WORKING PAPER

PLAYABLE DIFFERENTIAL GAMES

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August 1989
WP-89-60

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International Institute
for Applied Systems Analysis

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FOREWORD

Playability conditions of differential games are studied by using Viability Theory.

First, the results on playability of time independent differential games are extended to time dependent games. In fact, time is introduced in the dynamics of the game, in the state dependent contraints bearing on controls and in state contraints.

Second, some examples of pursuit games are studied. Necessary and sufficient conditions of playability of the game are provided. Here, pursuit games are directly considered as "games of kind" (in Isaacs's sense) and are not considered as "games of degree". The viability condition does not always provide the "optimal strategy" to be as close as possible to a certain goal, but it supplies strategies allowing the system to reach a given goal.

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Playable Differential Games

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1 Introduction

We consider a two players differential game whose dynamics are described by:

$$\begin{cases} (a) & \begin{cases} i) & x'(t) = f(t, x(t), y(t), u(t)) \\ ii) & u(t) \in U(t, x(t), y(t)) \end{cases} \\ \\ (b) & \begin{cases} i) & y'(t) = g(t, x(t), y(t), v(t)) \\ ii) & v(t) \in V(t, x(t), y(t)) \end{cases} \end{cases}$$

The constraints of the game are the time dependent **game rules** P and Q
$P : \mathbf{R} \times Y \mapsto X$ & $Q : X \times \mathbf{R} \mapsto Y$
The **playability tube** of this game is:
$K(t) := \{ (x, y) \ / \ x \in P(t, y) \ \text{ and } \ y \in Q(x(t), t) \}$

The **playability property** of the game holds when for all time t_0 and for every initial state (x_0, y_0) in $K(t_0)$, there exist solutions of the differential game starting at time t_0 for (x_0, y_0) such that :

$$\forall t \geq t_0 \ x(t) \in P(t, y(t)) \ \& \ y(t) \in Q(x(t), t)$$

We shall characterize it by constructing **the regulation map** $R_{P,Q}$ in which we could choose playable controls. This map is built thanks to contingent derivatives of the rules[1]. We can introduce the subset of **discriminating controls** which allow the players to associate to any control v played by the second player at least a control u such that (u, v) is playable.

$A_{P,Q}(x, t, y, v) := \{ u \in U(t, x, y); (u, v) \in R_{P,Q}(x, t, y) \}$

We also introduce the **pure control** map $B_{P,Q}$ which allows the first player to choose a control u such that (u, v) is playable for any $v \in V(t, x, y)$. Before going further, it may be use ful to relate these concepts to Isaacs-Hamilton-Jacobi equations.

Next, we adress the question of general pursuit games. A pursuer wants to catch an invader. Of course the meaning of "to catch" will depend on each example, but, generally, it means to be near enough as we shall see in the construction of the playability domains. At the beginning, we shall write a condition of playability for the famous R. Isaacs problem: The target and its guardian.

We solve the case of certain capture with playability rules of the form:

$P(t, y) = y + \varphi(t)C$ and $Q(x, t) = x - \varphi(t)C$

We then apply this to affine differential games:

$$\begin{cases} z'(t) = Az(t) - u(t) + v(t) \\ u(t) \in U(t, z(t)) \\ v(t) \in V(t, z(t)) \end{cases}$$

The regulation map of this game will be conducted in an example.

[1]Let us recall the **contingent cone** at x to a subset K:

$$T_K(x) := \{ v \ / \ \liminf_{h \to 0+} d(x + hv, K)/h = 0 \}$$

The **contingent derivative** of the set valued map $Q \ \ X \mapsto Y$ is the set valued map $DQ(x, y) \ : X \mapsto Y$ defined by:

$$\mathrm{Graph}(DQ(x, y)) := T_{\mathrm{Graph}(Q)}(x, y)$$

or, equivalently, by:

$$v \in DQ(x, y)(u) \iff \liminf_{h \to 0+, w \to u} (v, \frac{Q(x + hw) - y}{h}) = 0$$

Another problem is the end time of a capture. So, we give conditions for a time T to be the end time of the pursuit. For this purpose, we consider an increasing nonnegative function w and write the viability tube in the following way:

$$K(t) = \{ (x, y) \ / \ d(x, y) \leq \ w(T - t) \}$$

2 Time dependent differential games

Let us consider two players Xavier and Yves. Xavier acts on a state space X with a control u , and, Yves on a state space Y with a control v. The controls depend on players states and on time through the set valued maps U and V respectively. Their actions on their states are governed by the controlled system:

$$(1) \qquad \begin{cases} (a) & \begin{cases} i) & x'(t) = f(t, x(t), y(t), u(t)) \\ ii) & u(t) \in U(t, x(t), y(t)) \end{cases} \\ \\ (b) & \begin{cases} i) & y'(t) = g(t, x(t), y(t), v(t)) \\ ii) & v(t) \in V(t, x(t), y(t)) \end{cases} \end{cases}$$

Where X, Y are finite dimensional spaces, and where
f : Graph$U \mapsto X$ $\quad g$: Graph$V \mapsto Y$ are single valued maps.

