $\square$

Remark — When

$$\forall\, (x,y) \in K, \quad R(x,y) = U(x,y) \times V(x,y) \tag{16}$$

and when the set-valued maps

$$(x,y) \rightsquigarrow f(x,y;U(x,y)) \quad \text{and } g(x,y;V(x,y)) \tag{17}$$

are lipschitzean around K, then the playability domain K has the **winability property**: for all initial state (x_0, y_0), all solutions to the game (1) are playable.

Indeed, we deduce from [3, Theorem 4.6.2] that in this case, K is invariant by the differential inclusion associated with the differential game. $\square$

We shall investigate later several methods of selecting feedback controls (u,v) in the subsets $R(x,y)$, $A(x,y;v)$ and $B(x,y)$ by optimization and/or (static) game theoretical methods.

Remark—CONTINGENT HAMILTON-JACOBI-ISAACS EQUATION We can relate the above results to the original Isaacs's formulation of the Hamilton-Jacobi equations in the framework of differential games (see [18]). For that purpose, we characterize the rules P and Q by their indicator functions W_P and W_Q defined by

$$W_P(x,y) := \begin{cases} 0 & \text{if } x \in P(y) \\ +\infty & \text{if } x \notin P(y) \end{cases} \quad W_Q(x,y) := \begin{cases} 0 & \text{if } y \in Q(x) \\ +\infty & \text{if } y \notin Q(x) \end{cases}$$

We then introduce the **Hamiltonian** H associating with any lower semicontinuous function $W : X \times Y \mapsto \mathbf{R} \cup \{+\infty\}$ the function defined on by

$$H(W;x,y;u,v) := D_\uparrow(W)(x,y)(f(x,y;u), g(x,y;v)) \tag{18}$$

where $D_\uparrow W(x)$ denotes the **contingent epiderivative** of W at x.

Proposition 1.3 *We posit the assumptions of Theorem 1.1. Then the rules P and Q are playable if and only if their indicator functions W_P and W_Q are solutions to the contingent Hamilton-Jacobi equation*

$$(19) \qquad \inf_{u\in U(x,y), v\in V(x,y)} H(W_P+W_Q;x,y;u,v) \;=\; 0$$

and the regulation map is equal to

$$(20) \quad \begin{cases} R(x,y) = \{(u,v)\in U(x,y)\times V(x,y) \,|H(W_P+W_Q;x,y;u,v) \\ = \inf_{u'\in U(x,y), v'\in V(x,y)} H(W_P+W_Q;x,y;u',v')\} \end{cases}$$

Xavier has a discriminating control if and only if

$$(21) \qquad \sup_{v\in V(x,y)} \inf_{u\in U(x,y)} H(W_P+W_Q;x,y;u,v) \;=\; 0$$

and the feedback map A is equal to

$$(22) \quad \begin{cases} A(x,y;v) = \{u\in U(x,y) \,| \\ H(W_P+W_Q;x,y;u,v) = \inf_{u'\in U(x,y)} H(W_P+W_Q;x,y;u',v)\} \end{cases}$$

Finally, Xavier has a pure control if and only if

$$(23) \qquad \inf_{u\in U(x,y)} \sup_{v\in V(x,y)} H(W_P+W_Q;x,y;u,v) \;=\; 0$$

and the feedback map B is equal to

$$(24) \quad \begin{cases} B(x,y) = \{u\in U(x,y) \mid \sup_{v\in V(x,y)} H(W_P+W_Q;x,y;u,v) \\ = \inf_{u'\in U(x,y)} \sup_{v\in V(x,y)} H(W_P+W_Q;x,y;u',v)\} \end{cases}$$

Proof — The proof is based on the fact that the contingent cone to the epigraph of W_Q at (x,y), which is the epigraph of the contingent epiderivative $D_\uparrow W_Q(x,y;\cdot,\cdot)$, is the indicator function of the contingent cone to the graph of Q at (x,y), i.e., the indicator function of the graph of the contingent derivative $DQ(x,y)$. Therefore,

$$D_\uparrow W_Q(x,y)(u,v) \;=\; \begin{cases} 0 & \text{if } v\in DQ(x,y)(u) \\ +\infty & \text{if } v\notin DQ(x,y)(u) \end{cases}$$

and, in a similar way,

$$D_\uparrow W_P(x,y)(u,v) \;=\; \begin{cases} 0 & \text{if } u\in DP(y,x)(v) \\ +\infty & \text{if } u\notin DP(y,x)(v) \end{cases}$$

