WORKING PAPER

INERTIAL MANIFOLD FOR A REACTION DIFFUSION EQUATION MODEL OF COMPETITION IN A CHEMOSTAT

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Foreword

In this paper a reaction is studied which describes competition of two species for a common resource in the limit where the number of linearly coupled vessels goes to infinity. Using the theory of Foias, Sell and Teman the author proves the existence of an inertial manifold, this is, a finite dimensional manifold which exponentially attracts all solutions.

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Inertial Manifold for a Reaction Diffusion Equation Model of Competition in a Chemostat

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ABSTRACT

The existence of inertial manifold for a reaction-diffusion equation model of the chemostat is established.

1. Introduction

The purpose of this paper is to show that inertial manifolds exist for a system of reaction diffusion equations which was used to model competition in a chemostat (c.f. [So and Waltman]). The equations are:

$$
\begin{aligned}
S_t &= S_{xx} - f(S)u - g(S)v \\
u_t &= u_{xx} + f(S)u \\
v_t &= v_{xx} + g(S)v
\end{aligned}
\tag{1.1}
$$

where $S(t,x)$ (resp. $u(t,x)$, $v(t,x)$) denotes the concentration of the limiting substrate (resp. the competing micro-organisms) at time $t \geq 0$ and position $0 \leq x \leq L$. Here

$$
f(S) := \frac{mS}{a+S}, \qquad g(S) := \frac{nS}{b+S}
\tag{1.2}
$$

for $S \geq 0$, where m, a, n and $b > 0$. The boundary conditions are:

$$
\begin{aligned}
S_x(t,0) &= -S^{(0)} \\
u_x(t,0) &= v_x(t,0) = 0
\end{aligned}
\tag{1.3}
$$

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$$S_x(t,L) + \gamma S(t,L) = u_x(t,L) + \gamma u(t,L) = v_x(t,L) + \gamma v(t,L) = 0$$

where $S^{(0)}$ and $\gamma > 0$.

Let $z = S + u + v$. Then z satisfies

$$z_t = z_{xx} \tag{1.4}$$

with boundary conditions:

$$z_x(t,0) = -S^{(0)}\ , \qquad z_x(t,L) + \gamma z(t,L) = 0\ . \tag{1.5}$$

We will need the following form of the Poincaré inequality.

Proposition 1.1 (c.f. Theorem 11.11 of [Smoller]) Let $w \in W^{1,2}[0,L]$. Then

$$||w'||_2^2 + \gamma w(L)^2 \geq c||w||_2^2 \tag{1.6}$$

where $c > 0$ is the smallest eigenvalue of the boundary-value problem

$$-w'' = \lambda w\ , \qquad w'(0) = w'(L) + \gamma w(L) = 0\ . \tag{1.7}$$

Proof. Let $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ be the eigenvalues of (1.6) and let $\varphi_1, \varphi_2, \cdots$ be the corresponding orthonormal eigenfunctions. Let $w = \sum a_i\varphi_i$. Integrating by parts, we get

$$\begin{aligned}\gamma w(L)^2 + ||w'||_2^2 &= \gamma w(L)^2 + \int_0^L w'^2 = -\int_0^L ww'' \\ &= -<w,w''> = -<\sum a_i\varphi_i, -\sum a_i\lambda_i\varphi_i> = \sum a_i^2\lambda_i \\ &\geq c\sum a_i^2 = <\sum a_i\varphi_i, \sum a_i\varphi_i> = ||w||_2^2\ .\end{aligned}$$

Proposition 1.2. Let $z(t,x)$ be a solution of (1.4) and (1.5). Then $z(t,x)$ converges to the steady state solution $\hat{z}(x) := S^{(0)}(L + \frac{1}{\gamma} - x)$ of (1.4), (1.5) in the L^2 norm.