The influence between the two players is exerted through the **rules of the game**:
$P : [0,T] \times Y \mapsto X$ & $Q : X \times [0,T] \mapsto Y$
It means that constraints of the game are:
$\forall\, t \in [0,T] \quad x(t) \in P(t, y(t)) \quad$ and $\quad y(t) \in Q(x(t), t)$
So we can define a **playability tube**:
$K(t) =: \{(x, y) \;/\; (x, t, y) \in \text{Graph}Q \cap \text{Graph}P^{-1}\}$
For any (x_0, t_0, y_0), let's also introduce the solution map $\mathcal{S}(x_0, t_0, y_0)$ of solutions to (1) starting on (x_0, y_0) at t_0.

Now, we always assume that the playability domain is non empty and that graphs of P and Q are closed.

We need a suitable definition of playability:

Definition 2.1 *The game enjoys the playability property if and only if:*

$$(2) \qquad \begin{array}{l} \forall\, t_0 \in [0, T[\;\; \forall\, (x_0, y_0) \in K(t_0) \;\; \exists\, (x(\cdot), y(\cdot)) \in \mathcal{S}(x_0, t_0, y_0) \\ \begin{cases} i) & \forall\, t \in [t_0, T[\qquad (x(t), y(t)) \in K(t) \\ ii) & if\; T < \infty \qquad \forall\, t \geq T \;\; (x(t), y(t)) \in K(T) \end{cases} \end{array}$$

Before writting our first proposition, let us assume that:

$$(3)\begin{cases} i) & f \text{ and } g \text{ are continuous with linear growth and} \\ & \text{are affine with respect to } u \text{ and } v \\ ii) & \text{The controls maps } U \text{ and } V \text{ are upper semi continuous} \\ & \text{with compact convex images and with a linear growth.} \end{cases}$$

We have to define notions of transversality and sleek sets:

Definition 2.2 *A set K is sleek at x if and only if the set valued map $T_K(\cdot)$ is lower semi continuous. (All convex subsets are sleek.) A set valued map is sleek if and only if its graph is sleek.*

Definition 2.3 *The rules will be said "transversal" if and only if:*

$$\forall t \in [0,T] \;\; \forall (x,y) \in K(t)$$
$$T_{\mathrm{Graph}_Q}(x,t,y) \; - \; T_{(\mathrm{Graph}_{P(-1)})}(x,t,y) = X \times \mathbf{R} \times Y$$

Proposition 2.4 *Under assumptions (3) and if the rules P,Q are "sleek" and transversal, a necessary and sufficient condition of game playability is the following Haddad's contingent condition:*

$$\forall t \;\; \forall (x,y) \in K(t) \; \exists (u,v) \in U(t,x,y) \times V(t,x,y)$$

$$i) \quad \text{if } t \in [0,T[\quad \begin{cases} g(t,x,y,v) \in DQ(x,t,y)(f(t,x,y,u),1) \\ f(t,x,y,u) \in DP(x,t,y)(1,f(t,x,y,v)) \end{cases}$$

$$ii) \quad \text{if } T < \infty \quad \begin{cases} g(T,x,y,v) \in DQ(x,T,y)(f(T,x,y,u),0) \\ f(T,x,y,u) \in DP(x,T,y)(0,f(T,x,y,v)) \end{cases}$$

Remark — The transversality condition is assumed because it is useful to separate the rules following way:

$$(4)\, T_{\mathrm{Graph}_Q \cap \mathrm{Graph}(P^{-1})}(x,t,y) = T_{\mathrm{Graph}_Q}(x,t,y) \cap T_{\mathrm{Graph}_{P^{-1}}}(x,t,y)$$

It is an obvious consequence of (2.3).

A necessary and sufficient condition for the transversality of the rules is that for all perturbations (e, f, g), there exists $(u, \tau, v) \in X \times \mathbf{R} \times Y$ such that:

$$\begin{cases} i) & u \in DP(x,t,y)(\tau + f, v) + e \\ ii) & v \in DQ(x,t,y)(u,\tau) + g \end{cases}$$

See corollary 4-3 of [1] □

Proof —

Let us consider $H(x, s, y) =:$

$$\begin{cases} if \ \ s \in [0,T[\\ \{f(s,x,y,u)\} \times \{\ 1\ \} \times \{g(s,x,y,v)\} \ / \ (u,v) \in \ U(t,x,y) \times V(t,x,y)\} \\ \\ if \ \ s = T \\ \{f(T,x,y,u)\} \times [0,1] \times \{g(T,x,y,v)\} \ / \ (u,v) \in \ U(T,x,y) \times V(T,x,y)\} \\ \\ if \ \ s > T \\ \{f(T,x,y,u)\} \times \{\ 0\ \} \times \{g(T,x,y,v)\} \ / \ (u,v) \in \ U(t,x,y) \times V(t,x,y)\} \end{cases}$$

The system now becomes:

$(x'(t), s'(t), y'(t)) \in \ H(x, s, y)$

Applying Haddad's Viability Theorem (see [2]), there exist viable solutions if and only if:

$\forall s \ \forall (x,y) \in \ K(s)$

$H(x,s,y) \ \cap \ T_K(x,s,y) \neq \emptyset$ with $K := \text{Graph}Q \ \cap \ \text{Graph}P^{-1}$

According to the definition of the transversality and the contingent derivative, it is possible to write:

$T_{\text{Graph}Q\cap\text{Graph}(P^{-1})}(x,t,y) = \text{Graph}DQ(x,t,y) \ \cap \ \text{Graph}(DP(y,t,x)^{-1})$

With the expression of H we have proved the previous proposition . □

Remark — In some particular cases, we can compute directly the contingent cone $T_{\text{Graph}Q \,\cap\, \text{Graph}P^{-1}}$, without assuming the transversality condition. In fact, very often it is more simple to write $T_{\text{Graph}Q\cap\text{Graph}(P^{-1})}$ for instance when it is impossible to separate the constraints sets of the two players (see further the example of pursuit game with certain capture). □

We need to choose controls satisfying the previous proposition. For that purpose, let us define the **retroaction rules** C and D acting on the controls:

Definition 2.5 *Xavier's retroaction rule is the set-valued map :*

$$C(t,x,y;v) := \begin{cases} \text{if } t \leq T \\ \{ u \in U(t,x,y) / \quad f(t,x,y,u) \in DP(x,t,y)(1, g(t,x,y,v)) \} \\ \text{if } t = T \\ \{ u \in U(t,x,y) / \quad f(T,x,y,u) \in DP(x,T,y)(0, g(T,x,y,v)) \} \end{cases}$$

and Yves's retroaction rule is the set-valued map:

$$D(t,x,y;u) := \begin{cases} \text{if } t \leq T \\ \{ v \in V(t,x,y) / \quad g(t,x,y,v) \in DQ(x,t,y)(f(t,x,y,u),1) \} \\ \text{if } t = T \\ \{ v \in V(T,x,y) / \quad g(T,x,y,v) \in DQ(x,T,y)(f(T,x,y,u),0) \} \end{cases}$$

These maps allow to replace the initial differential game by a game on controls parametrized by the state and the rules through the following regulation map. With these retroaction rules, we can define subsets in which it is possible to choose playable controls, discriminating controls, pure controls, respectively.

Definition 2.6 *We associate with the retroaction rules C and D the regulation map R of playable controls defined by:*

$$R_{P,Q}(t,x,y) \quad = \quad \{ (u,v) \mid u \in C(t,x,y;v) \;\; \& \;\; v \in D(t,x,y;u) \}$$

The discriminating set valued map :

$$A_{P,Q}(t,x,y,v) := \{ u \in U(t,x,y); (u,v) \in R_{P,Q}(t,x,y) \}$$

The set valued map B:

$$B_{P,Q}(x,t,y) := \cap_{v \in V(t,x,y)} A_{P,Q}(t,x,y,v)$$

The concept of playable rules:(P and Q are playable iff:)

$$\forall\, t \in [0,T] \;\; \forall\, (x,y) \in K(t) \;\; R_{P,Q}(t,x,y) \neq \emptyset$$

Let's remark that $R_{P,Q}$ is the set of fixed-points to the set valued map $C \times D$.

An obvious consequence of these definition is the easy result:

Corollary 2.7 *If the domains of the retroaction rules are equal to the control set valued maps U and V, then the constraint set and the regulation set are nonempty.*

It can, then, be useful to translate the viability conditions of the game to Isaacs-Hamilton-Jacobi contingent equations[2]. Playability can be expressed by an Isaacs Hamilton Jacobi equation thanks to contingent epi-derivatives.Consequently,let us recall the definition of the contingent epi-derivative of V at x in the direction v:

Definition 2.8 *Let $V : X \to \mathbf{R} \cup \{\infty\}$*

$D_\uparrow V(x)(v) := \liminf_{h \longrightarrow 0^+, u \longrightarrow v}(V(x+hu) - V(x))/h)$

or in shorter way:

$T_{\text{EpigraphV}}(x, V(x)) = \text{Epigraph } (DV(x))$

Proposition 2.9 *The regulation map is nonempty if and only if:*

$\forall t \in [0, T[\quad \forall (x, y) \in K(t)$

i)

$$\inf_{\begin{cases} u \in U(t,x,y) \\ v \in V(t,x,y) \end{cases}} D_\uparrow(\max_{W_P, W_Q})(x,t,y)(f(t,x,y,u), 1, g(t,x,y,v)) = 0$$

ii)

if $T < \infty$

$$\inf_{\begin{cases} u \in U(T,x,y) \\ v \in V(T,x,y) \end{cases}} D_\uparrow(\max_{W_P, W_Q}))(x,T,y)(f(T,x,y,u), 0, g(T,x,y,v)) = 0$$

Here, the rules are characterized by indicators functions of their graphs W_P and W_Q.