Therefore u belongs to $\Phi(x, y; v)$ if and only if

$$D_\uparrow W_P(x,y)(f(x,y;u),g(x,y;v)) \;=\; 0$$

and that v belongs to $\Psi(x,y;u)$ if and only if

$$D_\uparrow W_Q(x,y)(f(x,y;u),g(x,y;v)) \;=\; 0$$

We finally recall that the sleekness and transversality assumptions imply that

$$D_\uparrow(W_P + W_Q)(x,y) \;=\; D_\uparrow W_P(x,y) + D_\uparrow W_Q(x,y)$$

With these formulas at hand, we can translate the definitions of the set-valued maps R, A and B into the formulas of our Proposition. $\square$

2 Playable feedbacks

Knowing the regulation law (11), **playing the game** amounts to choose for each pair $(x,y) \in K$ playable controls (u,v) in the regulation set $R(x,y)$ through **playable feedbacks**.

In particular, we shall look for single-valued playable feedbacks $(\tilde{u}, \tilde{v})$, which are selections of the regulation map R in the sense that

$$\forall\, (x,y) \;\in\; K, \;\; (x,y) \mapsto (\tilde{u}(x,y), \tilde{v}(x,y)) \in R(x,y)$$

or, equivalently, solutions to the system

$$\forall\, (x,y) \in K, \qquad \left\{ \begin{array}{lcl} \tilde{u}(x,y) & \in & \Phi(x,y;\tilde{v}(x,y)) \\ \text{and} & & \\ \tilde{v}(x,y) & \in & \Psi(x,y;\tilde{\tilde{u}}(x,y)) \end{array} \right.$$

For instance, continuous selections of the set-valued map R provide continuous playable feedbacks $(\tilde{u}, \tilde{v})$ such that the system of differential equations

$$\text{(25)} \qquad \left\{ \begin{array}{lcl} x'(t) & = & f(x(t), y(t), \tilde{u}(x(t), y(t))) \\ y'(t) & = & g(x(t), y(t), \tilde{v}(x(t), y(t))) \end{array} \right.$$

does have solutions which are playable.

Michael's Continuous Selection Theorem, as well as other selection procedures we shall use, requires the lower semicontinuity of the regulation map R.

Our next objective is then to provide criteria under which the regulation map is lower semicontinuous. For that purpose, we need to strengthen the concept of playable rules.

Definition 2.1 *We associate with any perturbation* (e, f) *the* **decision rules** $\Phi_{(e,f)}$ *and* $\Psi_{(e,f)}$ *defined by:*

$$(26)\quad \begin{cases} \Phi_{(e,f)}(x, y; v) \\ = \{ u \in U(x, y) \mid f(x, y; u) \in DP(y, x)(g(x, y, v) - f) + e \} \end{cases}$$

and

$$(27)\quad \begin{cases} \Psi_{(e,f)}(x, y; u) \\ = \{ v \in V(x, y) \mid g(x, y, v) \in DQ(x, y)(f(x, y; u) - e) + f \} \end{cases}$$

and **regulation map** $R_{(e,f)}$ *defined by*

$$(28)\quad R_{(e,f)}(x, y) \;=\; \{ (u, v) \mid u \in \Phi_{(e,f)}(x, y; v) \;\;\&\;\; v \in \Psi_{(e,f)}(x, y; u) \}$$

We shall say that the rules P *and* Q *are* **strongly playable** *if*

$$(29)\quad \begin{cases} \forall (x, y) \in K,\ \exists \gamma > 0, \delta > 0 \text{ such that } \forall (x', y') \in B_K((x, y), \delta), \\ \forall\, (e, f) \in \gamma B,\ \ R_{(e,f)}(x', y') \neq \emptyset \end{cases}$$