Proof. Let $w = z - \hat{z}$. Then w satisfies $w_t = w_{xx}$ and $w_x(t,0) = w_x(t,L) + \gamma w(t,L) = 0$. Now

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left[\int_0^L w^2 dx\right] &= \int_0^L w\frac{dw}{dt}dx = \int_0^L ww_{xx}dx \\ &= \Big[ww_x\Big]_0^L - \int_0^L w_x^2 dx = -\gamma w(t,L)^2 - \int_0^L w_x^2 dx\ .\end{aligned}$$

By Proposition (1.1),

$$\frac{1}{2}\frac{d}{dt}||w(t,.)||_2^2 \le -c||w(t,.)||_2^2$$

which in turn implies

$$||w(t,.)||_2 \le e^{-ct}||w(0,.)||_2 .$$

We now use $\hat{z}(x)$ to reduce (1.1), (1.3) to

$$u_t = u_{xx} + f(\hat{z}(x)-|u|-|v|)u \tag{1.8}$$

$$v_t = v_{xx} + g(\hat{z}(x)-|u|-|v|)v$$

with boundary conditions:

$$u_x(t,0) = v_x(t,0) = u_x(t,L) + \gamma u(t,L) = v_x(t,L) + \gamma v(t,L) = 0 , \tag{1.9}$$

where

$$f(S) := \begin{cases} \dfrac{mS}{a+|S|} & \text{for } S \ge -1 \\ -\dfrac{m}{a+1} & \text{for } S < -1 \end{cases} \tag{1.10}$$

$$g(S) := \begin{cases} \dfrac{nS}{b+|S|} & \text{for } S \ge -1 \\ -\dfrac{n}{b+1} & \text{for } S < -1 \end{cases}$$

Note that this definition of $f(S)$ and $g(S)$ for $S<0$ will not affect solutions $(S(t,x),u(t,x),v(t,x))$ of (1.1), (1.3) satisfying $S(t,x)$, $u(t,x)$, $v(t,x) \ge 0$ and $S(t,x) + u(t,x) + v(t,x) = \hat{z}(x)$. It is to (1.8), (1.9) for which we will show that inertial manifolds exist.

We will need the following simple estimates on f and g.

Proposition 1.3. For all S, S_1 and S_2,

$$|f(S)| \le m , \quad |g(S)| \le n ,$$

$$|f(S_1) - f(S_2)| \le \frac{m}{a}|S_1 - S_2| , \quad |g(S_1) - g(S_2)| \le \frac{n}{b}|S_1 - S_2| .$$

2. Inertial Manifolds. General Theory.

There are a number of existence theories for inertial manifolds (c.f. [Kamaev], [Mora], [Foias, Sell and Teman], [Mallet-Paret and Sell], [Chow and Lu] and [Teman]). In this section we will recall one that is immediately applicable to (1.8) and (1.9).

Consider an abstract evolution equation of the form

$$\frac{du}{dt} + Au = R(u) \tag{2.1}$$

on a Hilbert space H. A is a linear, unbounded, self-adjoint operator on H with dense domain, $D(A)$, in H. Moreover, A is assumed to be positive and that A^{-1} is compact. Under these assumptions on A, there exists an orthonormal basis $\{w_j\}$ of H consisting of eigenvectors of A, $Aw_j = \lambda_j w_j$, where the eigenvalues satisfy $0 < \lambda_1 \leq \lambda_2 \leq \cdots$, $\lambda_j \to \infty$ as $j \to \infty$. The nonlinear term $R : H \to H$ is assumed to be locally Lipschitz continuous.

Definition 2.1. A subset M of H is said to be an *inertial manifold* for (2.1) if it satisfies the following properties:

(i) M is a finite dimensional Lipschitz manifold,

(ii) M is positively invariant, and

(iii) M attracts exponentially all solutions of (2.1).

Assume that (2.1) is dissipative, i.e., there is a $\rho_0 > 0$ such that

$$limsup_{t\to\infty} \; ||u(t)||_2 \leq \rho_0 \,, \tag{2.2}$$

for all solution $u(t)$ of (2.1). In this case, one can modify (2.1) to the so-called prepared equation

$$\frac{du}{dt} + Au = \theta_\rho(|u|)R(u) \,, \tag{2.3}$$

Here, $\theta : [0,\infty) \to [0,1]$ is a fixed smooth function with $\theta(s) = 1$ for $0 \leq s \leq 1$, $\theta(s) = 0$ for $s \geq 2$ and $|\theta'(s)| \leq 2$ for $s \geq 0$. And $\theta_\rho(s) = \theta(\frac{s}{\rho})$ for $s \geq 0$, where $\rho = 2\rho_0$.

