WORKING PAPER

QUALITATIVE EQUATIONS: THE CONFLUENCE CASE

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IIASA
International Institute
for Applied Systems Analysis

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FOREWORD

This paper deals with a domain of Artificial Intelligence known under the name of "qualitative simulation" or "qualitative physics", to which special volumes of *Artificial Intelligence* (1984) and of *IEEE Transactions on Systems, Man and Cybernetics* (1987) have been devoted.

It defines the concept of "qualitative frame" of a set, which allows to introduce strict, large and dual confluence frames of a finite dimensional vector-space.

After providing a rigorous definition of standard, lower and upper qualitative solutions in terms of confluences introduced by De Kleer, it provides a duality criterion for the existence of a strict standard solution to both linear and non linear equations.

It also furnishes a dual characterization of the existence of upper and lower qualitative solutions to a linear equation.

These theorems are extended to the case of "inclusions", where single-valued maps are replaced by set-valued maps. This may be useful for dealing with qualitative properties of maps which are not precisely known, or which are defined by a set of properties, a requirement which is at the heart of qualitative simulation.

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Qualitative Equations: The Confluence Case

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Introduction

The purpose of this paper is to offer several theorems on the qualitative solutions to a linear or nonlinear equation

$$\text{given } y \in \mathbf{R}^m, \text{ find } \quad x \in \mathbf{R}^n \quad \text{such that } f(x) = y$$

in terms of "confluences", i.e., of the signs of the components of the vector y. (See[2,8,19,33,35,38,40,41,48,55,62,64] for a survey of the literature on this topic.)

For defining in a rigorous way qualitative solutions to an equation, we adopt, and slightly relax for duality purposes, the definition of qualitative frames proposed in [2]:

A **qualitative frame** $(\mathcal{X}, Q)$ of a set X is defined by

— a set $\mathcal{X}$, called the **qualitative set**[1]

— a set-valued map[2] $Q : \mathcal{X} \rightsquigarrow X$, called the **value map**

where we assume that

$$(1) \qquad \begin{cases} i) & \forall\, a \in \mathcal{X},\ Q(a) \neq \emptyset \\ ii) & Q \text{ is surjective } (\forall\, x \in X,\ \exists\, a \in \mathcal{X} \mid x \in Q(a)) \\ iii) & Q \text{ is injective } (\forall\, a, b \in \mathcal{X},\ a \neq b,\ Q_a \neq Q_b) \end{cases}$$

We shall say that the qualitative frame is **strict** if we assume further that

$$(2) \qquad \forall\, a, b \in \mathcal{X},\ a \neq b,\ Q_a \cap Q_b = \emptyset$$

In this paper, we shall use only the confluence frames of finite dimensional vector-spaces, defined as follow:

— **Strict Confluence Frame** We associate with $X := \mathbf{R}^n$ the n-dimensional **confluence space** $\mathcal{R}^n$ defined by

$$\mathcal{R}^n := \{-, 0, +\}^n$$

[1] which is generally assumed to be a finite set

[2] A set-valued map Q from $\mathcal{X}$ to X maps each $a \in \mathcal{X}$ to a subset $Q(a) \subset X$, possibly empty.

The **strict confluence frame** is defined by $(\mathcal{R}^n, Q_n)$ where the value map Q_n maps each qualitative value $a \in \mathcal{R}^n$ to the convex cone

$$Q_n(a) \ := \ \mathbf{R}^n_a \ := \ \{ \ v \in \mathbf{R}^n \ \mid \ \text{sign of } (v_i) = a_i \ \}$$

For $n = 1$, we have

$$Q_1(-) \ =]-\infty, 0[, \ \ Q_1(0) \ = \ \{0\} \ \ \& \ \ Q_1(+) \ =]0, +\infty[$$

— **Large Confluence Frame** We still associate with $\mathbf{R}^n$ the n-dimensional confluence space $\mathcal{R}^n$. The **large confluence frame** is then defined by the set-valued map $\overline{Q}_n$ associating with every $a \in \mathcal{R}^n$ the convex cone

$$\overline{Q}_n(a) \ := \ a\mathbf{R}^n_+ \ := \ \{ \ v \in \mathbf{R}^n \mid s(v_i) = a_i \text{ or } 0 \ \}$$

which is the closure of $Q_n(a) = \mathbf{R}^n_a$.

— **Dual Confluence Frame** Let $\mathbf{R}^{n^\star}$ denote the dual of $X := \mathbf{R}^n$. The **dual confluence frame** $(\mathcal{R}^n, Q^\star_n)$ is made of the n-dimensional confluence space $\mathcal{R}^n := \{-, 0, +\}^n$ and the set-valued map $Q^\star_n$ from $\mathcal{R}^n$ to $\mathbf{R}^{n^\star}$ which maps every $a \in \mathcal{R}^n$ to the closed cone $Q^\star(a)$ of elements $p := (p^1, \ldots, p^n)$ defined by

$$\begin{cases} p_i \geq 0 & \text{if } a_i \ = \ + \\ p_i \leq 0 & \text{if } a_i \ = \ - \\ p_i \in \mathbf{R} & \text{if } a_i \ = \ 0 \end{cases}$$

Let $(\mathcal{X}, Q_X)$ and $(\mathcal{Y}, Q_Y)$ be qualitative frames of two sets X and Y, $f : X \mapsto Y$ and $b \in \mathcal{Y}$ be a qualitative right-hand side. We define three types of qualitative solutions:

— **the standard qualitative solution** $a \in \mathcal{X}$ satisfying:

$$f(Q_X(a)) \cap Q_Y(b) \ \neq \ \emptyset$$

— **the upper qualitative solution** $a \in \mathcal{X}$ satisfying:

$$f(Q_X(a)) \ \subset \ Q_Y(b)$$

— **the lower qualitative solution** $a \in \mathcal{X}$ satisfying:

$$f^{-1}(Q_Y(b)) \ \subset \ Q_X(a)$$

and we see at once that any upper qualitative or lower qualitative solution is a standard solution.