Theorem 2.1 *Let us assume that the functions* f *and* g

$$(30)\quad \begin{cases} i) & \text{are continuous} \\ ii) & f(x, y, \cdot) \text{ and } g(x, y, \cdot) \text{ are affine} \\ iii) & \text{have a linear growth} \end{cases}$$

that the feedback maps U *and* V

$$(31)\quad \begin{cases} i) & \text{are continuous with compact convex images} \\ ii) & \text{have a linear growth} \end{cases}$$

and that the rules P *and* Q *are*

(32) *sleek, strongly transversal and strongly playable*

Then the regulation map R *is* **lower semicontinuous** *with closed convex images.*

In particular, there exist continuous playable feedbacks $(\tilde{u}, \tilde{v})$.

Proof — We use the Lower Semicontinuity Criterion of the intersection and the inverse image of lower semicontinuous set-valued maps[8]

First, we need to prove that the set-valued map

$$(x, y) \rightsquigarrow T(x, y) := \mathrm{Graph}(DP(y, x)^{-1}) \cap \mathrm{Graph}(DQ(x, y))$$

is lower semicontinuous. But this follows from the strong transversality of the rules P and Q and the Lower Semicontinuity Criterion.

We observe that $U \times V$ being upper semicontinuous with compact values, it maps a neighborhood of each point to a compact set. Since we can write

$$R(x, y) = \{(u, v) \in U(x, y) \in V(x, y) \mid (f(x, y; u), g(x, y; v)) \in T(x, y)\}$$

and since both $U \times V$ and T are lower semicontinuous with convex images, strong playability of the decision rules implies that the regulation map R is lower semicontinuous. □

Unfortunately, the proof of Michaels's Continuous Selection Theorem is not constructive. We would rather trade the continuity of the playable control with some explicit and computable property, such as $u^{\circ}(x, y)$ being the element of minimal norm in $R(x, y)$, or other properties. Hence we need to prove the existence of a solution to the differential equation (25) for such noncontinuous feedbacks.

We shall provide a general method of construction of such playable feedbacks. It is useful for that purpose to introduce the following definition:

[8] LOWER SEMICONTINUITY CRITERION—Let us consider a metric space X, two Banach spaces Y and Z, two set-valued maps T and U from X to Y and Z respectively and a (single-valued) map f from the graph of U to Y. We assume that

$$\begin{cases} i) & \text{T and U are lower semicontinuous} \\ & \text{with convex values} \\ ii) & \text{f is continuous} \\ iii) & \forall u, \quad u \mapsto f(x, u) \text{is affine} \end{cases}$$

We posit the following condition:
$\forall x \in X, \quad \exists \gamma 0, \ \delta > 0, \ c > 0$ such that $\forall x' \in B(x, \delta)$, we have

$$\begin{cases} i) & \gamma B_Y \subset f(x', U(x')) - T(x') \\ ii) & U(x') \subset cB_Z \end{cases}$$

Then the set-valued map $R : X \rightsquigarrow Z$ defined by

$$R(x) := \{u \in U(x) \quad | \quad f(x, u) \in T(x)\}$$

is lower semicontinuous with convex values (see [4, Theorem 3.1.16, p. 115]).

Definition 2.2 (Selection Procedure) *A selection procedure of the regulation map $R : X \times Y \rightsquigarrow U \times V$ is a set-valued map $S_R : X \times Y \rightsquigarrow U \times V$*

$$(33) \quad \left\{ \begin{array}{ll} i) & \forall (x,y) \in K, \ \ S(R(x,y)) := S_R(x,y) \cap R(x,y) \neq \emptyset \\ ii) & \textit{the graph of } S_R \textit{ is closed} \end{array} \right.$$

and the set-valued map $S(R) : (x,y) \rightsquigarrow S(R(x,y))$ is called the selection *of R.*

It is said convex-valued *or simply,* convex *if its values are convex and* strict *if moreover*

$$(34) \quad \forall (x,y) \in \mathrm{Dom}(R), \ S_R(x,y) \cap R(x,y) = \ \{\tilde{u}(x,y)), \tilde{v}(x,y)\}$$

is a singleton.