Theorem 2.2. (Theorem 2.2 of [Foias, Sell and Teman]) Under the above assumptions, there exist N_0, K_{12}, $K_{13} > 0$ such that if one has

$$N \geq N_0, \quad \lambda_{N+1} \geq K_{12}, \quad \lambda_{N+1} - \lambda_N \geq K_{13}, \tag{2.4}$$

then (2.3) possesses an inertial manifold of dimension N.

3. Inertial Manifolds. Our Model.

In order to show that (1.8), (1.9) possess an inertial manifold, we will first cast them in the form (2.1) and verify the hypotheses of Theorem 2.2. Let H be the Hilbert space $L^2[0,L]\times L^2[0,L]$. Let A be the linear operator $(-\frac{d^2}{dx^2}, -\frac{d^2}{dx^2})$ defined on the subspace of H consisting of all pairs (u,v), where $u,\ v \in C^2[0,L]$ satisfy the boundary conditions (1.9), i.e., $\frac{du}{dx}(0) = \frac{dv}{dx}(0) = 0$ and $\frac{du}{dx}(L)+\gamma u(L) = \frac{dv}{dx}(L)+\gamma v(L) = 0$. By Friedrichs extension theorem, we can extend A to a closed operator, again denoted by A. Then A is an unbounded, self-adjoint, positive operator from its domain $D(A)$ to H with A^{-1} compact. Moreover, if we denote the eigenvalues of A by: $0 < \lambda_1 \leq \lambda_2 \leq \cdots$, then $\lambda_{2n-1} = \lambda_{2n} = \mu_n^2$, where μ_n is the n-th positive root of the equation $\tan(\mu L) = \frac{\gamma}{\mu}$. Since $(n-1)\pi L^{-1} < \mu_n < (n-\frac{1}{2})\pi L^{-1}$, (2.4) can be satisfied with a large enough N.

Let $R : H \to H$ denote the Nemitski operator corresponding to the reaction term, i.e.

$$R(u,v)(x) = \Big[f(\hat{z}(x)-|u(x)|-|v(x)|)u(x),\ g(\hat{z}(x)-|u(x)|-|v(x)|)v(x)\Big]. \tag{3.1}$$

We will first show that R is globally Lipschitz continuous on H. Consider the integral

$$I := \int_0^L \Big|f(\hat{z}(x)-|u_1(x)|-|v_1(x)|)u_1(x)\ -\ f(\hat{z}(x)-|u_2(x)|-|v_2(x)|)u_2(x)\Big|^2\, dx$$

Then

$$I = \int_{M_1^+\cap M_2^+} + \int_{M_1^-\cap M_2^+} + \int_{M_1^+\cap M_2^-} + \int_{M_1^-\cap M_2^-},$$

where

$$M_i^+ := \{x \in [0,L] : \hat{z}(x)-|u_i(x)|-|v_i(x)| > -1\}$$

$$M_i^- := \{x \in [0,L] : \hat{z}(x)-|u_i(x)|-|v_i(x)| \leq -1\}$$

for $i = 1,2$. Denote these integrals by I_1, I_2, I_3 and I_4, resp.

For $x \in M_1^- \cap M_2^-$, the absolute value (i.e. without the square) in the integrand of I is (with the x suppressed):

$$\leq \left|(-\frac{m}{a+1}u_1) - (-\frac{m}{a+1}u_2)\right| \leq \frac{m}{a+1}\,|u_1 - u_2|\ .$$

Therefore, $I_4 \leq c_4\,||u_1-u_2||_2^2$, for some $c_4 > 0$.

For $x \in M_1^+ \bigcap M_2^+$, by Proposition 1.3, the absolute value is:

$$\leq \left| f(\hat{z}-|u_1|-|v_1|)(u_1-u_2) \right| + \left| (f(\hat{z}-|u_1|-|v_1|) - f(\hat{z}-|u_2|-|v_2|))u_2 \right|$$

$$\leq m|u_1-u_2| + \frac{m}{a}\left| |u_1|+|v_1|-|u_2|-|v_2| \right| |u_2|$$

$$\leq m|u_1-u_2| + \frac{m(\hat{z}(0)+1)}{a}\left(|u_1-u_2| + |v_1-v_2| \right)$$

Therefore, $I_1 \leq c_1 \left(||u_1-u_2||_2 + ||v_1-v_2||_2 \right)^2$, for some $c_1 > 0$.