Let the two quantitative spaces $X := \mathbf{R}^n$ and $Y := \mathbf{R}^m$ be finite dimensional vector-spaces.

In the case of standard solutions, it is more difficult to prove the existence of strict confluence solutions than large ones. We shall provide a criterion for the existence of strict standard solutions.

Let us begin by the case when $f := A \in \mathcal{L}(X,Y)$ is a linear operator. In this case, there exist always large qualitative solutions because $A(0) = 0$! But if we assume that the **dual condition**

$$(3) \qquad \begin{cases} (0,0) \text{ is the only solution } (p,q) \text{ to} \\ p \in A^\star(q) \cap Q_n^\star(a) \;\; \& \;\; q \in -Q_m^\star(b) \end{cases}$$

is satisfied, then there exists a standard strict qualitative solution $a \in \mathcal{R}^n$

$$A(Q_n(a)) \cap Q_m(b) \neq \emptyset$$

This theorem can be extended to the non linear case through linearization and duality.

Let $x_0 \in \overline{Q}_n(a)$ (where $y_0 := f(x_0) \in \overline{Q}_n(b)$) be a representative of a solution a to the large qualitative equation

$$f(\overline{Q}_n(a)) \cap \overline{Q}_m(b) \neq \emptyset$$

Assume that f is continuous and continuously differentiable at x_0. If

$$(4) \qquad \begin{cases} (0,0) \text{ is the only solution } (p,q) \text{ to} \\ p \in f'(x_0)^\star(q) \cap Q_n^\star(a) \;\; \& \;\; q \in -Q_m^\star(b) \end{cases}$$

then a is a solution to the qualitative equation

$$f(Q_n(a)) \cap Q_m(b) \neq \emptyset$$

For upper qualitative solutions, we shall prove in the linear case the following duality principle: Let $A \in \mathcal{L}(X,Y)$ be linear. Then the two following conditions are equivalent:

$$A(\overline{Q}_n(a)) \subset \overline{Q}_m(b)$$

and

$$A^\star(Q_m^\star(b)) \subset Q_n^\star(a)$$

as well as the equivalent statement for lower qualitative solutions: the two conditions

$$A^{-1}(\overline{Q}_m(b)) \subset \overline{Q}_n(a)$$

and

$$Q_m^\star(a) \subset \overline{A^\star(Q_m^\star(b))}$$

are equivalent.

As it can be seen from the definitions of qualitative solutions, our theorems rely on "set-valued analysis", which has been developed for various reasons from Painlevé's early works to recent results on graphical convergence and the differential calculus of set-valued maps, including an inverse function theorem we shall actually use.(See [1,3,4,6] for instance.)

Hence we can study right away qualitative analysis of set-valued maps $F : X \rightsquigarrow Y$. The mathematical cost will be slightly the same, and, furthermore, there is an important motivation to just do that at the onset in the framework of qualitative analysis.

Indeed, when the objective is the resolution of an equation $f(x) = y$, the single-valued map f is not exactly known, even when physicists and other scientists model them by classical and familiar "special functions" through their behavior. It is just enough to mention the favorite use of the exponential, logarithmic, logistic, trigonometric, ... functions in many models.

Actually, the choice of these functions is often made because there are "representatives" of a class of functions defined by a list of properties, whether this list is exhaustive or not, conscious or not.

Keeping in mind the philosophy of qualitative reasoning, it is more to the point to start with such a list of requirements on maps from X to Y and build from it the "largest" set-valued map F from X to Y which satisfy them.

Hence we are led to propose to solve right-away the qualitative solutions to "inclusions"

$$F(x) \ni y$$

We shall prove the set-valued versions of the results described above in the rest of the paper, after adapting to the set-valued case the above definitions.

We begin by defining precisely qualitative frames, and then, five types of qualitative solutions to inclusions. We then provide the duality results in the linear case and study the non linear case in the last section.

1 Qualitative Frames

The general features of qualitative analysis, and, more particularly, of confuences, can be captured in the mathematical framework we propose below.

Let X, called the **quantitative space**, denote the set of elements on which operates qualitative reasoning.

Definition 1.1 *A* **qualitative frame** $(\mathcal{X}, Q)$ *of a set* X *is defined by*

— *a set* $\mathcal{X}$*, called the* **qualitative set**[3]

— *a set-valued map*[4] $Q : \mathcal{X} \rightsquigarrow X$*, called the* **value map**

where we assume that

$$(5) \qquad \begin{cases} i) & \forall\, a \in \mathcal{X},\ Q(a) \neq \emptyset \\ ii) & Q \text{ is surjective } (\forall\, x \in X,\ \exists\, a \in \mathcal{X} \mid x \in Q(a)) \\ iii) & Q \text{ is injective } (\forall\, a, b \in \mathcal{X},\ a \neq b,\ Q_a \neq Q_b) \end{cases}$$

We shall say that the qualitative frame is **strict** *if we assume further that*

$$(6) \qquad \forall\, a, b \in \mathcal{X},\ a \neq b,\ Q_a \cap Q_b = \emptyset$$

We denote by $P := Q^{-1}$ *the inverse*[5] *of* P*, called the qualitative map.*

Remark — STRICT QUALITATIVE FRAMES. When the qualitative frame is strict, the qualitative map P is single-valued, and is then denoted by p.