Theorem 2.2 *We posit the assumptions of Theorem 2.1 and we suppose that K is a playability domain.*

Let S_R be a convex selection procedure of the regulation map R. Then, for all initial state $(x_0, y_0) \in K$, there exist a playable solution starting at (x_0, y_0) to the differential inclusion

$$(35) \quad \left\{ \begin{array}{llcl} i) & x'(t) & = & f(x(t), y(t); u(t)) \\ ii) & y'(t) & = & g(x(t), y(t); v(t)) \\ iii) & \text{for almost all } t, \ (u(t), v(t)) & \in & S(R(x(t), y(t))) \end{array} \right.$$

In particular, if the selection procedure is strict, then the controls

$$(\tilde{u}(x,y)), \tilde{v}(x,y)) \ \ \textit{defined by} \ \ (34)$$

are single-valued playable feedback controls.

Proof — Since the convex selection procedure S_R has a closed graph and convex values, we can replace the differential game (1) by the controlled system

$$(36) \left\{ \begin{array}{llcl} i) & x'(t) & = & f(x(t), y(t); u(t)) \\ ii) & y'(t) & = & g(x(t), y(t); v(t)) \\ ii) & \multicolumn{3}{l}{\text{for almost all } t,} \\ & \multicolumn{3}{l}{(u(t), v(t)) \ \in \ (U(x(t), y(t)) \times V(x(t), y(t))) \cap S_R(x(t), y(t))} \end{array} \right.$$

which satisfies the assumptions of the Viability Theorem. It remains to check that K is still a viability domain for this smaller system.

But by construction, we know that for all $(x,y) \in K$, there exists $(u,v) \in S(R(x,y))$, which belongs to the intersection $U(x,y) \times V(x,y) \cap S_R(x,y)$ and which is such that $(f(x,y;u), g(x,y;v))$ belongs to $T_K(x)$.

Hence the new controlled system (36) enjoys the viability property, so that, for all initial state $(x_0, y_0) \in K$, there exist a viable solution and a viable control to the controlled system (36) which, for almost all $t \geq 0$, are related by

$$(37) \begin{cases} i) & (u(t), v(t)) \in (U(x(t), y(t)) \times V(x(t), y(t))) \cap S_R(x(t), y(t)) \\ ii) & (f(x(t), y(t); u(t), g(x(t), y(t); v(t))) \in T_K(x(t), y(t)) \end{cases}$$

Therefore, for almost all $t \geq 0$, $(u(t), v(t))$ belongs to the intersection of $R(x(t), y(t))$ and $S_R(x(t), y(t))$, i.e., to the selection $S(R(x(t), y(t)))$ of the regulation map R. □

We can now multiply the possible corollaries, by giving several instances of selection procedures of set-valued maps.

We begin by cooperative procedures, where the players agree upon criteria $\sigma(x, y; \cdot, \cdot)$ for selecting controls in the regulation sets $R(x,y)$.

Example— COOPERATIVE BEHAVIOR

Let $\sigma : \text{Graph}(R) \mapsto \mathbf{R}$ be continuous.

Proposition 2.1 *We posit the assumptions of Theorem 2.1. Let σ be continuous on* Graph(R) *and convex with respect to the pair (u,v). Then, for all initial state $(x_0, y_0) \in K$, there exist a playable solution starting at (x_0, y_0) and a playable controls to the differential game (1) which are regulated by:*

$$(38) \quad \begin{cases} \text{for almost all } \geq 0, \\ \sigma(x(t), y(t); u(t), v(t)) = \inf_{u',v' \in R(x(t),y(t))} \sigma(x(t), y(t); u', v') \end{cases}$$

In particular, the game can be played by the **slow feedbacks** *of minimal norm:*

$$\begin{cases} (u^\circ(x,y), v^\circ(x,y)) \in R(x,y) \\ \|u^\circ(x,y)\|^2 + \|v^\circ(x,y)\|^2) = \min_{(u,v) \in R(x,y)} (\|u\|^2 + \|v\|^2) \end{cases}$$

Proof — We introduce the set-valued map S_R defined by:

$$(39) S_R(x,y) := \{(u,v) \in Y \mid \sigma(x,y;u,v) \leq \inf_{(u',v') \in R(x,y)} \sigma(x,y;u',v')\}$$

