WORKING PAPER

OPTIMAL SAVING WITH CAPITAL-EMBODIED TECHNICAL CHANGE AND NON-STATIONARY POPULATION

Evert van Imhoff

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IIASA
International Institute for Applied Systems Analysis

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FOREWORD

This paper was prepared in the course of the IIASA summer students program and deals with the modeling of dynamic economic phenomena.

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CONTENTS

1. INTRODUCTION

The theory of economic growth attempts to describe and to explain the long-run development over time of an economic system (or economy, for short). An economic system is essentially dynamic in nature. The three most important sources of dynamics in economics are: accumulation of capital (investment), population growth, and technical change. Moreover, some of these dynamic forces are, at least partly, endogenous to the economic system, i.e. determined by economic forces.

A concept of considerable interest in growth theory is the concept of the steady state. A steady state is a situation of economic development in which all variables grow at a constant rate. These rates can be different for different variables; it can also be zero, so that the corresponding variable is a constant in steady state. It should be pointed out that a necessary, though by no means sufficient, condition for a steady state to occur is that the relevant exogenous variables (like e.g. population) grow at a constant rate.

The theory of optimal economic growth assumes that one or more variables in the economic system can be controlled and is concerned with determining these control variables in such a way that the resulting economic development is optimal with respect to some objective or welfare function. Optimal economic growth has been pioneered by Ramsey in his seminal (1928) article.

Probably the most famous result of optimal economic growth theory is the so-called Golden Rule (Phelps, 1961; Robinson, 1962). It was originally derived within the context of comparative statics, i.e. comparing steady states. The Golden Rule states that the steady state with the highest level of consumption per capita is characterized by the equality of the marginal productivity of capital and the growth rate of the population. Cass (1965) has shown that the Golden Rule can alternatively be derived as the equilibrium position (singular solution) of an optimal control problem, with the integral of consumption per capita as the objective function. Since its first appearance in the literature numerous generalizations and extensions of the Golden Rule have been derived by various authors.

The main drawback of the whole concept of the Golden Rule is its tight link with the notion of the steady state. Although the latter is very appealing from a theoretical point of view it is hardly relevant for actual economic development. At noted above, steady states can come about only if the exogenous variables grow at a constant rate. Clearly, this condition is not satisfied in reality. This is most obvious for population, of which the growth rate is fluctuating quite strongly; typically, the growth rate of population is presently falling in almost all industrialized countries. Another example is technical change, although it should be added that it is not immediately clear whether this is a truly exogenous variable.

The purpose of my research is to analyze the effects of changes in the growth rate of population (g^P) on the optimal economic growth path. Among the most important variables that are directly affected by changes in g^P are the following:

1. the sheer size of the population, i.e. number of consumers and size of the labour force;
2. the labour force/population-ratio and its complement the dependency-ratio;
3. more generally, the age-structure of the population.

These and related demographic variables in turn affect many of the economic variables; in particular, they affect the optimal values of the control variables.

A population with a constant age-structure is said to be stationary. If the age-specific mortality rates are constant, then a population is stationary if and only if the growth rate of the number of newborns (g^B) has been constant for at least n years, where n is the maximum age that man can reach (say 100 years). Obviously, since the age-structure of the population is an important economic variable, a temporary change in the growth rate of newborns during m years results in a departure of the economic growth path from steady state for at least m+n years. The period during which the age-structure of the population is non-constant, population itself being non-stationary, can be labelled a period of demographic transition.

There are four broad groups of growth models that are analyzed in my dissertation. These are the following:

1. the simple neoclassical one-sector model of Solow (1956). This analysis can be regarded as a non-stationary generalization of the

classical Golden Rule case;

2. one-sector models with technical change;
3. one-sector models with education;
4. one-sector models with education and technical change.

For each model the analysis consists of four steps:

1. formulation of the model in mathematical terms;
2. the derivation of the necessary conditions for optimal economic growth, using the Maximum Principle of control theory;
3. characterization of steady states as equilibrium points (singular solutions) of the optimal control problem, as well as comparative statics, i.e. assessing the effects of changes in the long-run growth rate of the population on the steady-state values of the economic variables;
4. analysis of the non-stationary optimal economic growth path, i.e. the optimal growth path moving the economy from its initial steady state to its new steady state (after the period of demographic transition has come to an end).

Some results obtained thus far have been published in the form of working papers. The analysis of optimal growth in the basic one-sector model is given in Van Imhoff & Ritzen (1987). A model with education is considered in Van Imhoff (1985).

The present paper analyzes optimal economic growth in a model with technical change that is embodied in physical capital. For a discussion of this and other types of technical change, as well as of their relation to the production function, see Van Imhoff (1986a). If technical change is embodied in capital the model becomes one of capital vintages, i.e. capital goods (machines) are distinguished by their date of construction. Thus the development of the economy is an explicit function of its history, at least of its most recent history. This feature should lead one to expect that non-stationarities in the economic development triggered by the occurrence of demographic transition are particularly severe and persistent in this model.

The following assumptions will be made throughout this paper:

1. there is one single production sector that produces an aggregate

commodity. Production can be either consumed or added to the stock of physical capital which is distinguished according to its date of construction;

2. economic decisions are made by a central planning agency which seeks to maximize some social welfare function in terms of consumption per capita;
3. human capital (showing in labour efficiency) is a function of age only (i.e. investment in education will not be considered).

The plan of the remainder of this paper is as follows. Section 2 spells out the model. In section 3 a condition for optimal economic growth will be derived. This condition turn out to be in many respects similar to the well-known Golden Rule of Accumulation mentioned above. Section 4 gives some comparative statics results. In section 5 I analyze the stability of steady states while section 6 investigates some properties of the non-stationary optimal growth path. The final section summarizes the main results.

2. THE FIXED-COEFFICIENTS CAPITAL VINTAGE MODEL

The growth model consists of three building blocks: population and labour; production, investment, and technical change; and the social welfare function to be maximized by the central planning agency. Each block will be described in turn.

2.1. Population and labour

The model is one of overlapping generations in continuous time. The number of individuals born at time t is denoted by B(t). The number of newborns at time t is related to the number of newborns in the previous period by the rate of growth of births, denoted by $g^B(t)$:

$$\hat{B}(t) = \dot{B}(t)/B(t) = g^B(t) \tag{1}$$

The dynamic path of $g^B(t)$ will be assumed to be completely exogenously determined.

If people die according to some fixed age-specific survival schedule $\mu(v)$ and if maximum age is denoted by n, then total population at time t P(t) can be written as

$$P(t) = \int_0^n \mu(v) \cdot B(t-v)\, dv \tag{2}$$

where

$$\mu(0) = 1 \qquad \mu(n) = 0 \qquad \mu'(v) \leq 0 \tag{3}$$

Each individual is endowed with a stock of human capital $h(v)$ that is a function of age v only. This implies that all individuals of a given age are equal in their ability to produce. Typically the function $h(v)$ is assumed to be unimodal, with low values for v close to zero and v close to n and higher values for ages in the intermediate range. The labour force at time t $L(t)$ measured in units of human capital can now be written as

$$L(t) = \int_0^n h(v) \cdot \mu(v) \cdot B(t-v)\, dv \tag{4}$$

The rate of population growth $g^P(t)$ is defined by

$$g^P(t) = \hat{P}(t) = \dot{P}(t)/P(t) \tag{5}$$

Similarly, the rate of growth of the labour force $g^L(t)$ is defined by:

$$g^L(t) = \hat{L}(t) = \dot{L}(t)/L(t) \tag{6}$$

In general, given the survival schedule $\mu(v)$ and the age-ability profile $h(v)$, the growth rates $g^P(t)$ and $g^L(t)$ are completely determined by the dynamic path of $g^B(t)$. Equivalently, $g^P(t)$ and $g^L(t)$ are a function of $g^B(t)$ and the age-structure of the population. When $g^B(t)$ is constant for at least n successive periods then the age-structure of the population is also constant and we have:

$$g^P(t) = g^L(t) = g^B(t) = g \text{ , say} \tag{7}$$

In this case the population is said to be stationary.