In this case, the subsets $Q(a)$ do form a partition of X when a ranges over the qualitative set $\mathcal{X}$, so that they constitute the equivalence classes of

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[3] which is generally assumed to be a finite set

[4] A set-valued map Q from $\mathcal{X}$ to X maps each $a \in \mathcal{X}$ to a subset $Q(a) \subset X$, possibly empty. We say that $Q(a)$ is the *image* or the *value* of Q at a. The *image* $Q(M)$ of M is the union of the images (or values) $Q(a)$ when a ranges over M. One set $\mathrm{Im}(Q) := Q(\mathcal{X})$ (the **image** of Q) and $\mathrm{Dom}(Q)$, the **domain** of Q, the subset of $a \in \mathcal{X}$ such that $Q(a)$ is not empty.

A set-valued map Q is characterized by its **graph** $\mathrm{Graph}(Q)$, subset of the product space $\mathcal{X} \times X$ defined by

$$\mathrm{Graph}(Q) := \{(a, x) \in \mathcal{X} \times X \mid x \in Q(a)\}$$

[5] This means that $a \in P(x)$ if and only if $x \in Q(a)$. In particular,

$$\forall\, x \in X,\ x \in Q(P(x))$$

the the binary relation $\mathcal{R}$ defined on X by:

$$x \,\mathcal{R}\, y \iff p(x) = p(y)$$

which is an equivalence relation. Hence we can regard in this case the qualitative set $\mathcal{X}$ as the factor space $\mathcal{X} := X/\mathcal{R}$ and p as the canonical surjection[6]. □

Remark — CLOSURE OF A STRICT QUALITATIVE FRAME.

When X is a topological space, it is convenient to associate with a strict qualitative frame $(\mathcal{X}, Q)$ its closure $(\mathcal{X}, \overline{Q})$ where

$$(7) \qquad \forall\, a \in \mathcal{X}, \;\; \overline{Q}(a) \;:=\; \mathrm{cl}(Q(a)) \;=:\; \overline{Q(a)} \quad \square$$

Example — Strict Confluence Frame

We consider the usual finite dimensional vector-space $X := \mathbf{R}^n$ as a quantitative space and we associate with it the n-dimensional **confluence space** $\mathcal{R}^n$ defined by

$$\mathcal{R}^n \;:=\; \{\, -,\, 0,\, +\,\}^n$$

whose elements are denoted by $a := (a_1, \ldots, a_n)$.

The **strict confluence frame** is defined by $(\mathcal{R}^n, Q_n)$ where the value map Q_n maps each qualitative value $a \in \mathcal{R}^n$ to the convex cone

$$Q_n(a) \;:=\; \mathbf{R}^n_a \;:=\; \{\, v \in \mathbf{R}^n \;\mid\; \text{sign of } (v_i) = a_i \,\}$$

It is obviously a strict qualitative frame, so that the inverse of Q_n, denoted by s_n, is the single-valued map from $\mathbf{R}^n$ to $\mathcal{R}^n$ defined by:

$$\forall\, i \in \{1, \ldots, n\}, \;\; s_n(x)_i \;:=\; x_i \quad \square$$

Example — Large Confluence Frame

We still consider the finite dimensional vector-space $X := \mathbf{R}^n$ as a quantitative space and we associate with it the n-dimensional confluence space $\mathcal{R}^n$. The **large confluence frame** is then defined by the set-valued map $\overline{Q}_n$ associating with every $a \in \mathcal{R}^n$ the convex cone

$$\overline{Q}_n(a) \;:=\; a\mathbf{R}^n_+ \;:=\; \{\, v \in \mathbf{R}^n \mid s(v_i) = a_i \text{ or } 0 \,\}$$

[6]Conversely, we can associate with any equivalence relation $\mathcal{R}$ a strict qualitative frame, where the factor space $\mathcal{X} := X/\mathcal{R}$ is the qualitative set and the canonical surjection

$$p : X \mapsto \mathcal{X} \;:=\; X/\mathcal{R}$$

associating to each element $x \in X$ its equivalence class $p(x) \in \mathcal{X}$, the qualitative map.

which is the closure of $Q_n(a) = \mathbf{R}^n_a$, as well as the image of $\mathcal{R}^n_+$ by the map $a : x \mapsto ax = (a_1x_1, \ldots, a_nx_n)$.

