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FUNNEL EQUATIONS AND MULTIVALUED INTEGRATION PROBLEMS FOR CONTROL SYNTHESIS

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FOREWORD

This paper indicates a sequence of evolutionary *"funnel equations"* with set-valued solutions which are crucial for the construction of respective feed-back control strategies along the schemes introduced by N.N. Krasovskii. The integration of these funnel equations leads to a sequence of multivalued integrals that generalize some of those introduced earlier (Auman's integral, the Convolution integral, Pontryagin's Alternated integral).

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FUNNEL EQUATIONS AND MULTIVALUED INTEGRATION PROBLEMS FOR CONTROL SYNTHESIS

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This paper deals with the problem of synthesizing a feedback control strategy for a linear controlled system subjected to unknown but bounded input disturbances and convex state constraints (see [1-7]). While seeking for the solution in the form of an "extremal strategy" as introduced by N.N. Krasovski, it is shown that the respective sets of solubility states that are crucial for the solution of the control problem could also be treated as cross-sections of trajectory tubes for some specially designed "funnel equations". The set-valued solutions to these could be then presented in the form of specially derived multivalued integrals.

1. The Problem of Synthesizing "Guaranteed" Control Strategies.

Consider a controlled system

$$\dot{x} \in u - Q(t) , \tag{1.1}$$

$$t \in T = [t_0, t_1] , \quad x \in \mathbf{R}^n$$

under restrictions

$$u \in P(t) , \tag{1.2}$$

$$x(t) \in Y(t) , \tag{1.3}$$

$$x(t_1) \in \mathcal{M} . \tag{1.4}$$

Here (1.1) is the equation for the controlled process, u is the *control parameter* restricted by the set-valued function $P(t)$ as in (1.2), (1.3) is the *state constraint*, (1.4) is the *terminal condition*. The functions $P(t)$, $Q(t)$ are set-valued, with values in conv $(\mathbf{R}^n)$ and measurable in t, $Y(t)$ is set-valued with values in cl$(\mathbf{R}^n)$, continuous in t. The notions of continuity and measurability of multivalued maps are taken in the sense of [12]. The set $\mathcal{M} \in$ conv $(\mathbf{R}^n)$.

Here and further we assume the notations:

- conv$(\mathbf{R}^n)$ for the set of convex compact subsets of $\mathbf{R}^n$,
- cl$(\mathbf{R}^n)$ for the set of closed convex subsets of $\mathbf{R}^n$,
- comp$(\mathbf{R}^n)$ for the set of compact subsets of $\mathbf{R}^n$, I - for the unit matrix.

The set-valued function $Q(t)$ describes the range of uncertainty in the process assuming that the system is affected by unknown but bounded input disturbances $v(t)$, so that

$$\dot{x} = u - v(t)$$

with

$$v(t) \in Q(t) \ . \tag{1.5}$$

The aim of this paper is to describe a certain unified scheme for solving the following problem of *"guaranteed" control synthesis*:

Specify a *synthesizing strategy*

$$u = u(t,x)$$

so that for *every* solution to the system

$$\dot{x}[t] \in u(t,x[t]) - Q(t)$$

the inclusions

$$u(t,x[t]) \subseteq P(t) \ ,$$

$$x[t] \in Y(t) \ ,$$

$$x[t_1] \in \mathcal{M} \ ,$$

would be satisfied for any initial condition $x^0 = x[t_0]$, from a given set $X^0 : x^0 \in X^0$.

The class $\mathcal{U}$ of strategies $u(t,x)$ within which the problem is to be solved is taken to consist of set-valued functions $u(t,x)$ with values in conv $(\mathbf{R}^n)$, measurable in t and upper semicontinuous in x. The inclusion $u(t,x) \in \mathcal{U}$ here ensures the existence of a solution to the equation

$$\dot{x} \in u(t,x) - Q(t) \ , \ \ x^0 \in D$$

for any $D \in \text{conv}(\mathbf{R}^n)$. The solution control strategy $u(t,x)$ should thus guarantee the inclusions (1.1), (1.3), (1.4) whatever are the disturbance $v(t)$, and the vector $x^0 \in X^0$.

A crucial point in the solution of the control synthesis problem is to find the set $W[t_0] = \{x_0\}$ of all initial states x^0 that assure the solvability of the problem (so that $W[t_0]$ would be the largest of all sets X^0 that ensure the solution). A similar question may be posed for any instant of time $\tau \in (t_0,t_1)$. This leads to a multivalued function $W[\tau]$, $\tau \in T$.

As demonstrated in [1] the knowledge of function $W[t]$, in particular for the "linear-convex" problems (1.1), (1.2), (1.4), allows to devise a solution $u(t,x)$ in the form of an "extremal strategy" of control. However, the basic scheme also remains true for the problem (1.1)-(1.4) with a state constraint. The main accents of this paper are not on the discussion of the relevance of the extremal strategy (which will nevertheless be specified below) but rather on the unified formal scheme for describing $W(t)$. It will be shown in the sequel that $W[t]$ may be defined as the solution to an evolution "funnel equation" which allows a representation in the form of a special multivalued integral that generalizes some of the conventional multivalued integrals (Auman's integral, the convolution integral, Pontriagin's alternated integral [7]).

The treatment of equation (1.1) rather than

$$\dot{x} \in A(t)x + u - Q(t)$$

causes no loss of generality, provided $P(t)$ does not reduce to a constant transformation.