2.2. Production, investment, and technical change

The aggregate commodity is produced from labour and (physical) capital where capital goods are distinguished by their date of construction. Production obtained from capital of a certain vintage is described by a so-called vintage production function:

$$Q(v,t) = F[K(v,t),\ L(v,t);\ v] \tag{8}$$

Here v is the time at which the capital goods under consideration have been constructed; K(v,t) is the size of the capital stock installed at time v and still in existence at time t (this could be less than the amount originally invested as a result of depreciation); L(v,t) is the amount of labour allocated to work with the capital goods in question; and Q(v,t) is the resulting output. The fact that the vintage production function F[·] is parametrized with an index v reflects the presence of capital-embodied capital change: the productivity of given amounts of factor inputs K and L depends on the date at which the capital goods have been installed.

Total production at time t is given by the sum of all outputs produced from the different capital vintages, i.e.

$$Q(t) = \int_{-\infty}^{t} Q(v,t)\, dv \tag{9}$$

Physical capital is subject to depreciation at a constant rate δ:

$$K(v,t) = K(v,v)\cdot e^{\delta(v-t)} \tag{10}$$

In each period a fraction of total output is saved and added to the capital stock (invested):

$$K(t,t) = I(t) = s(t)\cdot Q(t) \tag{11}$$

The (gross) rate of savings s(t) cannot exceed one. It will be assumed that physical capital, once installed, is not fit for consumption which implies that the rate of savings cannot become negative. Output not invested in physical capital is consumed. Total consumption equals:

$$C(t) = Q(t) - I(t) \tag{12}$$

We are left with the specification of the production function (8). I assume the vintage production function to be characterized by fixed factor proportions ("clay-clay"):

$$Q(v,t) = \min \{k(v)\cdot K(v,t),\ l(v)\cdot L(v,t)\} \quad \text{for all } v \leq t \tag{13}$$

This model has been investigated extensively by Solow et alii (1966). $k(\cdot)$ and $l(\cdot)$ are indexes of capital-augmenting and labour-augmenting technology, respectively. The development over time of these indexes is assumed to satisfy:

$$k'(v) \geq 0 \quad ; \quad l'(v) \geq 0 \tag{14}$$

Most of the time I will assume that $k(\cdot)$ is constant and that $l(\cdot)$ grows exponentially over time, i.e. technical change is exponential and Harrod-neutral everywhere.

From (9), (13) and (14) it is evident that, given the stocks of physical capital of all different vintages, production at time t is maximized by allocating labour across capital vintages such that:

$$L(v,t) = \begin{cases} \frac{k(v)}{l(v)}\cdot K(v,t) & \text{for all } v \geq t - T(t) \\ 0 & \text{for all } v < t - T(t) \end{cases} \tag{15}$$

where $T(t)$ denotes the <u>age</u> of the oldest capital vintage in use at time t. $T(t)$ is restricted by the size of the labour force:

$$L(t) = \int_{t-T(t)}^{t} L(v,t)\, dv \tag{16}$$

(9)-(11), (13), (15) and (16) together imply:

$$L(t) = \int_{t-T(t)}^{t} \frac{k(v)}{l(v)}\cdot e^{\delta(v-t)}\cdot I(v)\, dv \tag{17}$$

$$Q(t) = \int_{t-T(t)}^{t} k(v)\cdot e^{\delta(v-t)}\cdot I(v)\, dv \tag{18}$$

It should be stressed that in (17), $L(t)$ is <u>exogenous</u> and $T(t)$ <u>endogenous</u>, not the other way round.

2.3. Social welfare

We will take the social welfare function, of which the maximization is the object of the central planning agency, to be simply the discounted sum of per capita consumption:

$$W = \int_0^\infty e^{-rt} \cdot \frac{Q(t)-I(t)}{P(t)}\, dt \tag{19}$$

where r is the social rate of time preference. For a discussion of this and related sociale welfare functions see Burmeister & Dobell (1970), pp. 398-400). One reason for choosing specification (19) is that it corresponds closely to the social welfare function in the earlier writings on the steady-state Golden Rule, maximizing long-run sustainable consumption per head.

3. OPTIMAL ECONOMIC GROWTH

The central planning agency maximizes the social welfare function (19) subject to (17), (18), and

$$0 \leq I(t) \leq Q(t) \qquad \text{(boundary restriction on the control)} \tag{20}$$

$$I(v) = I_0 \quad \text{for all } v<0 \quad \text{(initial conditions)} \tag{21}$$

The control variable is I(t). Although I(t) determines T(t) via (17), and T(t) determines Q(t) via (18), I treat I(t), T(t) and Q(t) as three independent control variables that are restricted by (17) and (18).

In the analysis that follows I have made use of some very valuable advice given to me by Onno van Hilten of Limburg University (cf. Malcomson, 1975; Nickell, 1975; Verheyen & Lieshout, 1978).

Linking restrictions (17) and (18) to the maximand (19) with the use of the Lagrange multipliers $w_L(t)$ and $w_Q(t)$ yields:

$$W = \int_0^\infty \Bigg[e^{-rt} \cdot \frac{Q(t)-I(t)}{P(t)} + w_L(t) \cdot \left[\int_{t-T(t)}^{t} \frac{k(v)}{l(v)} \cdot e^{\delta(v-t)} \cdot I(v)\, dv - L(t) \right] +$$

$$+ w_Q(t) \cdot \left[\int_{t-T(t)}^{t} k(v) \cdot e^{\delta(v-t)} \cdot I(v)\, dv - Q(t) \right] \Bigg] dt =$$

$$= \int_0^\infty \left[e^{-rt} \cdot \frac{Q(t)-I(t)}{P(t)} - w_L(t)\cdot L(t) - w_Q(t)\cdot Q(t) \right] dt \; +$$

$$+ \int_0^\infty \left[\int_{t-T(t)}^{t} k(v)\cdot e^{\delta(v-t)}\cdot I(v)\cdot \left[w_Q(t) + w_L(t)/l(v)\right] dv \right] dt \quad (22)$$

The last term on the RHS of (22) is a double integral. The area over which the integration is performed is the shaded area in Figure 1.

If the function T(t) is such that

$$T'(t) > -1 \quad \text{for all } t \qquad (23)$$

(i.e. capital once out of use remains out of use forever), then the following inverse funcion of t-T(t) exists:

$$t+Z(t) = INV[t-T(t)] \qquad (24)$$

From (24):

$$t = t + Z(t) - T[t+Z(t)] \implies Z(t) = T[t+Z(t)] \qquad (25)$$

Thus, Z(t) is the age at which capital installed at time t will become obsolete. From (23) and (25) we find that:

$$Z'(t) = T'[t+Z(t)].(1+Z'(t)) \implies Z'(t) > -1 \quad \text{for all } t \qquad (26)$$

Using the definition of Z(t), a double integral of some function f(v,t) over the shaded area in Figure 1 can be rewritten by changing the order of integration as follows:

$$\int_0^\infty \left[\int_{t-T(t)}^{t} f(v,t)\, dv \right] dt =$$

$$= \int_0^\infty \left[\int_v^{v+Z(v)} f(v,t)\, dt \right] dv + \int_{-T(0)}^{0} \left[\int_0^{v+Z(v)} f(v,t)\, dt \right] dv \quad (27)$$

In Figure 2 the shaded area corresponds to the first integral on the RHS of (27) while the cross-hatched area corresponds to the second integral.

Using (27) after interchanging the symbols v and t, the integral in (22) can be written as:

$$W = \int_0^\infty \left[e^{-rt} \cdot \frac{Q(t)-I(t)}{P(t)} - w_L(t) \cdot L(t) - w_Q(t) \cdot Q(t) \right] dt +$$

$$+ \int_0^\infty k(t) \cdot e^{\delta t} \cdot I(t) \cdot \left[\int_t^{t+Z(t)} e^{-\delta v} \cdot \left[w_Q(v) + w_L(v)/l(t)\right] dv \right] dt +$$

$$+ \int_{-T(0)}^{0} k(t) \cdot e^{\delta t} \cdot I_t \cdot \left[\int_0^{t+Z(t)} e^{-\delta v} \cdot \left[w_Q(v)+w_L(v)/l(t)\right] dv \right] dt \quad (28)$$

In writing the third integral in (28) use has been made of the initial conditions in (21).