---

[2]See [2]

$$W_P(x,t,y) := \begin{cases} 0 \ if \ x \in P(t,y) \\ \infty \ else \end{cases} \quad W_Q(x,t,y) := \begin{cases} 0 \ if \ y \in Q(x,t) \\ \infty \ else \end{cases}$$

Proof — It is only the translation of (2.4), if we notice that:
$0 \geq D_1 W_Q(x,t,y)(a,1,b)$ if and only if:
$(a,1,b) \in T_{graphQ}(x,t,y)$ □

To proceed further, it is convenient to write the differential game in a more compact form. The state (x,y) is now $z \in X \times Y$ and this system includes the playability rules in the set valued maps U and V:
$U(t,z) := \emptyset$ if $(t,z) \notin$ Graph$P$
$V(t,z) := \emptyset$ if $(t,z) \notin$ Graph$Q$
This is given by the following equations with the single valued map $h(t,z,u,v)$ describing the evolution:

$$(5) \qquad \begin{cases} z'(t) = h(t,z(t),u(t),v(t)) \quad t \in [0,T] \\ u(t) \in U(t,z(t)) \\ v(t) \in V(t,z(t)) \end{cases}$$

with constraints:
$\forall t \in [0,T] \ z(t) \in K(t) := \{z \ / \ U(t,z(t)) \neq \emptyset \ \& \ V(t,z(t)) \neq \emptyset \ \}$
We assume that:

$$(6) \qquad \begin{cases} i) & h : X := \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R}^q \mapsto \mathbf{R}^n \\ & \text{is continuous with a linear growth and is affine} \\ & \text{with respect to } u \text{ and } v. \\ ii) & K \text{ is sleek.} \\ iii) & U,\ V \text{ are upper semi continuous with compact} \\ & \text{convex images and with a linear growth.} \end{cases}$$

Under assumptions (6), we can write the Haddad's contingent condition for the game playability:

$$(7) \qquad \begin{cases} \forall t \in [0,T[\ \ \forall z \in K(t) \ \exists (u,v) \in U(t,z) \times V(t,z) \\ \quad i) \ \ if \ t \in [0,T] \quad h(t,z,u,v) \in DK(t,z)(1) \\ \quad ii) \ \ if \ \ T < \infty \quad h(T,z,u,v) \in DK(T,z)(0) \end{cases}$$

We can, then, translate this viability condition into the following Isaacs-Hamilton-Jacobi contingent equation:

$$
(8) \qquad \forall\, t \in [0,T] \quad \forall\, z \in K(t)
\begin{cases}
i) & \inf_{u \in U(t,z)\, v \in V(t,z)} D_{\uparrow} W_K(t,z)(1, h(t,z,u,v)) = 0 \\
ii) & \text{if } T < \infty \\
 & \inf_{u \in U(T,z)\, v \in V(T,z)} D_{\uparrow} W_K(T,z)(0, h(T,z,u,v)) = 0
\end{cases}
$$

with $W_K(t,z) = \begin{cases} 0 \text{ if } z \in K(t) \\ \infty \text{ else} \end{cases}$

In the same way, we can associate to the control system four Isaacs-Hamilton-Jacobi contingent equations.

$$
(9) \quad
\begin{cases}
\alpha) \begin{cases} i) \ \inf_{v \in V(t,z)} \inf_{u \in U(t,z)} D_{\uparrow}\Phi(t,z)(1, h(t,z,u,v)) \leq 0 \\ ii) \ \text{if } T < \infty \\ \quad \inf_{v \in V(T,z)} \inf_{u \in U(T,z)} D_{\uparrow}\Phi(T,z)(0, h(T,z,u,v)) \leq 0 \end{cases} \\
\beta) \begin{cases} i) \ \sup_{v \in V(t,z)} \sup_{u \in U(t,z)} D_{\uparrow}\Phi(t,z)(1, h(t,z,u,v)) \leq 0 \\ ii) \ \text{if } T < \infty \\ \quad \sup_{v \in V(T,z)} \sup_{u \in U(T,z)} D_{\uparrow}\Phi(T,z)(0, h(T,z,u,v)) \leq 0 \end{cases} \\
\gamma) \begin{cases} i) \ \sup_{v \in V(t,z)} \inf_{u \in U(t,z)} D_{\uparrow}\Phi(t,z)(1, h(t,z,u,v)) \leq 0 \\ ii) \ \text{if } T < \infty \\ \quad \sup_{v \in V(T,z)} \inf_{u \in U(T,z)} D_{\uparrow}\Phi(T,z)(0, h(T,z,u,v)) \leq 0 \end{cases} \\
\delta) \begin{cases} i) \ \inf_{u \in U(t,z)} \sup_{v \in V(t,z)} D_{\uparrow}\Phi(t,z)(1, h(t,z,u,v)) \leq 0 \\ ii) \ \text{if } T < \infty \\ \quad \inf_{u \in U(T,z)} \sup_{v \in V(T,z)} D_{\uparrow}\Phi(T,z)(0, h(T,z,u,v)) \leq 0 \end{cases}
\end{cases}
$$

Theorem 2.10 *We assume that :*

The function h is continuous with linear growth, set valued maps U and V are closed with linear growth,and that