Its inverse is the set-valued map from $\mathbf{R}^n$ to $\mathcal{R}^n$ denoted by S_n, which is defined by:

$$\forall\, i \in \{1, \ldots, n\}, \;\; S_n(x)_i \;:=\; \begin{cases} - & \text{if } x_i \leq 0 \\ \{\,-, 0, +\,\} & \text{if } x_i = 0 \\ + & \text{if } x_i \geq 0 \end{cases} \quad \square$$

Example — Dual Confluence Frame

Let $\mathbf{R}^{n^\star}$ denote the dual of $X := \mathbf{R}^n$. The **dual confluence frame** $(\mathcal{R}^n, Q^\star_n)$ is made of the n-dimensional confluence space $\mathcal{R}^n := \{-, 0, +\}^n$ and the set-valued map $Q^\star_n$ from $\mathcal{R}^n$ to $\mathbf{R}^{n^\star}$ which maps every $a \in \mathcal{R}^n$ to the closed cone $Q^\star(a)$ of elements $p := (p^1, \ldots, p^n)$ defined by

$$\begin{cases} p_i \geq 0 & \text{if } a_i = + \\ p_i \leq 0 & \text{if } a_i = - \\ p_i \in \mathbf{R} & \text{if } a_i = 0 \end{cases}$$

Its inverse is the set-valued map from $\mathbf{R}^{n^\star}$ to $\mathcal{R}^n$ denoted by $S^\star_n$ and defined by

$$S^\star(p)_i \;=\; \begin{cases} \{\,+, 0\,\} & \text{if } p_i > 0 \\ \{\,-, 0\,\} & \text{if } p_i < 0 \\ \{\,-, 0\, +\,\} & \text{if } p_i = 0 \end{cases}$$

The reason why this qualitative frame is called the dual confluence frame is given by the following lemma:

Lemma 1.1 *The positive polar cone to the convex cone $\overline{Q}(a) := a\mathcal{R}^n_+$ is the cone $Q^\star(a) := a^{-1}(\mathcal{R}^n_+)$*

Proof — We associate with any $a \in \mathcal{R}^n$ the subsets

$$\begin{cases} I_0(a) & := \{\, i \mid a_i = 0 \,\} \\ I_+(a) & := \{\, i \mid a_i > 0 \,\} \\ I_-(a) & := \{\, i \mid a_i < 0 \,\} \end{cases}$$

of $I := \{\, 1, \ldots, n \,\}$.

We observe that

$$p \in (a\mathcal{R}^n_+)^+ \text{ if and only if } p_i \geq 0 \text{ if } i \in I_+(a) \;\&\; p_i \leq 0 \text{ if } i \in I_-(a)$$

since this is equivalent to

$$\forall\, y \in \mathcal{R}^n_+,\ \sum_{i\in I} a_i p_i y_i \ = \ \sum_{i\in I_+(a)} p_i y_i - \sum_{i\in I_-(a)} p_i y_i \ \geq \ 0 \ \Box$$

2 Qualitative Inclusions

Let us consider two quantitative spaces X and Y, a single-valued map f : $X \mapsto Y$ and the equation

$$\text{(8)} \qquad \text{find } x \in X \ \text{ such that } f(x) \ = \ y$$

which we shall call the "quantitative equation". More generally, we can also start with a set-valued map $F : X \rightsquigarrow Y$ and the "quantitative inclusion"

$$\text{(9)} \qquad \text{find } x \in X \ \text{ such that } F(x) \ \ni \ y$$

In order to make a qualitative analysis of such an equation or an inclusion, we introduce two qualitative frames $(\mathcal{X}, Q_X)$ and $(\mathcal{Y}, Q_Y)$ and their qualitative maps P_X and P_Y.

There are many ways to associate with F "projections" which map $\mathcal{X}$ to $\mathcal{Y}$. We shall only mention three of them.

— **Standard Projection**

It is the set-valued map $\pi_0(F) : \mathcal{X} \rightsquigarrow \mathcal{Y}$ defined by

$$\text{(10)} \qquad \widehat{F} \ := \ \pi_0(F) \ := \ P_Y \circ F \circ P_X^{-1} \ := \ P_Y \circ F \circ Q_X$$

where $\circ$ denotes the usual product of set-valued maps[7].

[7]One can conceive two dual ways for defining composition products of set-valued maps (which coincide when G is single-valued):

Let X, Y, Z be Banach spaces and $G : X \rightsquigarrow Y$, $H : Y \rightsquigarrow Z$ be set-valued maps:

1 — the usual composition product (called simply the **product**) $H \circ G : X \rightsquigarrow Z$ of H and G at x is defined by

$$(H \circ G)(x) \ := \ \bigcup_{y \in G(x)} H(y)$$

2 — the **square product** $H \Box G : X \rightsquigarrow Z$ of H and G at x introduced in [5], is defined by

$$(H \Box G)(x) \ := \ \bigcap_{y \in G(x)} H(y)$$

There are also two manners to define the inverse image by a set-valued map G of a subset

Hence, we can associate with the quantitative inclusion the standard qualitative inclusion:

$$\text{(11)} \qquad \text{find } a \in X \quad \text{such that } \pi_0(F)(a) \ni b$$

which is equivalent to either formulations[8]

$$\text{(12)} \qquad \left\{ \begin{array}{ll} i) & F(Q_X(a)) \cap Q_Y(b) \neq \emptyset \\ ii) & Q_X(a) \cap F^{-1}(Q_Y(b)) \neq \emptyset \\ iii) & \mathrm{Graph}(F) \cap (Q_X(a) \times Q_Y(b)) \neq \emptyset \\ iv) & a \in P_X(F^{-1}(Q_Y(b))) \end{array} \right.$$

The last property follows from the observation that

$$\text{(13)} \qquad (\pi_0(F))^{-1} = \pi_0(F^{-1})$$

which is obvious when we remark that

$$\mathrm{Graph}(\pi_0(F)) = (P_X \times P_Y)\mathrm{Graph}(F)$$

Solving the standard qualitative equation means that, given a "qualitative right-hand side" $b \in \mathcal{Y}$, there exist a quantitative right-hand side $y \in Q_Y(b)$ and a solution $x \in F^{-1}(y)$ which belongs to $Q_X(a)$.