Necessary conditions for the maximization of W are that the integrand in (28) be maximized with respect to the controls $I(\cdot)$, $Q(\cdot)$ $Z(\cdot)$ and be minimized with respect to the multipliers $w_Q(\cdot)$ and $w_L(\cdot)$, at each point in time. If attention is restricted to time periods later than $Z(0)$, then the third integral in (28) vanishes and the necessary conditions are the following:

$$\frac{\partial W}{\partial Q(t)} = \frac{e^{-rt}}{P(t)} - w_Q(t) = 0 \quad (29)$$

$$\frac{\partial W}{\partial I(t)} = - \frac{e^{-rt}}{P(t)} + k(t) \cdot e^{\delta t} \cdot \int_t^{t+Z(t)} e^{-\delta v} \cdot \left[w_Q(v)+w_L(v)/l(t)\right] dv =$$

$$= \begin{cases} \geq 0 & \text{if} \quad I(t) = Q(t) \\ = 0 & \text{if} \quad 0 < I(t) < Q(t) \\ \leq 0 & \text{if} \quad I(t) = 0 \end{cases} \quad (30)$$

$$\frac{\partial W}{\partial Z(t)} = k(t) \cdot e^{\delta \cdot Z(t)} \cdot I(t) \cdot \left[w_Q[t+Z(t)] + w_L[t+Z(t)]/l(t)\right] = 0 \quad (31)$$

$$\frac{\partial W}{\partial w_Q(t)} = - Q(t) + \int_{t-T(t)}^{t} k(v) \cdot e^{\delta(v-t)} \cdot I(v)\, dv = 0 \quad (32)$$

$$\frac{\partial W}{\partial w_L(t)} = - L(t) + \int_{t-T(t)}^{t} \frac{k(v)}{l(v)} \cdot e^{\delta(v-t)} \cdot I(v)\, dv = 0 \quad (33)$$

From now on I will concentrate on singular arcs. In other words: I will assume an interior solution to optimal investment, such that $0<I(t)<Q(t)$ and the RHS of (30) is identically zero.
Under this assumption (31) implies:

$$w_Q[t+Z(t)] + w_L[t+Z(t)]/l(t) = 0 \quad (34)$$

or, equivalently, lagging (34) by Z(t) periods and using (24):

$$w_Q[t] + w_L[t]/l[t-T(t)] = 0 \tag{35}$$

On the other hand we have from (29) and the observation that the conditions (29) through (33) must hold for longer than a single instant along a singular arc:

$$\frac{d}{dt}\frac{W}{Q(t)} = - r\cdot\frac{e^{-rt}}{P(t)} - \frac{\dot{P}(t)}{P(t)}\cdot\frac{e^{-rt}}{P(t)} - \dot{w}_Q(t) = 0 \tag{36}$$

from which, using (29) and definition (5):

$$\dot{w}_Q(t) = - [r + g^P(t)]\cdot w_Q(t) \tag{37}$$

Integrating (37):

$$w_Q(v) = w_Q(t)\cdot \exp\Big[- \int_t^v [r + g^P(u)]\, du \Big] \tag{38}$$

Substitution of (35) and (38) into (30), using (29), yields:

$$0 = \frac{e^{-rt}}{P(t)}\cdot\Big[- 1 + \int_t^{t+Z(t)} \exp\Big[- \int_t^v [r+d+g\ (u)]\, du \Big]\cdot k(t)\cdot \\ \cdot\Big[1 - \frac{l[v-T(v)]}{l(t)} \Big]\, dv \Big] \tag{39}$$

Bearing in mind the inverse relationship between Z(·) and T(·), equation (39) is a <u>condition for the occurrence of a singular arc in terms of the lifetimes of subsequent capital vintages</u>.

I will now show that condition (39) is equivalent to the <u>non-stationary Golden Rule</u> for the simple (non-vintage) neoclassical model as derived in Van Imhoff & Ritzen (1987). The marginal productivity of capital of some vintage v can be obtained from (18):

$$\frac{dQ(t)}{dI(v)} = J_{[t-T(t),t]}(v)\cdot k(v) + k[t-T(t)]\cdot e^{-\delta\cdot T(t)}\cdot I[t-T(t)]\cdot\frac{dT(t)}{dI(v)} \tag{40}$$

where the <u>indicator function</u> $J_A(x)$ is defined by

$$J_A(x) = \begin{cases} 1 & \Longleftrightarrow\ x \in A \\ 0 & \Longleftrightarrow\ x \notin A \end{cases} \tag{41}$$

From (17) we have:

$$0 = J_{[t-T(t),t]}(v)\cdot\frac{k(v)}{l(v)}\cdot d[I(v)] +$$

$$+ \frac{k[t-T(t)]}{l[t-T(t)]}\cdot e^{-\delta\cdot T(t)}\cdot I[t-T(t)]\cdot d[T(t)] \qquad (42)$$

from which

$$\frac{dT(t)}{dI(v)} = - J_{[t-T(t),t]}(v)\cdot\frac{k(v)}{l(v)}\cdot\frac{l[t-T(t)]}{k[t-T(t)]}\cdot e^{\delta\cdot T(t)}\cdot\frac{1}{I[t-T(t)]} \qquad (43)$$

and thus, from (40) and (43):

$$\frac{dQ(t)}{dI(v)} = J_{[t-T(t),t]}(v)\cdot k(v)\cdot\left[1 - \frac{l[t-T(t)]}{l(v)} \right] \qquad (44)$$

From (44) and (24) it follows that:

$$\frac{dQ(v)}{dI(t)} = J_{[t,t+Z(t)]}(v)\cdot k(t)\cdot\left[1 - \frac{l[v-T(v)]}{l(t)} \right] \qquad (45)$$

(cf. Solow e.a., 1966).

Thus it can be seen that the integral in the RHS of (39) is equal to the present value of all future returns to investment made at time t, discounted at a rate equal to the sum of the rates of social impatience (r), depreciation (δ), and population growth (g^P). On the other hand, the marginal costs of investment (in terms of consumption foregone) equal unity. Thus condition (39) simply says that the singular arc is characterized by the familiar equality of marginal costs of and returns to investment.

The condition spelled out in the previous paragraph is easily seen to be the finite-lifetime equivalent of the non-stationary Golden Rule of Van Imhoff & Ritzen (1987). With infinite lifetime of capital (no obsolescence) the condition becomes:

$$1 = \int_t^\infty \exp\left[- \int_t^v [r+\delta+g^P(u)]\, du \right]\cdot\frac{Q(v)}{K(t)}\, dv \qquad (46)$$

Differentiating (46) with respect to time t:

$$0 = - \frac{dQ(t)}{dK(t)} + [r+\delta+g^P(t)]\cdot\int_t \exp\left[- \int_t^v [r+\delta+g^P(u)]\, du \right]\cdot\frac{dQ(v)}{dK(t)}\, dv \quad (47)$$

from which, using (46):

$$\frac{dQ(t)}{dK(t)} = r + \delta + g^{P}(t) \tag{48}$$

which is the non-stationary Golden Rule.

4. COMPARATIVE STATICS

If population grows at a constant rate g, and if technical progress is exponential and Harrod-neutral everywhere, i.e.

$$k(t) = k_0 \quad ; \quad l(t) = l_0 \cdot e^{lt} \quad \text{with } k_0,\ l_0,\ l \text{ constant} \tag{49}$$

then the optimal growth path could well lead the economy into a steady state.

In steady state the optimal savings rate is constant. As Solow e.a. (1966) have shown a constant savings rate for this model implies the maximum age of capital to be constant too, i.e. $Z(t) = T(t) = T^*$, say.