There are similar estimates on I_2 and I_3 as well as on the second component of R. Hence, R is globally Lipschitz continuous.

Next we show that the dissipative condition (2.2) is satisfied. Integrating

$$uu_t = uu_{xx} + f(\hat{z}-|u|-|v|)u^2$$

we get

$$\frac{1}{2}\frac{d}{dt}\int_0^L u^2 = -\gamma u(.,L)^2 - \int_0^L u_x^2 + \int_0^L f(\hat{z}-|u|-|v|)u^2$$
$$\leq -c\int_0^L u^2 + \int_0^L f(\hat{z}-|u|-|v|)u^2 \ ,$$

by Proposition 1.1. Fix any t and consider the integral

$$I := \int_0^L f(\hat{z}-|u|-|v|)u^2 = \int_{M+} f(\hat{z}-|u|-|v|)u^2 + \int_{M-} f(\hat{z}-|u|-|v|)u^2$$

where

$$M^+ := \{x \in [0,L] : \hat{z}(x)-|u(t,x)|-|v(t,x)| > -1\}$$

$$M^- := \{x \in [0,L] : \hat{z}(x)-|u(t,x)|-|v(t,x)| \leq -1\} \ .$$

Denote these integrals by I_1 and I_2 resp. The first integral I_1 is bounded above by

$$m\int_{M+} u^2 \leq m(\hat{z}(0)+1)^2 L := K \ .$$

Let $\bar{\rho} > 0$ be such that $\bar{\rho}^2 = \max\{\frac{(a+1)K}{m}, \frac{(b+1)K}{n}\} + (\hat{z}(0)+1)^2 L$. and pick any $\rho_0 > \bar{\rho}$. Suppose $||u(\bar{t},.)||_2 \geq \rho_0$ for some $\bar{t}$. Then for $t = \bar{t}$, we have

$$\int_{\{x\in[0,L]:|u(t,x)|<\hat{z}(x)+1\}} u^2 + \int_{\{x\in[0,L]:|u(t,x)|\geq\hat{z}(x)+1\}} u^2 \geq \rho_0^2$$

which implies

$$\int_{\{x\in[0,L]:|u(t,x)|\geq\hat{z}(x)+1\}} u^2 \geq \rho_0^2 - \int_{\{x\in[0,L]:|u(t,x)|<\hat{z}(x)+1\}} u^2$$

$$\geq \rho_0^2 - (\hat{z}(0)+1)^2 L > \frac{(a+1)K}{m}$$

Therefore, at $t = \bar{t}$,

$$I_2 \leq -\frac{m}{a+1} \int_{M^-} u^2 \leq -\frac{m}{a+1} \int_{\{x \epsilon [0,L]:|u(t,x)| \geq \hat{z}(x)+1\}} u^2 < -K .$$

Hence, $I < 0$ and consequently $\frac{d}{dt}||u(t,.)||^2 \leq -2c||u(t,.)||^2$, whenever $||u(t,.)||_2 \geq \rho_0$. Similarly, $\frac{d}{dt}||v(t,.)||^2 \leq -2c||v(t,.)||^2$, whenever $||v(t,.)||_2 \geq \rho_0$.

Thus, by Theorem 2.2, we have proved that the prepared equation for (1.8), (1.9) possesses an inertial manifold M_{ρ_0}.

Actually the above argument shows a little more. If we let

$$B := \{(u,v) \in H : ||u||_2, ||v||_2 < \rho_1 \} ,$$

where $\rho_1 > \bar{\rho}$ then B is is positively invariant and absorbing, i.e., if we denote the solution operator for (1.8), (1.9) by $T(t)$ then $T(t)B \subseteq B$ and for each bounded set B_1, there exists t_1 such that $T(t)B_1 \subseteq B$ for all $t \geq t_1$. Moreover, $T(t)$ maps bounded sets to bounded sets. Hence, by Theorem 4.2.4 of [Hale], (1.8), (1.9) possess a global attractor which lies in B. If we now pick $\rho_0 > \rho_1$ so large that the ball in H with radius ρ_0 and centered at the origin contains the B, then $B \bigcap M_{\rho_0}$ is an inertial manifold for (1.8), (1.9). Thus, we have proved that

Theorem 3.1. Under the above assumptions, (1.8), (1.9) possess an inertial manifold.

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