— **The Upper Projection**

M:

$$\left\{ \begin{array}{lllll} a) & G^{-1}(M) & := & \{ x \mid G(x) \cap M \neq \emptyset \} & \text{the \textbf{inverse image} of } M \\ b) & G^{+1}(M) & := & \{ x \mid G(x) \subset M \} & \text{the \textbf{core} of } M \end{array} \right.$$

The formulas which state that the inverse of a product is the product of the inverses (in reverse order) become:

$$\left\{ \begin{array}{lllll} i) & (H \circ G)^{-1}(z) & = & G^{-1}(H^{-1}(z)) & = (G^{-1} \circ H^{-1})(z) \\ ii) & (H \square G)^{-1}(z) & = & G^{+1}(H^{-1}(z)) & \end{array} \right.$$

[8] When the quantitative spaces X and Y are vector-spaces, we can associate with F the set-valued map $\Phi : X \times Y \rightsquigarrow Y$ defined by

$$\Phi(x, y) := F(x) - y$$

Hence $\pi_0(F)(a) \ni b$ if and only if

$$\exists\, (x, y) \in Q_X(a) \times Q_Y(b) \text{ such that } \Phi(x, y) = 0$$

It is the set-valued map $\pi_+(F) : \mathcal{X} \rightsquigarrow \mathcal{Y}$ defined by

$$(14) \qquad \pi_+(F) := P_Y \square (F \circ P_X^{-1}) := P_Y \square (F \circ Q_X)$$

Hence, we can also associate with the quantitative inclusion the upper qualitative inclusion:

$$(15) \qquad \text{find } a \in X \quad \text{such that } \pi_+(F)(a) \ni b$$

which is equivalent to either properties[9]

$$(16) \qquad \begin{cases} i) & F(Q_X(a)) \subset Q_Y(b) \\ ii) & Q_X(a) \subset F^{+1}(Q_Y(b)) \end{cases}$$

In other words, solving the upper qualitative equation means that, given a "qualitative right-hand side" $b \in \mathcal{Y}$, a is a an upper qualitative solution if for all representative $x \in Q_X(a)$, every $y \in F(x)$ is a representative of b.

— **The Anti Lower Projection**

It is the set-valued map $\varpi_-(F) : \mathcal{X} \rightsquigarrow \mathcal{Y}$ defined by

$$(17) \qquad \varpi_-(F) := (P_Y \circ F) \square P_X^{-1} := (P_Y \circ F) \square Q_X$$

Hence, we can also associate with the quantitative inclusion the anti lower qualitative inclusion:

$$(18) \qquad \text{find } a \in X \quad \text{such that } \varpi_-(F)(a) \ni b$$

which can be written in the following equivalent form:

$$(19) \qquad Q_X(a) \subset F^{-1}(Q_Y(b))$$

Therefore, solving the anti lower qualitative equation means that, given a "qualitative right-hand side" $b \in \mathcal{Y}$, a is a anti lower qualitative solution

[9]We use the following observations:

$$\begin{cases} i) & x \in (H \square G)^{-1}(z) \iff G(x) \subset H^{-1}(z) \\ ii) & x \in (G^{-1} \square H^{-1})(z) \iff H^{-1}(z) \subset G(x) \end{cases}$$

and

$$\begin{cases} i) & (H \circ G)^{-1}(M) = G^{-1}(H^{-1}(M)) \\ ii) & (H \circ G)^{+1}(M) = G^{+1}(H^{+1}(M)) \end{cases}$$

if for all representative $x \in Q_X(a)$, there exists a representative $y \in F(x)$ of b.

We can also inverse first the set-valued map F and "project" the set-valued map F^{-1}. We then obtain the two following concepts of projections and qualitative solutions:

— **The Lower Projection**

It is the set-valued map $\pi_-(F) : \mathcal{Y} \rightsquigarrow \mathcal{X}$ defined by

$$(20) \qquad \pi_-(F) := P_X \square (F^{-1} \circ P_Y^{-1}) := P_X \square (F^{-1} \circ Q_Y)$$

Hence, we can also associate with the quantitative inclusion the lower qualitative inclusion:

$$(21) \qquad \text{find } a \in \pi_-(F)(b)$$

which is equivalent to property

$$(22) \qquad F^{-1}(Q_Y(b)) \subset Q_X(a)$$

Therefore, to say that a is a lower qualitative solution amounts to saying that for all representatives $y \in Q_Y(b)$ of the qualitative right-hand side b, all solutions to the inclusion $F(x) \ni y$ are representatives of a.

— **The Anti Upper Projection**

It is the set-valued map $\varpi_+(F) : \mathcal{Y} \rightsquigarrow \mathcal{X}$ defined by

$$(23) \qquad \varpi_+(F) := (P_X \circ F^{-1}) \square P_Y^{-1} := (P_X \circ F^{-1}) \square Q_Y$$

Hence, we can also associate with the quantitative inclusion the anti upper qualitative inclusion:

$$(24) \qquad \text{find } a \in \varpi_+(F)(b)$$

which can be written in the following equivalent form:

$$(25) \qquad Q_Y(b) \subset F(Q_X(a))$$

Therefore, solving the anti upper qualitative equation means that, given a "qualitative right-hand side" $b \in \mathcal{Y}$, for all representative $y \in Q_Y(b)$, there exists a solution x to the inclusion $F(x) \ni y$ which is a representative of a.