The value of T^* can be obtained by solving (39). Carrying out the integration yields:

$$1 = \int_t^{t+T^*} e^{a\cdot(t-v)} \cdot \left[1 - e^{l\cdot(v-T^*-t)} \right] dv =$$

$$= (1/a) \cdot \left[1 - e^{-a\cdot T^*} \right] - \frac{e^{-l\cdot T^*}}{a-l} \cdot \left[1 - e^{-(a-l)\cdot T^*} \right]$$

$$\Longrightarrow (a-1)\cdot(a-l) + a\cdot e^{-l\cdot T^*} - l\cdot e^{-a\cdot T^*} = 0 \tag{50}$$

where I write

$$a = r + \delta + g \tag{51}$$

for notational convencience.
Equation (50) cannot be explicitly solved for T^*. If we define

$$G(T^*) = (a-1)\cdot(a-l) + a\cdot e^{-l\cdot T^*} - l\cdot e^{-a\cdot T^*} \tag{52}$$

a steady-state value for T^* exists if $G(\cdot)$ has a finite positive root. Since

$$G'(T^*) = a\cdot b\cdot \left[e^{-a\cdot T^*} - e^{-b\cdot T^*} \right] \tag{53}$$

is monotonous on R^+, a positive root of $G(\cdot)$ is unique if it exists. The existence of such a root depends on the values of a and l. Analysis of the function $G(\cdot)$ yields the following tableau:

parameter values	number of positive roots of $G(\cdot)$
$a = 0\ ,\ l \neq 0$	none
$l = 0\ ,\ a \neq 0$	none
$a = 1$	infinite (identity)
$a > 1 > 0$	one root (if a<1)
$1 > a > 0$	one root (if a<1)
$a > 0 > 1$	none
$1 > a > 0$	one root
$0 > a > 1$	none
$0 > 1 > a$	none

Thus we have the following existence condition:

an optimal steady-state value for T^* exists only if
$l>0$, $a<1$, $a\neq0$ and $a\neq1$ (54)

However, a steady state must also be feasible. That is, the savings rate $s^* = I^*/Q^*$ corresponding to the steady-state value of T^* must be between zero and unity (cf. condition (20)). From (17) and (49) it is seen that in steady state investment $I(\cdot)$ grows at an exponential rate g+l; and from (18) so does production $Q(\cdot)$. Then we have from (18):

$$Q(t) = k_0\cdot \int_{t-T^*}^{t} e^{\delta\cdot(v-t)}\cdot I(v)\ dv = k_0\cdot \int_{t-T^*}^{t} s^*\cdot Q(t)\cdot e^{(g+l)\cdot(v-t)}\ dv$$

$$\Longrightarrow\quad 0 \leq s^* = (1/k_0)\cdot \frac{\delta + g + l}{1 - e^{-(\delta+g+l)\cdot T^*}} \leq 1 \tag{55}$$

Given the form of condition (50) and expression (55) it is very difficult to obtain general comparative-statics results, that is to sign the partial derivatives of s^* and T^* with respect to the parameters g, l, r and δ. Some numerical calculations of steady states are given in Table 1. These results suggest that for reasonable values of the parameters the signs of the partial derivatives are as in Table 2.

It is interesting to note that these comparative-statics results, as far as the savings rate is concerned, are essentially the same as for the simple neoclassical model with dis<u>embodied</u> technical change (see Van Imhoff, 1986b). Moreover, if one is prepared to interpret an increase in T as a decrease in the "capital/labour-ratio", then (with the exception of the effect of l) the results of the two models are similar too for the capital variable.

5. STABILITY OF STEADY STATES

Along a singular arc the endogenous variable T(t), being the age of the oldest capital vintage in operation at time t, develops over time according to equation (39). The question can now be raised: does the optimal economic growth path under suitable external conditions <u>converge</u> towards a steady state? Particularly, if population grows at a constant rate g and if technical progress is exponential and Harrod-neutral everywhere, does then a trajectory T(·) satisfying (39) converge towards the constant value T*? This question is important as it relates to the <u>stability</u> of the steady state.
From (39) and (49) we have:

$$1/k_0 = \int_t^{t+Z(t)} \exp\Big[-\int_t^v [r+\delta+g^P(u)]\,du\Big]\cdot\Big[1 - e^{l\cdot[v-t-T(v)]}\Big]\,dv \qquad (56)$$

Differentiation of (56) with respect to t yields:

$$0 = [1+\dot{Z}(t)]\cdot\exp\Big[-\int_t^{t+Z(t)} [r+\delta+g^P(u)]\,du\Big].\Big[1-e^{l\cdot\{Z(t)-T[t+Z(t)\}}\Big] +$$

$$- \Big[1 - e^{-l\cdot T(t)}\Big] +$$

$$+ [r+\delta+g^P(t)].\int_t^{t+Z(t)} \exp\Big[-\int_t^v [r+d+g^P(u)]\,du\Big]\cdot\Big[1-e^{l\cdot[v-t-T(v)]}\Big]\,dv$$

$$+ l\cdot\int_t^{t+Z(t)} \exp\Big[-\int_t^v [r+\delta+g^P(u)]\,du\Big]\cdot e^{l\cdot[v-t-T(v)]}\,dv \qquad (57)$$

The first term in (57) is equal to zero because of (25). The third term is equal to $[r+\delta+g^P(t)]/k_0$, using (56). The fourth term equals

$$l \cdot \left[- 1/k_o + \int_t^{t+Z(t)} \exp \left[- \int_t^v [r+\delta+g^P(u)]\, du \right] dv \right] \tag{58}$$

also using (56). Thus, equation (57) can be written as:

$$\int_t^{t+Z(t)} \exp \left[- \int_t^v [r+\delta+g^P(u)]\, du \right] dv =$$

$$= \frac{1 - e^{-l \cdot T(t)} - [r + \delta + g^P(t) - l]/k_o}{l} \tag{59}$$

Equation (59), together with definition (25), is an interesting type of difference equation linking T(t) and Z(t): it gives a relationship along the singular arc between the oldest age of capital at time t on the one hand, and the oldest age that capital installed at time t will ever reach on the other hand.

It is easily seen that a necessary and sufficient condition for the difference equation (59) to converge is given by

$$\left| \frac{dZ(t)}{dT(t)} \right| < 1 \tag{60}$$

Carrying out the differentiation yields:

$$\frac{dZ(t)}{dT(t)} = \exp \left[-l \cdot T(t) + \int_t^{t+Z(t)} [r+\delta+g^P(u)]\, du \right] \tag{61}$$

which in the case of stationary population reduces to:

$$\frac{dZ(t)}{dT(t)} = e^{(r+\delta+g) \cdot Z(t) - l \cdot T(t)} \tag{62}$$

This expression is always positive. In the neighbourhood of the steady state we have $Z(t) \approx T(t)$, so that a necessary condition for local convergence (local stability of the steady state) is:

$$l > r+\delta+g \tag{63}$$

This is quite an uncomfortable result as it implies that the integral of the social welfare function (19) <u>diverges</u> for a locally stable singular arc.

A difference equation similar in kind to (59) can also be derived for s(·). From (17) and (49) we have:

$$L(t) = \int_{t-T(t)}^{t} (k_o/l_o)\cdot e^{-l\cdot t}\cdot e^{\delta\cdot(v-t)}\cdot I(v)\,dv \qquad (64)$$

Differentiation of (64) with respect to time yields:

$$\dot{L}(t) = -\delta\cdot\int_{t-T(t)}^{t} (k_o/l_o)\cdot e^{-lt}\cdot e^{\delta(v-t)}\cdot I(v)\,dv + (k_o/l_o)\cdot e^{-lt}\cdot I(t) +$$

$$- [1-\dot{T}(t)]\cdot(k_o/l_o)\cdot e^{-l\cdot t}\cdot e^{(1-\delta)\cdot T(t)}\cdot I[t-T(t)] \qquad (65)$$

After substitution of (64) and (6) and some rearranging (65) reduces to:

$$I(t) = (l_o/k_o)\cdot[g(t)+\delta]\cdot e^{lt}\cdot L(t) + [1-\dot{T}(t)]\cdot e^{(1-\delta)T(t)}\cdot I[t-T(t)] \qquad (66)$$

or, equivalently:

$$s(t) = (l_o/k_o)\cdot[g(t)+\delta]\cdot\frac{e^{l\cdot t}\cdot L(t)}{Q(t)} +$$

$$+ [1-\dot{T}(t)]\cdot e^{(1-\delta)\cdot T(t)}\cdot s[t-T(t)]\cdot\frac{Q[t-T(t)]}{Q(t)} \qquad (67)$$

Expression (67) is a kind of difference equation linking the savings rate at time t to the savings rate at the timeat which the oldest capital in use at time t has been installed.