It is a convex *selection procedure* of R. Indeed, since R is lower semicontinuous, the function

$$(x,y;u,v) \mapsto \sigma(x,y;u,v) + \sup_{(u',v') \in R(x,y)} (-\sigma(x,y;u',v'))$$

is lower semicontinuous thanks to the Maximum Theorem. Then the graph of S_R is closed because

$$(40) \quad \begin{array}{l} \text{Graph}(S_R) = \\ \{(x,y) \mid \sigma(x,y;u,v) + \sup_{(u',v')\in R(x,y)}(-\sigma(x,y;u',v')) \leq 0\} \end{array}$$

The images are obviously convex. Consequently, the graph of R being also closed, so is the selection $S(R)$ equal to:

$$S(R(x,y)) = \{(u,v) \in R(x,y) \mid \sigma(x,y;u,v) \leq \inf_{(u',v')\in R(x,y)} \sigma(x,y;u',v'))\}$$

We then apply Theorem 2.2. We observe that when we take

$$\sigma(x,y;u,v) \quad := \quad \|u\|^2 + \|v\|^2$$

the selection procedure is strict and yields the elements of minimal norm. □

Example— NONCOOPERATIVE BEHAVIOR We can also choose controls in the regulation sets $R(x,y)$ in a non cooperative way, as saddle points of a function $a(x,y;\cdot,\cdot)$.

Proposition 2.2 *We posit the assumptions of Theorem 2.1 and we suppose that K is a playability domain. Let us assume that $a : X \times Y \times U \times V \to \mathbf{R}$ satisfies*

$$(41) \quad \left\{ \begin{array}{ll} i) & a \text{ is continuous} \\ ii) & \forall (x,y,v) \in X \times V, \quad u \mapsto a(x,y;u,v) \text{ is convex} \\ iii) & \forall (x,y;u) \in X \times U, \quad v \mapsto a(x,y;u,v) \text{ is concave} \end{array} \right.$$

Then, for all initial state $(x_0, y_0) \in K$, there exist a playable solution starting at (x_0, y_0) and a playable controls to the differential game (1) which are regulated by:

$$\text{for almost all } t \geq 0, \quad \left\{ \begin{array}{ll} i) & (u(t),v(t)) \in R(x(t),y(t)) \\ ii) & \forall\,(u',v') \in R(x(t),y(t)), \\ & a(x(t),y(t);u(t),v') \leq a(x(t),y(t);u(t),v(t)) \\ & \leq a(x(t),y(t);u',v(t)) \end{array} \right.$$

Proof — We prove that the set-valued map S_R associating to any $(x,y) \in K$ the subset

$$(42) \quad \begin{array}{l} S_R(x,y) \quad := \quad \{\ (u,v) \in U \times V \text{ such that} \\ \forall (u',v') \in R(x,y), \quad a(x,u,v') \leq a(x,u',v)\ \} \end{array}$$

is a convex selection procedure of R. The associated selection map $S(R(\cdot))$ associates with any $x \in X$ the subset

$$(43) \qquad \begin{array}{lcl} S(R(x,y)) & := & \{ (u,v) \in R(x,y) \text{ such that} \\ \forall (u',v') \in R(x,y), & & a(x,y;u,v') \le a(x,y;u',v) \ \} \end{array}$$

of saddle-points of $a(x,y;\cdot,\cdot)$ in $R(x,y)$. Von Neumann' Minimax Theorem states that the subsets $S(R(x,y))$ of saddle-points are not empty since $R(x,y)$ are convex and compact. The graph of S_R is closed thanks to the assumptions and the Maximum Theorem because it is equal to the lower section of a lower semicontinuous function:

$$(44)\mathrm{Graph}(S_R) = \{(x,y) \mid \sup_{(u',v') \in R(x,y)} (a(x,y;u,v') - a(x,y;u',v)) \le 0\}$$

We then apply Theorem 2.2. □

3 Discriminating and Pure Feedbacks

We address now the question of finding for Xavier feedback controls which are selection of the set-valued map

$$(x,y,v) \rightsquigarrow A(x,y,v) \ \subset \ U(x,y)$$

defined by

$$(45) \qquad A(x,y;v) \ := \ \{ \ u \in U(x,y) \mid (u,v) \in R(x,y) \ \}$$

Such feedbacks are called **discriminating feedbacks**. If we assume that Xavier has access to the controls chosen by Yves, he can keep the states of the system playable by "playing" a discriminating control whatever the choice of Yves through a discriminating feedback.