We observe at once that

Lemma 2.1 *Let F be a set-valued map from X to Y.*

— ***Any upper qualitative solution is a anti lower qualitative solution,***

— ***any lower qualitative solution is an anti upper qualitative solution,***

— ***both any anti lower and anti upper qualitative solutions are standard qualitative solutions.***

holds true

We shall say that a is a "strong lower qualitative solution" if it is both a lower and anti lower qualitative solution, i.e., if

$$(26) \qquad F^{-1}(Q_Y(b)) = Q_X(a)$$

and that a is a "strong upper qualitative solution" if it is both a upper and anti upper qualitative solution, i.e., if

$$(27) \qquad Q_Y(b) = F(Q_X(a))$$

Remark — Naturally, when F is a single-valued map f, the inverse image $f^{-1}(M)$ and the core $f^{+1}(M)$ do coincide, so that

$$(28) \qquad \pi_+(f) = \varpi_-(f) = \tilde{f} \quad \square$$

Remark — We could also have considered the projection of a set-valued map F to the set-valued map $P_Y \square (F \square Q_X)$, but its inverse has not interesting properties for our concern: Indeed, a belongs to $(P_Y \square (F \square Q_X))^{-1}(b)$ if and only if

$$\text{whenever } Q_X(a) \subset F^{-1}(y), \text{ then } y \in Q_Y(b) \quad \square$$

3 Qualitative Duality of Linear Confluences

Let the two quantitative spaces $X := \mathbf{R}^n$ and $Y := \mathbf{R}^m$ be finite dimensional vector-spaces and $A \in \mathcal{L}(X, Y)$ be a linear operator from X to Y.

We introduce the strict and large confluence frames $(\mathcal{R}^n, Q_n)$ and $(\mathcal{R}^m, Q_m)$, their dual confluence frames and the transpose $A^\star \in \mathcal{L}(X^\star, Y^\star)$ of A.

We consider the upper large projection of A and the anti lower projection of the transpose $A^\star$ defined by:

$$(29) \qquad \begin{cases} i) & \overline{\pi_+}(A) := S_m \square (A \circ \overline{Q}_n) \\ ii) & \varpi_-(A^\star) := (S_n^\star \circ A^\star) \square Q_m^\star \end{cases}$$

and the lower and anti upper projections

$$(30) \qquad \begin{cases} i) & \pi_-(A) := S_n \square (A^{-1} \circ \overline{Q}_m) \\ ii) & \varpi_+(A^\star) := (S_m^\star \circ A^{\star^{-1}}) \square Q_n^\star \end{cases}$$

Theorem 3.1 (Qualitative Duality) *Let us consider a linear operator $A \in \mathcal{L}(X,Y)$ and its transpose. The following conditions are equivalent:*

$$(31) \qquad \begin{cases} i) & a \in \mathcal{R}^n \text{ solves } \overline{\pi_+}(A)(a) = b \text{ where } b \in \mathcal{R}^m \\ ii) & b \in \mathcal{R}^m \text{ solves } \varpi_-(A^\star)(b) \ni a \text{ where } a \in \mathcal{R}^n \end{cases}$$

as well are the two conditions

$$(32) \qquad \begin{cases} i) & a \in \mathcal{R}^n \text{ is equal to } \pi_-(A)(b) \text{ where } b \in \mathcal{R}^m \\ ii) & b \in \mathcal{R}^m \text{ belongs to } \varpi_+(A^\star)(a) \text{ where } a \in \mathcal{R}^n \end{cases}$$

Proof — Indeed, to say that $b \in \overline{\pi_+}(A)(a)$ amounts to saying that

$$A(a\mathbf{R}^n) \subset b\mathbf{R}^m_+$$

or, by polarity, that

$$(b\mathbf{R}^m_+)^+ \subset (A(a\mathbf{R}^n))^+$$

By the "Bipolar Theorem"(see for instance [3, Chapter I]), this is equivalent to say that

$$b^{-1}(\mathbf{R}^m_+) \subset A^{\star^{-1}}(a^{-1}(\mathbf{R}^n_+))$$

which we can write in the form:

$$Q_m^\star(b) \subset A^{\star^{-1}}(Q_n^\star(a))$$

This means that $a \in \varpi_-(A^\star)(b)$.

The proof of the second statement is analogous. □

We can extend these theorems to the set-valued analogues of continuous linear operators, which are the closed convex processes.

Definition 3.1 (Closed Convex Process) *Let $F : X \rightsquigarrow Y$ be a set-valued map form a Banach space X to a Banach space Y. We shall say that F is*

— **convex** *if its graph is convex*[10]

[10]This means that

$$\begin{cases} \forall x_1, x_2, \in \mathrm{Dom}(F), \ \lambda \in [0,1], \\ \lambda F(x_1) + (1-\lambda)F(x_2) \subset F(\lambda x_1 + (1-\lambda)x_2) \end{cases}$$

— **closed** *if its graph is closed*

— *a* **process** *(or* **positively homogeneous***) if its graph is a cone*[11]. *Hence a* **closed convex process** *is a set-valued map whose graph is a closed convex cone.*

(See for instance [3, Chapter III]). Since the graphs of continuous linear operators from a Banach space to another are closed vector subspaces, we justify our statement that closed convex processes are their set-valued analogues. Actually, most of the properties of continuous linear operator are enjoyed by closed convex processes.