In the neighbourhood of the steady state we have:

$$\frac{e^{l\cdot t}\cdot L(t)}{Q(t)} \approx \text{constant}$$

$$\dot{T}(t) \approx 0$$

$$\frac{Q[t-T(t)]}{Q(t)} \approx e^{-(g+l)\cdot T(t)}$$

so that

$$0 < \frac{d\ s(t)}{d\ s[t-T(t)]} \approx e^{-(g+\delta)\cdot T(t)} < 1 \qquad (68)$$

Thus the non-stationary time path of s(·) is locally convergent as the economy approaches its new steady-state growth path.

6. THE NON-STATIONARY OPTIMAL ECONOMIC GROWTH PATH

Along the singular arc the time path of the lifetime of subsequent capital vintages is governed by condition (39). Clearly this condition is too complicated to allow the derivation of general characteristics of the non-stationary optimal economic growth path (as in Van Imhoff & Ritzen, 1987). For this particular vintage-model, therefore, I must be content with the more modest target of trying to simulate an optimal economic growth path given some fixed values for the external parameters.

The simulation problem can be described as follows. Given are specified values for:

- r, the social rate of impatience;
- δ, the rate of capital depreciation;
- l, the rate of labour-augmenting technical progress.

It is assumed that initially the growth rate of population has been constant (g_0) for a long time, and that the economy is in its optimal steady state corresponding to this population growth rate g_0. At a certain point in time (t_a) the growth rate of population begins to fall (linearly for the sake of convenience) until at time t_b it reaches a new level g_1 which it will keep forever afterwards. Now what is the optimal economic growth path for these external conditions? The parameters used in the simulation problem are listed in Table 3.

I have used two different approaches to solve the dynamic optimization problem described in the previous paragraph. The first method maximizes the social welfare function directly over a finite time interval, subject to the terminal conditions that the investments made in the last time periods be such that the economy is left (after the planning period) in the steady state corresponding to the population growth rate g_1. The second method maximizes the social welfare function indirectly by simulating the difference equation (59).

A simple check of either method is to run a simulation with g_1=g_0. If everything is well the simulation method should find that the optimal policy is to remain in the initial steady state forever.

Both simulation methods are discussed below.

6.1. The direct method

The direct method maximizes the welfare function (19) over some finite time interval $[t_0, t_1]$ where t_0 and t_1 are chosen such that $t_0 \ll t_a < t_m \ll t_1$, i.e. the simulation period contains the period of demographic transition. The welfare function is maximized with respect to the savings rates $s(t_0)$, $s(t_0+1)$, ... , $s(t_1-1)$, $s(t_1)$, and subject to the conditions

$$I(t) = I_1^*(t) \qquad t = t_1 - \text{int}(T_1^*) \text{ , ... , } t_1 \tag{69}$$

where I_1^* and T_1^* refer to the steady-state values of $I(\cdot)$ and $T(\cdot)$, respectively, corresponding to g_1.

Obviously, in order to keep the number of control variables within manageable limits a discretization of the model is required. For the simulation of the non-stationary growth path itself this discretization is fairly straightforward. In computing the steady-state values of the endogenous variables use has been made of the following approximation:

$$\int_t^{t+T} e^{-x\cdot(v-t)}\,dt \approx \sum_{i=1}^{\text{int}(T)} (1+x)^{-i} + \text{frac}(T)\cdot(1+x)^{-\text{int}(T)-1} \equiv$$

$$\equiv D(t, T, 1+x) \tag{70}$$

T_0^* and T_1^* are computed from the following discrete-time equivalent of (50):

$$1 = D[t, T_i^*, (1+g_i)\cdot(1+\delta)\cdot(1+r)] -$$

$$- (1+l)^{-T_i^*}\cdot D[t, T_i^*, (1+g_i)\cdot(1+\delta)\cdot(1+r)/(1+l)] \qquad i=0,1 \tag{71}$$

Similarly s_0^* and s_1^* are computed from (cf. (55)):

$$s_i^* = \frac{1}{k_0\cdot D[t, T_i^*, (1+g_i)\cdot(1+l)]} \qquad i=0,1 \tag{72}$$

Finally the absolute steady-state level of investment at time t is obtained from the discrete approximation to the steady-state version of (17):

$$L_t = I_t\cdot(k_0/l_0)\cdot(1+l)^{-t}\cdot D[t, T_i^*, (1+g_i)\cdot(1+\delta)] \qquad i=0,1 \tag{73}$$

Along the non-stationary optimal economic growth path T_t is computed from the condition (cf. (17)):

$$L_t = \sum_{i=1}^{int(T_t)} (k_o/l_o)\cdot(1+l)^{i-t}\cdot(1+\delta)^{-i}\cdot I_{t-i} +$$

$$+ frac(T_t)\cdot(k_o/l_o)\cdot(1+l)^{int(T_t)+1-t}\cdot(1+\delta)^{-int(T_t)-1}\cdot I_{t-int(T_t)-1} \quad (74)$$

where L_t and past investments I_{t-i}, $i=1,\ldots$ are known at time t. Given T_t from (74), production at time t is computed as (cf. (18)):

$$Q_t = \sum_{i=1}^{int(T_t)} k_o\cdot(1+d)^{-i}\cdot I_{t-i} +$$

$$+ frac(T_t)\cdot k_o\cdot(1+d)^{-int(T_t)-1}\cdot I_{t-int(T_t)-1} \quad (75)$$

The welfare function to be maximized is (cf. (19)):

$$\max_{s_t} \quad W = \sum_{t=t_o}^{t_1} (1+r)^{-t}\cdot \frac{Q_t\cdot(1-s_t)}{P_t} \quad (76)$$

subject to initial conditions, (74)-(75), the accumulation equations

$$I_t = s_t\cdot Q_t \quad , \quad (77)$$

the control constraints

$$0 \le s_t \le 1 \quad , \quad (78)$$

and the terminal constraints (69).

In order to numerically solve the optimization problem described above I tried several algorithms. None of these, however, produced satisfactory results. It turned out that the numerical solution is highly unstable. This instability can be partly attributed to the non-smoothness of the discretized model (cf. equations (74) and (75)). More generally, the objective function, in combination with the nonlinear constraints (69), appears to be badly-behaved, rendering convergence of any algorithm very difficult to achieve.

The MINOS-program (Murtagh & Saunders, 1983) handles the nonlinear constraints (69) in a direct way, using a so-called projected augmented Lagrangian algorithm. An optimal solution was reported by the program but the time path of s(·) was highly irregular. Further analysis of the solution reveals that the corresponding values of T(·) are for most periods very close to being integer, points at which the equations (74) and (75) are nondifferentiable. Apparently, the nonsmoothness of the discretized problem is so severe that the MINOS-algorithm breaks down.

A very flexible and convenient way of interactively controlling the maximization process is offered by the SQG/PC-program developed by Alexei Gaivoronski at IIASA/SDS. It is actually intended to solve stochastic optimal control problems but it can handle deterministic problems as the one under consideration equally well (for a description of an earlier mainframe-version of the program see Ermoliev & Gaivoronski, 1984). The nonlinear constraints (69) were taken into account by adding a penalty term to the objective function (76).

However, for the present problem convergence of the solution was very difficult to achieve. More seriously, the various algorithms reproduced only a rough approximation to the optimal steady state when the program was run with g_1 set equal to g_0.

Finally I tried a very simple although rather time-consuming simplex algorithm due to Nelder and Mead (described in Churchhouse, 1981). Here the findings were essentially the same as for SQG/PC: very slow convergence, solutions sensitive to starting positions, and high numerical instability in general.

Thus, the conclusion of this sub-section is a simple and disappointing one: for the capital-vintage model under consideration it is very difficult to find the non-stationary optimal growth path by direct methods.