$\Phi : \mathbf{R} \times X \to \mathbf{R} \cup \{\infty\}$ *is nonnegative, contingently epidifferentiable (see (2.8)) and that its domain is contained in the intersection of domains of U and V.*

Then the equation (9) is equivalent to :

$-\alpha)$ *If U and V have convex values and h is affine with respect to the two controls .* $\begin{cases} \forall\,(s,z) \in \text{Dom}(\Phi) \ \exists\, z(.) \text{ solution to (5)} \\ \forall\, t \in [0,T] \ \Phi(t,z(t)) \le \Phi(s,z) \end{cases}$

$-\beta)$ *if h is uniformely lipsitzchean*

$$\begin{cases} \forall\,(s,z) \in \text{Dom}(\Phi), \ \forall\, z(.) \text{ solution to (5),} \\ \forall\, t \in [0,T]\ \Phi(t,z(t)) \le \Phi(s,z) \end{cases}$$

$-\gamma)$ *If V is lower semi continuous U and V with convex values and h affine with respect to the two controls.*

$$\begin{cases} \text{For any closed loop control } \tilde{v}(s,z) \in V(s,z) \\ \forall\,(s,z) \in \text{Dom}(\Phi)\ \exists\, z(.) \text{ solution to (5) } \textit{with } \tilde{v}(t,z(t)) \\ \text{such that } \forall\, t \in [0,T]\ \Phi(t,z(t)) \le \Phi(s,z) \end{cases}$$

$-\delta)$ *V is lower semi continuous with convex values and $T = \infty$.*
$B = \{\ \bar{u} \in U(s,z),\ inf_{u \in U(t,z)} \sup_{v \in V(t,z)}$
$D_{\uparrow}\Phi(t,z)(1,h(t,z,u,v)) = \sup_{v \in V(t,z)} D_{\uparrow}\Phi(t,z)(1,h(t,z,\bar{u},v))\ \}$
is lower semi continuous with convex values.
The equation (9) δ is satisfied if and only if:

$$\begin{cases} \text{There exists } \tilde{u}(s,z) \in U(s,z) \text{ played by Xavier} \\ \text{such that for any closed loop strategy} \tilde{v}(s,z) \in V(s,z) \\ \forall\,(s,z) \in \text{Dom}(\Phi)\ \exists\, z(.) \text{ solution to (5) with fixed } \tilde{v} \text{ and } \tilde{u} \\ \text{such that: } \forall\, t \in [0,T]\ \Phi(t,z(t)) \le \Phi(s,z) \end{cases}$$

Remark — If $\Phi = \max_{W_P,W_Q}$ the case α means that $\text{Dom}(\Phi) = K$ is a playability tube; the game has the playability property. In the case β, K is an invariant tube; the game has the winnability property.

The case γ define **Xavier's discriminating property**

$(\forall\, \tilde{v}\ A_{P,Q}(t,z,\tilde{v}) \neq \emptyset)$. The last case define **Xavier's leading property** $(B_{P,Q}(t,z) \neq \emptyset)$. □

Proof — For sake of simplicity, we only prove this theorem when $T = \infty$.

Let be:

$$H(s,z) =: \{\ (1, h(s,z,u,v))\ /\ u \in\ U(s,z)\ \ v \in\ V(s,z)\ \}$$

First, it's convenient to notice the following Lemma.

Lemma 2.11 *We have* $D_{\uparrow}\Phi(t,z)(1,h(t,z,u,v)) \leq 0$ *if and only if:*

$$(10) \qquad \left\{ \begin{array}{c} \forall\, w \geq \Phi(t,z) \\ (1, h(t,z,u,v), 0)\ \in\ T_{\text{Epigraph}}\Phi(t,z,w) \end{array} \right.$$

Proof — It's the obvious consequence of the definition of Hamilton Jacobi contingent equations for the system :

$(s', z', w') \in\ H(s,z) \times \{0\}$ with Epigraph Φ as a viability tube. □

Equivalences α and β are the application of the invariant tube and viability tube theorems . The lemme (2.11) shows that implications

(2.10)$\Longrightarrow$ (9) are a simple translation.

Let's prove the third implication. According to Michael's selection Theorem, for any $(s, z,) \in$ Dom(Φ), for any $v_0 \in\ V(s,z)$

there exists continuous $\tilde{v}$ in the set valued map V such that $\tilde{v}(s,z) =\ v_0$.

Hence,$\inf_u(D_{\uparrow}\Phi(t,z)(1, h(t,z,u,\tilde{v}))\ \leq\ 0)$ means that we can applie a similar lemme (that (2.11)) to $H_{\tilde{v}}$. Consequently, Viability tubes Theorem proves the implication.

Finally, let's prove the last result:

According to Michael's selection Theorem (B is lower semi continuous with closed convex values), there exists continuous $\bar{u}$ in the set valued map B and for $H_{\tilde{v}}$ thanks to the lemme we can conclude.