Then, we shall investigate whether we can find (possibly, single-valued) selections of such a set-valued map A, and for that, provide sufficient conditions for A to be lower semicontinuous.

We first observe that A can be written in the form

$$(46) \qquad A(x,y;v) \ := \ \Phi(x,y;v) \cap (\Psi(x,y))^{-1}(v)$$

The first assumption we have to make for obtaining discriminating feedbacks for Yves is that the domain of the set-valued maps $A(x,y;\cdot))$ are not empty. i.e., that

$$\begin{cases} \forall \, v \in V(x,y), \ \exists \, u \in U(x,y) \ \text{ such that} \\ f(x,y;u) \in DP(y,x)(g(x,y;v)) \cap DQ(x,y)^{-1}(g(x,y;v)) \end{cases}$$

We shall actually strengthen it a bit to get the lower semicontinuity of A, by assuming that

$$(47)\begin{cases} \forall\,(x,y)\in K, \forall\, v\in V(x,y), \exists\,\delta>0,\ \exists\,\gamma>0 \quad \text{such that} \\ \forall\,(x',y')\in B_K(x,y,\delta),\ \forall\, v'\in B(v,\delta)\cap V(x',y'), \\ \forall\,\|e_i\|\le\gamma\ (i=1,2)\ \exists\, u\in U(x',y') \quad \text{such that } f(x',y';u) \\ \in (DP(y',x')(g(x',y';v'))-e_1)\cap(DQ(x',y')^{-1}(g(x',y';v'))-e_2) \end{cases}$$

Proposition 3.1 *We posit the assumptions of Theorem 2.1, where we replace strong playability by assumption (47), and we assume further that the norms of the closed convex processes $DP(y,x)$ and $DQ(x,y)^{-1}$ are bounded. Then the set-valued map A is lower semicontinuous.*

Proof — First, we have to prove that Φ is lower semicontinuous, and, for that purpose, that $(x,y;w)\rightsquigarrow DP(y,x)(w)$ is lower semicontinuous.

By a generalization of the Banach-Steinhauss Theorem to closed convex process of [7], we know that it is sufficient to prove that

$$(x,y)\rightsquigarrow \mathrm{Graph}(DP(y,x)) \text{ is lower semicontinuous}$$

and that

$$\|DP(y,x)\| := \sup_{\|w\|\le 1}\ \inf_{u\in DP(y,x)(w)} \|u\| < +\infty$$

This is the case because P is assumed to be sleek and because we have assumed that the norms of the derivatives are bounded.

Therefore, the set-valued map

$$(x,y,v)\rightsquigarrow DP(y,x)(g(x,y;v))$$

is also lower semicontinuous.

The Lower Semicontinuity Criterion and assumption (47) imply that $(x,y,v)\rightsquigarrow \Phi(x,y;v)$ is lower semicontinuous.

The same proof shows that the set-valued map $(x,y,v)\rightsquigarrow \Psi(x,y)^{-1}(v)$ is also lower semicontinuous. Since A is the intersection of these two set-valued maps, we apply again the Lower Semicontinuity Criterion to deduce that A is lower semicontinuous, which is possible thanks to assumption (47). □

Theorem 3.1 *We posit the assumptions of Proposition 3.1. For any continuous feedback control $(x,y)\mapsto \tilde{v}(x,y)$ played by Yves, there exits a continuous single-valued feedback $\tilde{u}(x,y)$ played by Xavier such that the differential equation (25) has playable solutions for any initial state $(x_0,y_0)\in K$.*

More generally, let S_A be a convex selection procedure of set-valued map A. Then, for any continuous feedback control $(x,y) \mapsto \tilde{v}(x,y)$ played by Yves, for all initial state $(x_0,y_0) \in K$, there exists a playable solution starting at (x_0,y_0) to the differential game

$$(48) \qquad \begin{cases} i) & x'(t) = f(x(t),y(t);u(t)) \\ ii) & y'(t) = g(x(t),y(t);\tilde{v}(x(t),y(t))) \\ iii) & u(t) \in S(A(x(t),y(t);\tilde{v}(x(t),y(t)))) \end{cases}$$

In particular, if the selection procedure is strict, then the control $\tilde{u}(x,y)$ defined by

$$\tilde{u}_{\tilde{v}}(x,y) \ := \ S(A(x,y;\tilde{v}(x,y)))$$

is a single-valued feedback controls.