6.2. The indirect method

The indirect method computes the optimal growth path from the singularity condition / difference equation (59). The computation consists of two steps: computation of T(·) from (59); and computation of s(·) given T(·). Contrary to the direct method where, in order to keep the number of variables to be determined optimally within reasonable limits discretization was necessary, the indirect method works with a discretization that can be made as close to continuous time as one wishes.

The computation of T(·) starts from the initial steady state given by

$$T(t) = T_0^* \; ; \quad s(t) = s_0^* \qquad t = t_0 \, , \, \ldots \, , \, t_0 + \text{int}(T_0^*/dt) + 1 \quad (79)$$

where dt is the discretization parameter. Now for $t=t_0$, t_0+dt, ... the variable Z(t) is obtained by numerically solving (59); the intregral is approximated using the Trapezium Rule followed by Romberg Integration (e.g. Churchhouse, 1981). The result is saved as T[t+Z(t)]. Since the index t is necessarily discrete and the solution Z(t) is generally not, the values of T[t+dt·int(Z(t)/dt)] are approximated by parabolic interpolation between three consecutive values of t+Z(t).

The second step involves computing the time path of s(·) corresponding to the simulated path of T(·). There are several ways in which this can been done. Originally I used the method which is illustrated in Figure 3. Here, along the Y-axis is plotted the quantity

$$E(v,t) \equiv \frac{k(v)}{I(v)} \cdot e^{\delta \cdot (v-t)} \cdot I(v) \qquad v \le t$$

i.e. employment at time t on capital installed at time v. At time t all values of E(v,t) for v=t-dt, t-2·dt, ... are known. Also T(t) and L(t) are known. The quantity E[t-T(t),t] is approximated by linear interpolation (cf. point D).

The integral

$$\int_{t-T(t)}^{t} E(v,t) \, dv$$

is approximated according to the Trapezium Rule. Its approximated value is equal to the sum of the known area ABED and the unknown area BCFE. The value of E(t,t) is found by requiring that the approximated integral be equal to the size of the labour force, i.e.

$$L(t) = ABED + BCFE$$

The computed value of E(t,t) automatically yields the value of I(t). Using this value, Q(t) is determined by a similar Trapezium Rule approximation. Finally the ratio of I(t) and Q(t) determines s(t).

This procedure, however, turned out not to be very successful in practice. The computed non-stationary time path of the optimal savings rate became wildly oscillating with an ever-increasing amplitude, finally exploding out of the control space. No matter how small the discretization parameter was chosen, the explosive property of the solution remained.

I therefore decided to use the difference equation (66) for the computation of s(·). Here the time derivative of T(·) was approximated by central differences while I[t-T(t)] was obtained by exponential interpolation between two consecutive values of I(·). Once I(t) has been found Q(t) and s(t) follow easily. The results of this approach are summarized in Table 4 and Figure 4.

The results of the simulation confirm the point raised in the introduction, viz. that because of the fact that the state of the economy is a function of its history non-stationarities are particularly severe and persistent. The oscillations in the optimal trajectories of T(·) and s(·) are quite strong (taking into account that the demographical disturbance of the original steady state is relatively small) and take a very long time to dampen out. However, gradually the optimally growing economy converges to a new stationary growth path, in which both the optimal savings rate s and the optimal lifetime of capital equipment T are once more constant.

7. SUMMARY AND CONCLUSIONS

In this paper I have investigated optimal economic growth in a model with technical progress that is embodied in physical capital. The production function corresponding to each capital vintage has been taken to be of the fixed-coefficients type, as in Solow e.a. (1966).

A suitable transformation of the Lagrangian allows the derivation of necessary conditions for optimal economic growth. These necessary conditions are in terms of two key variables which are inversely related to each other, viz.:

- T(t), the age of the oldest capital in use at time t; and
- Z(t), the age at which capital installed at time t will become obsolete.

Along a singular trajectory the necessary conditions reduce to a

Generalized Golden Rule. It is shown that this Generalized Golden Rule is nothing more than a disguised version of the Golden Rule for more traditional growth models.

A comparative-statics analysis bears out that the optimal savings rate in steady state varies positively with the growth rate of population (g), the rate of labour-augmenting technical progress (l), and the rate of depreciation (δ); and negatively with the social rate of impatience (r). These results are essentially the same as for models with disembodied technical change.

Investigation into the stability of steady states yields the conclusion that a necessary condition for the optimal economic growth path to converge is that $l > r+g+\delta$. This is a puzzling result, as the integral in the social welfare function is divergent if this stability condition is satisfied. For T and s two difference-type of equations have been derived which describe the dynamics of the optimally controlled economy.

Direct methods of actually computing non-stationary optimal growth paths for this model turn out to be unsuccessful, due to the high numerical instability of the optimization problem. An indirect method, however, which simply integrates the two difference equations referred to above, yields a plausible and theoretically satisfying optimal growth path. The results of the simulation show that non-stationarities are particularly severe and persistent in this model, as a result of the fact that the state of the economy is a function of its history.