3 Some applications to pursuit Games:

Let us study some cases to which we can apply last results:

3.1 The target guardian problem.

[see Isaacs I.9 p.18]
We consider a game between a guardian (Xavier) and an invader (Yves). The guardian's task is to guarantee that no one can go near some target (a set C) and the invader has the opposite goal. The guardian's coordinates are x and, his opponent's coordinates are y. The evolution of the state (x, y) is given by equations (1). If the distance between Yves's state and C is lower than $l(\cdot)$ the invader wins; if the distance between Xavier's state and Yves's state is lower than $w(\cdot)$ the Guardian wins. These cases determinate the end of the game.
We can write this, using a viability tube, in the following way:

$$K(t) := \{ (x,y) \;/\; d(x,y) \geq w(t) \text{ and } d(C,y) \geq l(t) \}$$

We immediately give a viability condition for this system:

Proposition 3.1 *If w and l are two nonnegative single valued C^1 differentiable maps , if the set C is reduced to a point $\{p\}$, then the game is playable if and only if:*

$$\forall (x,y) \in K(t) \, \exists \, (u,v) \in U(t,x,y) \times V(t,x,y) \text{ such that}$$
$$\begin{cases} i) & \text{if } d(x,y) = w(t) \\ & < x-y, (f(x,y,u) - g(x,y,v) > -w'(t).w(t) \geq 0 \\ ii) & \text{if } \|y-p\| = l(t) \\ & < (y-p), g(x,y,v) > -w'(t).w(t) \geq l(t) \end{cases}$$

Before prooving this proposition let's write the following proposition for tangent cones calculus, it is an obvious consequence of (corollary 4-1 in [1]).

Proposition 3.2 *Let be X, Y two finite dimensional Banach spaces, A $X \mapsto Y$ a map C^1-differentiable around x.*
If $\nabla A(x)(X) = Y$, and if M is sleek,
then
$(\nabla A(x))^{-1}.T_M(A(x)) = T_{A^{-1}(M)}(x)$

Proof — We have to compute the contingent-derivative of the set valued map K when the set C is a point p .

Let be : $A(x,y,t) := (t, \|x-y\|^2 - w(t)^2, \|y-p\|^2 - l(t)^2)$
The map A is obviously C^1 (because w and l are C^1 too) and
Graph $K = A^{-1}(\mathbf{R}_+ \times \mathbf{R}_+ \times \mathbf{R}_+)$.
As $w(t)$ and $l(t)$ are nonnegative, $\nabla A(x,y,t)$ is surjective and we can apply (3.2):
Consequently:
$T_{\mathrm{Graph}K}(t,x,y) = (\nabla A(t,x,y)^{-1}.T_{\mathbf{R}\times\mathbf{R}_+\times\mathbf{R}_+}A(t,x,y))$ □

Remark — In the case *i*, for instance, the condition means that if Xavier is near the prey, the game will be playable if and only if the relative velocity $\vec{v_x} - \vec{v_y}$ has with the vector $\vec{yx}$ an angle less than or equal to 90^o. □

Here, it's easier to compute directly the cone, without separating rules.

Now, with these formulas, it will be possible to choose open loop and closed loop controls, in practical cases.

3.2 Pursuit game with certain capture:

Let us consider a pursuit between two players, Xavier the pursuer and Yves the quarry. We know that the evader can escape from Xavier if he is far enough : outside of a set which may depend on time (this is realistic, for example Xavier can have less and less energy in a two planes pursuit). We shall study the case with a certain capture. For this, let's introduce a set C of final states and a single valued map $\varphi(t)$ which defines a tube. Players have to move in this tube. Here, for sake of simplicity let's assume that the end time $T = \infty$. This is not very important because we can always modify the function φ such that it is constant (=1) as soon as $t \geq T$.

The viability constraint is then:

$$(x,y) \in K(t) := \{(x,y)/\ x - y \in \varphi(t)C\}$$
$$\text{with } P(t,y) := \varphi(t)C + y \text{ and } Q(x,t) := (-\varphi(t))C + x$$

A reasonable assumption is to have φ larger than or equal to 1 and C^1 differentiable.

Proposition 3.3 *Let us posit the same assumptions as in first section. If C is locally compact, the evader cannot escape if and only if :*

$$\begin{cases} \forall\, t \in \mathbf{R}_+ \ \forall\, (x,y) \in K(t) \ \exists\, (u,v) \text{ such that:} \\ \varphi'(t)(x-y) + \varphi(t)(f(t,x,y,u) - g(t,x,y,v)) \in T_C(\frac{x-y}{\varphi(t)}) \end{cases}$$

Proof — We can calculate in fact the contingent derivative of the tube thanks to (3.2). As GraphP^{-1} = Graph$Q$ the consequence (4) of the transversality is satisfied.