Definition 3.2 (Transpose of a Process) *Let $F : X \leadsto Y$ be a process. Its* **left-transpose** *(in short, its* **transpose***) $F^\star$ is the closed convex process from $Y^\star$ to $X^\star$ defined by*

$$(33) \qquad \begin{cases} p \in F^\star(q) \quad \text{if and only if} \\ \forall\, x \in X, \;\; \forall\, y \in F(x), \;\; < p, x > \; \leq < q, y > \end{cases}$$

(See for instance [3, Chapter III]). The graph of the transpose $F^\star$ of F is related to the polar cone of the graph of F in the following way: The following conditions are equivalent:

$$(34) \qquad \begin{cases} (q, p) \in \mathrm{Graph}(F^\star) \quad \text{if and only if} \\ (p, -q) \in (\mathrm{Graph}(F))^- \end{cases}$$

The Qualitative Duality Theorem 3.1 can be extended to closed convex processes because we can adapt the Bipolar Theorem[12].

Theorem 3.2 (Set-Valued Qualitative Duality) *Let us consider a closed convex process F from X to Y and its transpose. Assume that*

$$\mathrm{Dom}(F) - \overline{Q}_n(a) \;=\; X$$

[11] this means that

$$\forall\, x \in X, \; \lambda > 0, \; \lambda F(x) \;=\; F(x) \quad \text{and } 0 \in F(0)$$

[12] **Theorem (Bipolar Theorem)** Let $F : X \leadsto Y$ be a closed convex process and $K \subset X$ be a closed convex cone satisfying

$$\mathrm{Dom}(F) - K \;=\; X$$

Then

$$(F(K))^+ \;=\; F^{\star^{-1}}\left(K^+\right)$$

(See for instance [3, Chapter III])

Then the following conditions are equivalent:

$$(35) \qquad \begin{cases} i) & a \in \mathcal{R}^n \text{ solves } \overline{\pi_+}(F)(a) \ni b \text{ where } b \in \mathcal{R}^m \\ ii) & b \in \mathcal{R}^m \text{ solves } \varpi_-(F^\star)(b) \ni a \text{ where } a \in \mathcal{R}^n \end{cases}$$

If we assume that

$$\mathrm{Im}(F) - \overline{Q}_m(b) = Y$$

then the two following conditions are equivalent:

$$(36) \qquad \begin{cases} i) & a \in \mathcal{R}^n \text{ belongs to } \pi_-(F)(b) \text{ where } b \in \mathcal{R}^m \\ ii) & b \in \mathcal{R}^m \text{ solves } \varpi_+(F^\star)(a) \text{ where } a \in \mathcal{R}^n \end{cases}$$

4 Standard Strict Solutions to Confluence Equations

Let the two quantitative spaces $X := \mathbf{R}^n$ and $Y := \mathbf{R}^m$ be finite dimensional vector-spaces and f be a continuous single-valued map from X to Y.

We consider the qualitative spaces $\mathcal{R}^n$ and $\mathcal{R}^m$, the strict and large projections

$$\begin{cases} \pi_0(f) & = s_m \circ f \circ Q_n \\ \overline{\pi_0}(f) & = S_m \circ f \circ \overline{Q}_n \end{cases}$$

of f and the qualitative equations

$$(37) \qquad \begin{cases} i) & \pi_0(f)(a) \ni b \\ ii) & \overline{\pi_0}(f)(a) \ni b \end{cases}$$

Let us assume that we know a solution to the large qualitative equation (37)ii) which can be written in the form:

$$(38) \qquad f(\overline{Q}_n(a)) \cap \overline{Q}_m(b) \neq \emptyset$$

We shall provide a sufficient condition for a solution to the strict qualitative equation (37)i) to exist.

Theorem 4.1 *Let $x_0 \in \overline{Q}_n(a)$ (where $y_0 := f(x_0) \in \overline{Q}_n(b)$) be a representative of a solution a to the large qualitative equation (37)ii).*

Assume that f is continuous and continuously differentiable at x_0. If

$$(39) \qquad \begin{cases} (0,0) \text{ is the only solution } (p,q) \text{ to} \\ p \in f'(x_0)^\star(q) \cap Q_n^\star(a) \;\&\; q \in -Q_m^\star(b) \end{cases}$$

then a is a solution to the qualitative equation (37)i).

Proof — By assumption, we know that (x_0, y_0) belongs to the intersection of the graph of f and the closed convex cone $\overline{Q}_n(a) \times \overline{Q}_m(b)$, so that $((x_0, y_0), (x_0, y_0))$ is a solution in $\mathrm{Graph}(f) \times (\overline{Q}_n(a) \times \overline{Q}_m(b))$ to the equation $(x_1, y_1) - (x_2, y_2) = 0$.

Let $\mathbf{1} \in \mathbf{R}^n$ denote the unit vector

$$\mathbf{1} \;:=\; (1, \ldots, 1)$$

We shall prove that there exist a solution $(x_1, y_1) \in \mathrm{Graph}(f)$ and a solution $(x_2, y_2) \in Q_n(a) \times \overline{Q}_m(b)$ to the equation

$$(x_1, y_1) - (x_2, y_2) \;=\; \epsilon(a\mathbf{1}, b\mathbf{1})$$

for some $\epsilon > 0$, so that $x_1 = x_2 + \epsilon a\mathbf{1}$ belongs to $Q_n(a)$ and $f(x_1) = y_1 = y_2 + \epsilon b\mathbf{1}$ belongs to $Q_m(b)$.