Figure 1: The double integral before changing the order of integration

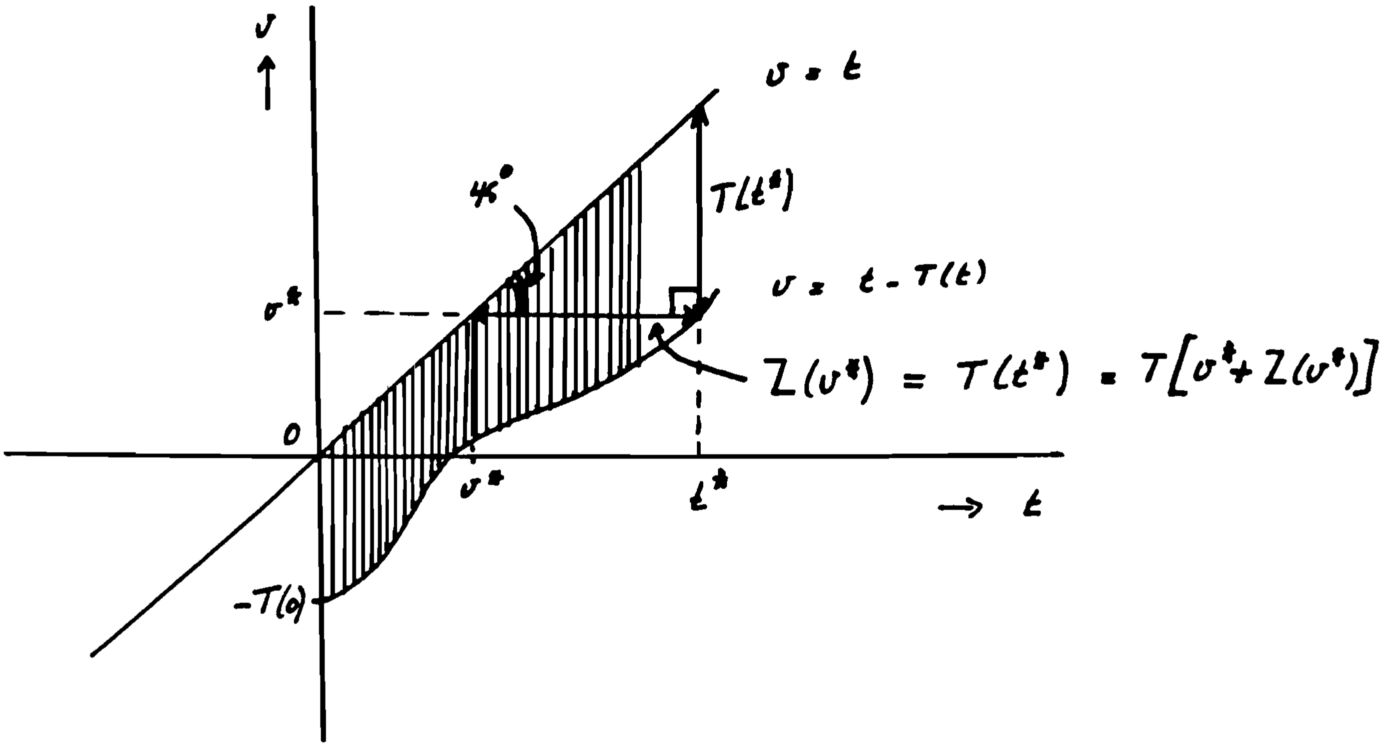

Figure 2: The double integral after changing the order of integration

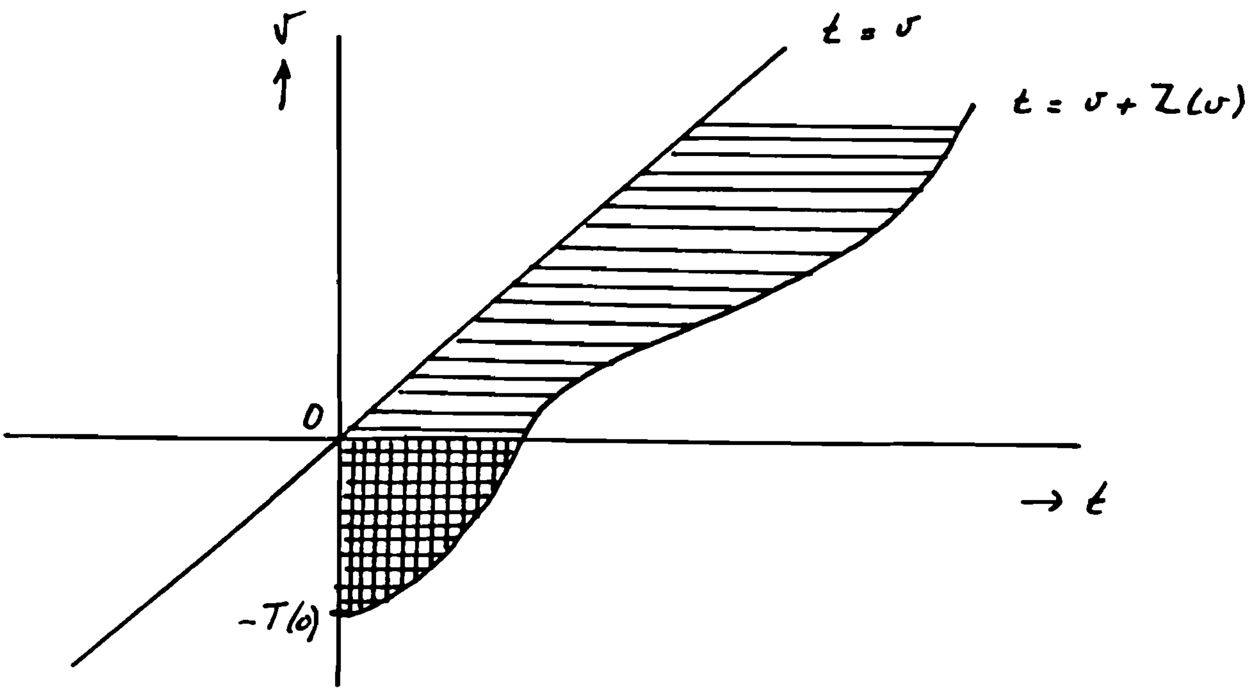

Figure 3: Computation of s(t)