This is the case, for instance, when Xavier plays the feedback control $u^{\circ}_{\tilde{v}}(x,y)$ of minimal norm in the set $A(x,y;\tilde{v}(x,y))$.

Proof — Whenever Yves plays a continuous feedback $\tilde{v}(x,y)$, K remains a playability domain for the system

$$(49) \qquad \begin{cases} i) & x'(t) = f(x(t),y(t);u(t)) \\ ii) & y'(t) = g(x(t),y(t);\tilde{v}(x(t),y(t))) \\ iii) & u(t) \in A(x(t),y(t);\tilde{v}(x(t),y(t))) \end{cases}$$

Since the set-valued map $(x,y) \rightsquigarrow A(x,y;\tilde{v}(x,y))$ is lower semicontinuous, it contains continuous selections $\tilde{u}(x,y)$ which therefore yield playable selections.

We can also think of using more constructive convex selection procedures of the set-valued map $(x,y) \rightsquigarrow A(x,y;\tilde{v}(x,y))$ and deduce, as in the proof of Theorem 2.2, that Xavier can implement playable solutions by playing controls $u(t)$ in the selection $S(A(x(t),y(t);\tilde{v}(x(t),y(t))))$. □

A much better situation for Xavier occurs when he can find feedback controls $\tilde{u}$ which are selections of the set-valued map B defined by

$$B(x,y) \ := \ \bigcap_{v \in V(x,y)} A(x,y;v)$$

In other words, such a feedback allows him to implement playable solutions whatever the control $v \in V(x,y)$ chosen by Yves, since in this case the pair (u,v) belongs to the regulation set $R(x,y)$ for any v.

Such feedbacks are called **pure feedbacks**.

In order to obtain continuous single-valued feedbacks, we need to prove the lower semicontinuity of the set-valued map B, which is an infinite intersection of lower semicontinuous set-valued maps.

Theorem 3.2 *We posit the assumptions of Proposition 3.1. We assume further that there exist positive constants δ and γ such that for all $(x', y') \in B_K((x,y),\delta)$, we have*

$$(50)\begin{cases} \forall\, v \in V(x',y'),\ \forall\, e_v^i \in \gamma B,\ (i=1,2),\ \exists\, u \in U(x',y') \quad \textit{such that} \\ f(x',y';u) \in DP(y',x';v) + e_v^1 \\ g(x',y';v) \in DQ(x',y';u) + e_v^2 \end{cases}$$

Then the set-valued map B is lower semicontinuous and there exist continuous single-valued pure feedback controls for Xavier.

Proof — We observe that V is upper semicontinuous with compact values, that A is lower semicontinuous and has its images in a fixed compact set, and that assumption (50) implies obviously that there exist positive constants δ and γ such that for all $(x', y') \in B_K((x,y),\delta)$, we have

$$(51) \qquad \forall\, v \in V(x',y'),\ \forall\, e_v \in \gamma B,\ \bigcap_{v \in V(x',y')} (A(x',y';v) - e_v) \neq \emptyset$$

This theorem follows then from the following general criterion on the lower semicontinuity of an infinite intersection of lower semicontinuous set-valued maps. □

Theorem 3.3 *Let us consider set-valued maps $F : X \times Y \rightsquigarrow Z$ and $H : X \rightsquigarrow Y$. We assume that*

$$(52) \qquad \begin{cases} i) & F \textit{ is lower semicontinuous with convex values} \\ ii) & H \textit{ is upper semicontinuous with compact values} \end{cases}$$

and that there exist positive constants γ, δ and c such that for all $x' \in B(x,\delta)$, we have

$$(53) \quad \forall\, y \in H(x'),\ \forall\, e(y) \in \gamma B,\ cB \cap \left(\bigcap_{y \in H(x')} F(x',y) - e(y) \right) \neq \emptyset$$

Then the set-valued map $G : X \rightsquigarrow Z$ defined by

$$(54) \qquad \forall\, x \in X,\ G(x) := \bigcap_{y \in H(x)} F(x,y)$$

is lower semicontinuous.