In fact, here: Graph$K = A^{-1}(C)$ with $A(t,x,y) := \frac{x-y}{\varphi(t)}$ a C^1 differentiable function.
Hence :
if $(x,y) \in K(t)$
$(u,v) \in DK(t,x,y)(\tau)$ if and only if
$\frac{1}{\varphi(t)^2}(\varphi'(t)(x-y).\tau + \varphi(t)(u-v)) \in T_C(\frac{x-y}{\varphi(t)})$
because
$T_{A^{-1}(C)}(t,x,y) = (\nabla A(t,x,y))^{-1}.T_C(\frac{x-y}{\varphi(t)})$
In fact, $\nabla A(x,y,t)$ is surjective because $\varphi > 1$ and we can use (3.2). $\square$

We study more concrete cases:
For instance, in $\mathbf{R}^3$, if $C = \{\ x \ /\ \|x\| \leq 1\ \}$
this equation can easily be interpreted:
$(\varphi'(t)(x-y) + \varphi(t)(f(t,x,y,u) - g(t,x,y,v))).(x-y) \leq 0$
It means that there is an angle less than 90^0 between the vector $\vec{yx}$ and $\varphi'(t)(x-y) + \varphi(t)(f(t,x,y,u) - g(t,x,y,v))$
Very often, it is necessary to specify the function φ , for instance a "good one" is :

$\varphi(t) = 1 + ae^{-bt}$

and, of course we should be able to choose a and b allowing the pursuit is possible for every pair of controls (u,v) just solving (if $f := u$

and $g := v$) :

$\forall\, (u,v) - ab\|x-y\|^2 + (1+ae^{-bt})(u-v)(x-y) \leq 0$

This is not very useful because this condition is depending on time, we shall try, now, to have a condition independent on time. A way to do this is to determinate all suitable functions $\varphi(\cdot)$ to describe the tube $K(t)$. Let 's find such functions solving the following system :

(11) $$\varphi'(t) = W(\varphi(t)) \text{ and } (1)$$

In this case, it means that

$K'(t) := \{ (x, y, \varphi) \in X \times Y \times \mathbf{R}^+ \ / \ \|x - y\| \leq \varphi\}$ is a viability tube of this new system. We can write a necessary and sufficient condition on W for this :

Lemma 3.4 *The function W will provide solutions if and only if:*

If $\|x - y\| = \varphi$

$\exists\,(u, v)$ such that $(x - y, u - v) - W(\varphi)\varphi \leq 0$

Proof— Let's define $B(x, y, \varphi) := \|x - y\|^2 - \varphi^2$ and let's notice that $B^{-1}(\mathbf{R}^-) = \mathrm{Graph}K'$ and thanks to (3.2) the lemme is proved. In fact, as soon as $((x - y), (x - y), \varphi) \neq (0, 0, 0)$, $\nabla B(x, y, \varphi)$ is surjective. □

Let us study a case when f and g have explicit forms.

3.3 An affine differential pursuit game:

3.3.1 General case:

We are in the case when two players act on the same state $z(\cdot)$. The first player tries to brake the system and the second player tries to accelerate it by using two controls u and v.

The evolution of the system is given by the following differential equation:

(12) $$\begin{cases} i) & z' = Az(t) - u(t) + v(t) \\ ii) & u(t) \in U(t, z(t)) \quad v(t) \in V(t, z(t)) \end{cases}$$

The goal is to drive the system near a given target C. Consequently, let us consider the following constraints:

(13) $$\forall t \ \ z(t) \in K(t) := \ \{z \ / \ z \in e^{-\lambda t}(r + C)\ \}$$

With $C := \{ z \in \mathbf{R}^n \ / \ Mb = Mz \}$ and $(b, r, \lambda) \in \mathbf{R}^n \times \mathbf{R}_+ \times \mathbf{R}_+$

Let us write the playability condition of this game:

Proposition 3.5 *Let A: $\mathbf{R}^n \mapsto \mathbf{R}^n$, M: $\mathbf{R}^n \mapsto \mathbf{R}^k$ be linear .*
The pursuit is possible if and only if :

$$\text{whenever } z \in e^{-\lambda t}(r + C)$$
$$\begin{cases} \forall t \ \ \exists (u, v) \in U(t, z) \times V(t, z) \text{ such that:} \\ M[\lambda z + A.z - u + v] = 0 \end{cases}$$

Proof — According to (3.2), the necessary and sufficient condition of playability is:
$\forall z \in K(t) \ \exists (u, v) \in U(t, z) \times V(t, z) \ / \ Az - u + v \in DK(t, z)(1)$
But, we know that (thanks to (3.2)):
$T_{\text{Graph}_K}(t, z) = \text{Graph } DK(t, z) = T_{L^{-1}(C)} = \nabla L(t, z)^{-1}.T_C(L(t, z))$
with $L(t, z) := e^{\lambda t}z - r$
And we can notice that:
$\nabla L(t, z) = (\lambda e^{\lambda t}z, \ e^{\lambda t})$
is obviously always surjective. Hence:
$x \in DK(t, z)(\tau) \iff [\lambda e^{\lambda t}.z\tau + e^{\lambda t}.x] \in kerM$ □

We can write the discriminating set valued map:
$A(t, z, u) := \{ v \in V(z, t) \ / \ v \in kerM - (\lambda + A)z \}$
Now, let us apply this proposition to the following example:

3.3.2 An example in a two dimensional space:

$$\begin{cases} x'(t) = y(t) - u(t) \\ y'(t) = v(t) \\ C = \{ (x, y) \in \mathbf{R}^2 \ / \ x = y \} \\ (u, v) \in [-1, 1] \end{cases}$$

As we just saw:

Proposition 3.6 *This game is playable if and only if:*

$$\lambda x + (1 - \lambda)y = u + v$$

Proof — It is just the translation of previous proposition with:

$$A =: \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad M =: (\,1,\ -1\,)$$
$$u =: \begin{pmatrix} u \\ 0 \end{pmatrix}. \quad v =: \begin{pmatrix} 0 \\ v \end{pmatrix}. \ \square$$

3.4 End time of a capture:

3.4.1 A general model:

Let us consider a two player pursuit game in which one player has to catch the other one in a finite time. The evolution of the game is governed by (1).