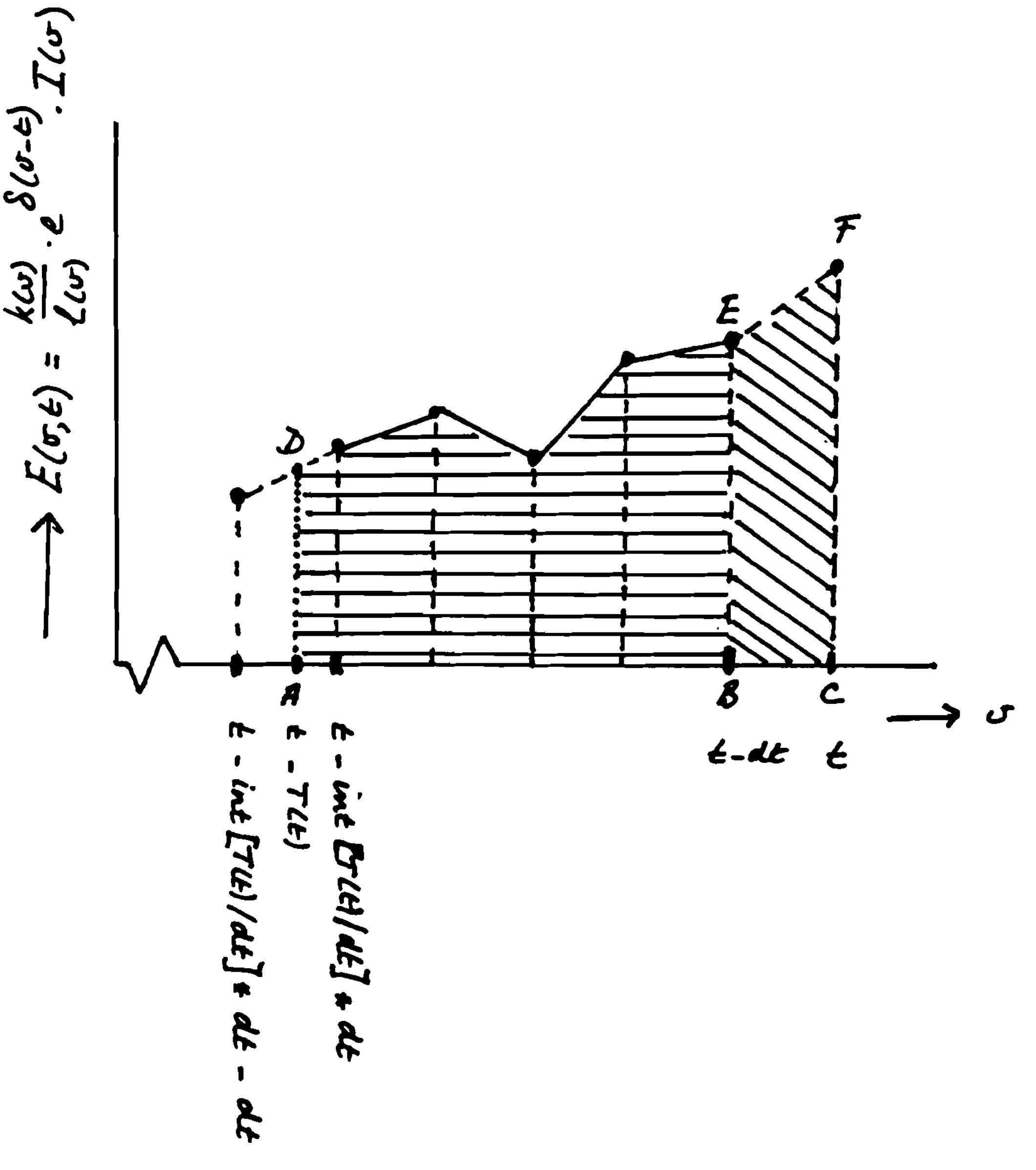

Figure 4: The simulated non-stationary optimal economic growth path

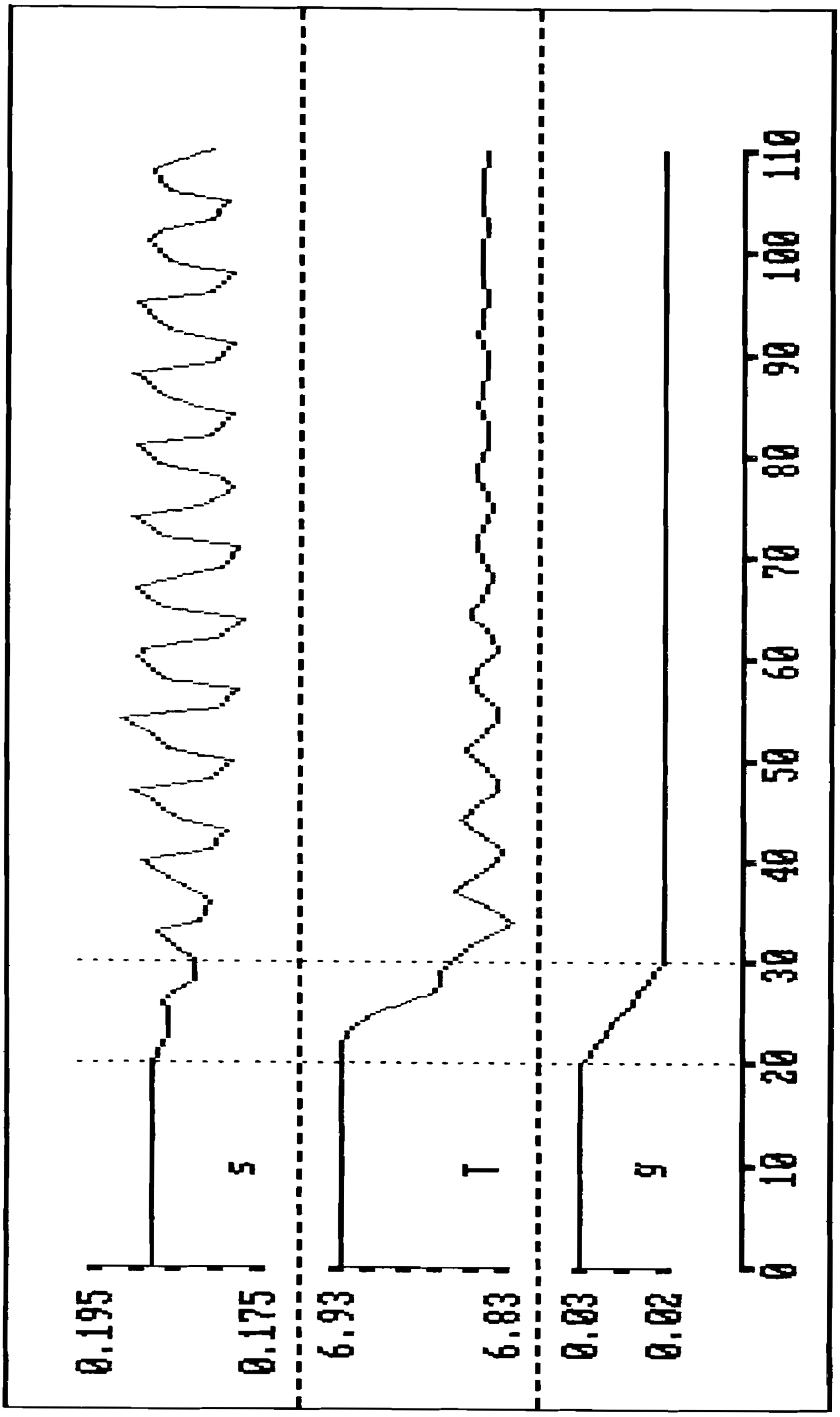

Table 1: Selected numerical steady-state values

r	g	δ	l	T*	s*
0. 02	0. 04	0. 02	0. 04	8.3115	0.1772
0. 02	0. 04	0. 02	0. 00	n.a.	n.a.
0. 02	0. 04	0. 02	0. 02	12.6149	0.1289
0. 02	0. 04	0. 02	0. 04	8.3115	0.1772
0. 02	0. 04	0. 02	0. 06	6.7333	0.2165
0. 02	0. 04	0. 00	0. 04	8.0682	0.1682
0. 02	0. 04	0. 02	0. 04	8.3115	0.1772
0. 02	0. 04	0. 04	0. 04	8.5706	0.1868
0. 02	0. 04	0. 06	0. 04	8.8468	0.1971
0. 02	0. 00	0. 02	0. 04	n.a.	n.a.
0. 02	0. 02	0. 02	0. 04	8.0682	0.1682
0. 02	0. 04	0. 02	0. 04	8.3115	0.1772
0. 02	0. 06	0. 02	0. 04	8.5706	0.1868
0. 00	0. 04	0. 02	0. 04	8.0682	0.1806
0. 02	0. 04	0. 02	0. 04	8.3115	0.1772
0. 04	0. 04	0. 02	0. 04	8.5706	0.1737
0. 06	0. 04	0. 02	0. 04	8.8468	0.1703

Table 2: Comparative-statics effects

effect on: \ of:	r	g	δ	l
T*	+	+	+	–
s*	–	+	+	+

Table 3: Parameters used in the simulations

parameter	value
r	0.00
δ	0.00
l	0.05
g_0	0.03
g_1	0.02
t_a	20
t_b	30
k_0	1.00
l_0	1.00

Table 4: The simulated non-stationary optimal economic growth path

t	g(t)	T*(t)	s*(t)
0	0.03	6.9278	0.1880
10	0.03	6.9278	0.1880
20	0.03	6.9278	0.1880
21	0.029	6.9276	0.1870
22	0.028	6.9264	0.1863
23	0.027	6.9232	0.1860
24	0.026	6.9167	0.1859
25	0.025	6.9060	0.1862
26	0.024	6.8899	0.1867
27	0.023	6.8700	0.1852
28	0.022	6.8690	0.1830
29	0.021	6.8666	0.1827
30	0.02	6.8624	0.1827
31	0.02	6.8554	0.1842
32	0.02	6.8457	0.1858
33	0.02	6.8334	0.1875
34	0.02	6.8248	0.1819
35	0.02	6.8327	0.1812
36	0.02	6.8447	0.1806
37	0.02	6.8581	0.1829
38	0.02	6.8518	0.1856
39	0.02	6.8436	0.1873
40	0.02	6.8330	0.1892
41	0.02	6.8289	0.1811
42	0.02	6.8358	0.1801
43	0.02	6.8460	0.1791
44	0.02	6.8543	0.1847
45	0.02	6.8490	0.1866
46	0.02	6.8420	0.1880
47	0.02	6.8330	0.1903
48	0.02	6.8320	0.1806
49	0.02	6.8380	0.1794
50	0.02	6.8467	0.1781
51	0.02	6.8514	0.1857
52	0.02	6.8469	0.1874
53	0.02	6.8408	0.1892
54	0.02	6.8331	0.1917
55	0.02	6.8345	0.1802
56	0.02	6.8397	0.1788
57	0.02	6.8470	0.1775
58	0.02	6.8491	0.1863
59	0.02	6.8452	0.1879
60	0.02	6.8401	0.1897
61	0.02	6.8339	0.1890
62	0.02	6.8364	0.1799
63	0.02	6.8409	0.1785
64	0.02	6.8472	0.1767
65	0.02	6.8473	0.1866
66	0.02	6.8440	0.1883
67	0.02	6.8396	0.1900
68	0.02	6.8353	0.1847
69	0.02	6.8379	0.1797
70	0.02	6.8417	0.1783
71	0.02	6.8470	0.1777
72	0.02	6.8459	0.1869
73	0.02	6.8431	0.1885
74	0.02	6.8393	0.1902
75	0.02	6.8367	0.1821
76	0.02	6.8390	0.1796
77	0.02	6.8423	0.1782
78	0.02	6.8464	0.1804
79	0.02	6.8448	0.1870
80	0.02	6.8424	0.1886
81	0.02	6.8392	0.1901
82	0.02	6.8379	0.1811
83	0.02	6.8399	0.1796
84	0.02	6.8427	0.1783
85	0.02	6.8455	0.1832
86	0.02	6.8439	0.1871
87	0.02	6.8419	0.1886
88	0.02	6.8392	0.1903
89	0.02	6.8388	0.1809
90	0.02	6.8405	0.1797
91	0.02	6.8429	0.1784
92	0.02	6.8446	0.1851
93	0.02	6.8433	0.1872
94	0.02	6.8415	0.1885
95	0.02	6.8392	0.1896
96	0.02	6.8395	0.1809
97	0.02	6.8410	0.1798
98	0.02	6.8431	0.1785
99	0.02	6.8439	0.1858
100	0.02	6.8428	0.1872
101	0.02	6.8413	0.1884
102	0.02	6.8394	0.1875
103	0.02	6.8401	0.1810
104	0.02	6.8414	0.1799
105	0.02	6.8431	0.1788
106	0.02	6.8434	0.1860
107	0.02	6.8424	0.1871
108	0.02	6.8411	0.1882
109	0.02	6.8398	0.1851
110	0.02	6.8405	0.1811
∞	0.02	6.8417	0.1839

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