For that purpose, we can apply the "Constrained Inverse Function Theorem" (see [3,4]), which states that a solution to the above equation does exist provided that the assumption

$$C_{\mathrm{Graph}(f)}(x_0, y_0) - C_{\overline{Q}_n(a)}(x_0) \times C_{\overline{Q}_m(b)}(y_0) = X \times Y$$

(where $C_K(z)$ denotes the Clarke tangent cone[13] to a subset K at a point $z \in K$), is satisfied.

Since x_0 belongs to $\overline{Q}_n(a) := a\mathcal{R}^n_+$, the Clarke tangent cone $C_{\overline{Q}_n(a)}(x_0)$ coincides with the tangent cone, which contains[14] $Q_n(a)$. In the same way,

$$C_{\overline{Q}_m(b)}(y_0) \;\supset\; \overline{Q}_m(b)$$

On the other hand, f being continuously differentiable at x_0, the Clarke tangent cone to the graph of f is the graph of the derivative $f'(x_0)$. Hence the above assumption can be rewritten in the form

$$\mathrm{Graph}(f'(x_0)) - (\overline{Q}_n(a) \times \overline{Q}_m(b)) \;=\; X \times Y$$

[13] An element v belongs to the Clarke tangent cone $C_K(z)$ if and only if

$$\lim_{h \to 0+, K \ni y \to z} d(y + hv, K)/h \;=\; 0$$

This is always a closed convex cone, which coincides with the tangent space when K is a differentiable manifold and with the tangent cone of convex analysis when K is a convex subset.(See for instance [3, Chapter VII])

[14] If Q is a convex cone,

$$T_Q(x) \;=\; \mathrm{cl}(Q + x\mathbf{R}) \;\supset\; Q$$

By polarity, this is equivalent to the condition

$$(\mathrm{Graph}(f'(x_0)))^- \cap (\overline{Q}_n(a) \times \overline{Q}_m(b))^+ \;=\; 0$$

which is nothing other than condition (40). □

When $f = A \in \mathcal{L}(X,Y)$ is a linear operator from X to Y, we can take $(x_0, y_0) = (0,0)$ and, observing that $A = f'(0)$, deduce the following consequence:

Theorem 4.2 *Let $A \in \mathcal{L}(X,Y)$ be a linear operator. If the dual standard qualitative equation is "singular" in the sense that*

$$(40) \qquad \begin{cases} (0,0) \text{ is the only solution } (p,q) \text{ to} \\ p \in A^\star(q) \cap Q_n^\star(a) \;\; \& \;\; q \in -Q_m^\star(b) \end{cases}$$

then there exists a solution a to the standard qualitative equation

$$(41) \qquad \pi_0(A)(a) \;\ni\; b$$

We can use exactly the same proof for extending this theorem to the case of set-valued maps F, since we only used the fact that the Clarke tangent cone to the graph of f is the graph of $f'(x_0)$.

When F is a set-valued map from X to Y, we define its "circatangent derivative" $CF(x_0, y_0)$ at a point (x_0, y_0) of its graph by the formula

$$\mathrm{Graph}(CF(x_0, y_0)) \;=\; C_{\mathrm{Graph}(F)}(x_0, y_0)$$

(See for instance [3, Chapter VII]). Since the Clarke tangent cone is always a closed convex cone, we deduce that the circatangent derivative $CF(x_0, y_0)$ is always a closed convex process. Hence Theorem 4.1 can be extended in the following way:

Theorem 4.3 *Let $(x_0, y_0) \in \mathrm{Graph}(F) \cap (\overline{Q}_n(a) \times \overline{Q}_m(b))$ be a representative of a solution a to the large qualitative inclusion*

$$\overline{\pi_0}(F)(a) \;:= S_m \circ F \circ \overline{Q}_n \;\ni\; b$$

If

$$(42) \qquad \begin{cases} (0,0) \text{ is the only solution } (p,q) \text{ to} \\ p \in CF(x_0, y_0)^\star(q) \cap Q_n^\star(a) \;\; \& \;\; q \in -Q_m^\star(b) \end{cases}$$

then a is a solution to the qualitative inclusion

$$\pi_0(F)(a) \;:=\; s_m \circ F \circ Q_n \ni \; b$$

When F is a closed convex process, we can take $(x_0, y_0) = (0,0)$. Since $CF(0,0) = F$ (because the Clarke tangent cone to the closed convex cone Graph(F)) at the origin is equal to this closed convex cone, we deduce from the above theorem the following consequence:

Theorem 4.4 (Standard Set-Valued Qualitative Duality) *Let F be a closed convex process from a finite dimensional vector-space X to a finite dimensional vector-space Y. If the dual qualitative inclusion is singular in the sense that*

$$\left\{ \begin{array}{l} (0,0) \text{ is the only solution } (p,q) \text{ to} \\ p \in F^\star(q) \cap Q_n^\star(a) \;\; \& \;\; q \in -Q_m^\star(b) \end{array} \right. \tag{43}$$

then there exists a solution a to the qualitative inclusion

$$\pi_0(F)(a) \ni b \tag{44}$$

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