For this, let us introduce a constraint function $w(\cdot)$ which is the largest possible distance between the two players at time t. Let us consider the end time T as a variable related to t. Hence, we can write a condition for the existence of solutions to this game.
Then solutions have to belong to :
$K := \{\ (x, y, t, T)\ /\ d(x, y)\ \leq\ w(T - t)\ \}$
With the following assumption on w :
$\forall\, s \leq 0 \;\; w(s) \; = \; 0$ and $\forall\, s \;\; w(s) > 0$
It means that the distance between two players is equal to zero after capture.

We need another assumption because the two players coordinates do not have to change after Xavier has caught Yves. Consequently f and g are such that:

$$\begin{cases} \forall\, x \in X & f(.,x,x,.) = 0 \\ \forall\, x \in X & g(.,x,x,.) = 0 \end{cases}$$

It means that as soon as $x = y$ (the capture) the system does not evolve, forever the state will remain constant forever.

The set $\{(x, y)\ /\ x = y\} \times \mathbf{R}^+ \times \mathbf{R}^+$ is a viability tube of the game.

Proposition 3.7 *Under assumptions ($\mathcal{S}$), a necessary and sufficient condition for the game playability is :*
forall (x, y, t, T) such that $d(x, y) = w(T - t)$
there exists (u, v) such that

$$(14) \quad \begin{cases} u(t) \in U(t, x(t), y(t)) \; v(t) \in V(t, x(t), y(t)) \\ (f(t, x, y, T) - g(t, x, y, T))(x - y) + w(T - t)w'(T - t) \leq 0 \end{cases}$$

Proof— We now write the inclusion to which we shall apply Haddad's theorem in the form:

$$\begin{cases} (a) & \begin{cases} i) \; x'(t) = f(s, x(t), y(t), u(t)) \\ ii) \; u(t) \in U(s, x(t), y(t)) \end{cases} \\ (b) & \begin{cases} i) \; y'(t) = g(s, x(t), y(t), v(t)) \\ ii) \; v(t) \in V(s, x(t), y(t)) \end{cases} \\ & \begin{cases} s' = 1 \\ T' = 0 \end{cases} \end{cases}$$

The viability set is:
$K := \{ \, (x, y, s, T) \; / \; d(x, y) \leq w(T - s) \, \}$

It is necessary to calculate the contingent cone at K in (x, y, t, T); it is easy with assumption of C^1 differentiability of w.
For this calculus let introduce the following C^1 differentiable map:
$A(x, y, t, T) := \|x - y\|^2 - w(T - t)^2$
then $K = A^{-1}(\mathbf{R}_-)$

Hence (See 3.2)
$T_K(x, y, t, T) = \nabla(A(x, y, t, T))^{-1}.T_{\mathbf{R}_-}(A(x, y, t, T))$
because $\nabla A(x, y, t, T)$ is surjective as soon as $x \neq y$ or $w'(T - t) \neq 0$

We can calculate the cone:

a- if $d(x, y) > w(t - T)$ then: $T_K(x, y, t, T) = \emptyset$(it is outside K)
because $T_{\mathbf{R}_-}(A(x, y, t, T)) = \emptyset$.

b- if $d(x, y) < w(t - T)$ then: $T_K(x, y, t, T) = X \times X \times \mathbf{R} \times \mathbf{R}$ (it is in the interior of K)
because $T_{\mathbf{R}_-}(A(x, y, t, T)) = \mathbf{R}$

c- if $d(x,y) = w(t-T)$ then:
$T_K(x,y,t,T) = \{(u,v,\sigma,\tau)/(x-y).(u-v) - w(T-t)w'(T-t)(\tau-\sigma) \leq 0\}$
(on the boundary of K)
because $T_{\mathbf{R}_-}(0) = \mathbf{R}_-$ □

3.4.2 A very simple example:

The two players can only choose their velocities u and v the norms of which have to be less than or equal to respectively α and β (nonnegative numbers).

$(u,v) \in B(0,\alpha) \times B(0,\beta)$

The playability condition now becomes:
If $d(x,y) = w(t-T)$
$(u-v).(x-y) + w(T-t)w'(T-t) \leq 0$

It is always possible if :

$-(\alpha+\beta)\|x-y\| + w(T-t)w'(T-t) \leq 0$ i.e $\alpha+\beta \geq w'(T-t)$

This condition means that the two players have to move faster than the "slope " of the tube when they are on its boundary.

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