Remark — When the set-valued map F is locally compact, i.e., maps an adequate neighborhood of each point to compact subsets, we do not need the constant c and we can replace (53) by

$$(55) \quad \forall\, y \in H(x'),\ \forall\, e(y) \in \gamma B,\ \left(\bigcap_{y \in H(x')} F(x', y) - e(y) \right) \neq \emptyset \quad \square$$

Proof — Let us choose any sequence of elements x_n converging to x and $z \in G(x)$. We have to approximate z by elements $z_n \in G(x_n)$.

We introduce the following numbers:

$$(56) \qquad e_n \;:=\; \sup_{y \in H(x_n)} d(z, F(x_n, y))/2$$

Now, let us choose in each $y \in H(x_n)$ an element $u_n(y) \in F(x_n, y)$ satisfying

$$\|z - u_n(y)\| \;\leq\; 2d(z, F(x_n, y)) \;\leq\; e_n$$

We set $\theta_n \;:=\; \gamma/(\gamma + e_n)$. Consequently,

$$\theta_n(z - u_n) \in \theta_n e_n B = (1 - \theta_n)\gamma B$$

So that there exists $a_n(y) \in \gamma B$ such that $\theta_n(z - u_n) = (1 - \theta_n)a_n(y)$

Therefore, assumption (53) implies the existence for all n large enough of elements $w_n \in cB$ and elements $v_n(y) \in F(x_n, y)$ such that $a_n(y) = v_n(y) - w_n$ for all $y \in H(x_n)$.

Hence we can write

$$\theta_n(z - u_n) = (1 - \theta_n)(v_n(y) - w_n)$$

so that the common value:

$$(57) \qquad z_n := \theta_n z + (1 - \theta_n)w_n = \theta_n u_n(y) + (1 - \theta_n)v_n(y)$$

does not depend of y, belongs to all $F(x_n, y)$ (by convexity) and converges to z because

$$\|z - z_n\| \;=\; (1 - \theta_n)\|z - w_n\| \;\leq\; (1 - \theta_n)(\|z\| + c)$$

and because $(1 - \theta_n) = e_n/(\gamma + e_n)$ converges to 0 for e_n converges to 0 thanks to the following lemma. $\square$

Lemma 3.1 *Let us assume that F is lower semicontinuous and that H is upper semicontinuous with compact images. Then the numbers e_n defined by (56) converge to* 0.

Proof — Since F is lower semicontinuous, the Maximum Theorem implies that the function

$$(x, y, z) \mapsto d(z, F(x, y))$$

is upper semicontinuous. Therefore, for any $\epsilon > 0$ and any $y_i \in H(x)$, there exist an integer N_i and a neighborhood $\mathcal{V}_i$ of y_i such that

$$\forall\, y \in \mathcal{V}_i,\ \forall\, n \geq N_i,\ d(z, F(x_n, y)) \leq \epsilon \tag{58}$$

because $d(z, F(x, y_i)) = 0$. Hence the compact set $H(x)$ can be covered by n neighborhoods $\mathcal{V}_i$ and there exists an integer N_0 such that, H being upper semicontinuous,

$$\forall\, n \geq N_0,\ H(x_n) \subset \bigcup_{i=1,\ldots,n} \mathcal{V}_i \tag{59}$$

Set $N := \max_{i=0,\ldots,n} N_i$. Then, for all $n \geq N$ and $y \in H(x_n)$, y belongs to some $\mathcal{V}_i$, so that, by (58), $d(z, F(x_n, y)) \leq \epsilon$. Thus, for all $n \geq N, e_n := \sup_{y \in H(x_n)} d(z, F(x_n, y))/2 \leq \epsilon/2$, i.e., our lemma is proved. □

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