2. The Basic "Funnel" Equation

In this paragraph we introduce a formal evolution equation whose solutions are set-valued functions which will later be shown to describe the required sets $W[t]$.

Assume the multivalued functions $P(t)$, $Q(t)$, $Y(t)$ are to be given as in § 1. With P', $P'' \in \text{conv}(\mathbf{R}^n)$ we introduce the standard "Hausdorff semidistance" as

$$h_+(P',P'') = \min_{r \geq 0}\{r : P' \subseteq P'' + rS\}$$

where $S = \{x : (x,x) \leq 1,\ x \in \mathbf{R}^n\}$ is a unit ball in $\mathbf{R}^n$ ((x,x) is the inner product in $\mathbf{R}^n$).

Definition 2.1. A multivalued map $Z(t)$ with compact values will be said to be h_+ - ***absolutely continuous*** on an interval $[t_0,t_1]$ if $\forall \varepsilon > 0$, $\exists \delta > 0$:

$$\sum_i (t_i'' - t_i') < \delta \Longrightarrow \sum_i h_+(Z(t_i'),\ Z(t_2'')) < \varepsilon$$

where $\{(t_i',t_i'')\}$ - is a finite or countable number of nonintersecting subintervals of $[t_0,t_1]$.

Consider the *"funnel equation"*

$$\lim_{\sigma\to 0} \sigma^{-1} h_+(Z(t-\sigma) - \sigma Q(t), (Z(t)\cap Y(t)) - \sigma P(t)) = 0 \tag{2.1}$$

with boundary condition

$$z(t_1) \subseteq \mathcal{M} \;, \mathcal{M} \in \mathrm{comp}(\mathbf{R}^n) \;. \tag{2.2}$$

Definition 2.2. An h_+*-solution* to equation (2.1) will be defined as an h_+-absolutely continuous set-valued function $Z(t)$ with values in comp$(\mathbf{R}^n)$ that satisfies (2.1) almost everywhere on $[t_0,t_1]$.

In general the h_+-solution to (2.1), (2.2) is nonunique. The unicity may be achieved by selecting a "maximal" solution to (2.1), (2.2) in the sense of the partial order $\leq$ for the set of all h_+-solutions $\{Z(\cdot)\}$ to (2.1), (2.2), $t\in[\tau,t_1]$ introduced by assuming that $Z_1(\cdot) \leq Z_2(\cdot)$ iff $Z_1(t) \subseteq Z_2(t)$ for all $t\in[\tau,t_1]$.

Lemma 2.1. If $W[\tau)\neq\phi$ for some $\tau\in(t_0,t_1)$, then the variety of all solutions to (2.1), (2.2), $t\in[\tau,t_1]$ is nonvoid and has a unique maximal element with respect to the partial order $\leq$.

3. The Formal Solution

The solution to the problem of synthesizing controls that guarantee the restrictions (1.1), (1.2), (1.4) is given by the following theorem:

Theorem 3.1 (i) *The solution to the problem of control synthesis for the system (1.1)-(1.4) in the class $u\in\mathcal{U}$ from the initial position $x_\tau^0 = x[\tau]$ does exist if and only if $W[\tau]\neq\phi$ and*

$$x_\tau^0 \in W[\tau] \;, \tag{3.1}$$

(ii) *The condition $W[\tau]\neq\phi$ is fulfilled if and only if on the interval $[\tau,t_1]$ there exists an h_+-solution $Z(t)$ to equation (2.1) with boundary condition (2.2); then $W[t]$ is the unique maximal solution to (2.1) with respect to the partial order $\leq$,*

(iii) *the guaranteed synthesizing strategy that resolves (1.1)-(1.4) is given in the form*

$$u(t,x) = \begin{cases} P(t), \text{if } x\in W[t] \\ \partial_\ell\, \rho(-\ell^0|P(t)), \text{ if } x\bar{\in} W[t] \end{cases} \tag{3.2}$$

Here $\rho(\ell|Z) = \max \{(\ell,z)|z\in Z\}$ *is the support function* for set Z, $\partial_\ell f(\ell,t)$ stands for the *subdifferential* of $f(\ell,t)$ in ℓ, $\ell^0 = \ell^0(t,x)$ is a unit vector that solves the problem

$$d(x,Z(t)) = (\ell^0,x) - \rho(\ell^0|Z(t)) = \max_{\|\ell\|=1}\{(\ell,x) - \rho(\ell|Z(t))\} , \tag{3.3}$$

and the symbol $d(x,Z(t))$ stands for the Euclid distance from point x to set $Z(t)$.

Extremal strategies of type (3.2), (3.3) were introduced by N.N. Krasovski (see [1] , [6]).

The proof of Theorem 3.1 is based on the following assertions.

Lemma 3.1. Suppose $Z(t)$ is an h_+-solution to equation (2.1), (2.2) for the interval $[\tau,t_1]$. Then

$$Z(t) \subseteq Y(t), \quad t \in [\tau,t_1] .$$

For any h_+-solution $Z(t)$ to (2.1), (2.2) we may define an extremal strategy $u_z(t,x)$ according to (3.2), (3.3), (substituting W for Z).

Lemma 3.2. The multivalued map $u = u_z(t,x)$ is such that $u \in U$, i.e. $u_z(t,x)$ is measurable in t and upper semicontinuous in x.

This ensures the existence of solutions to equation

$$\dot{x} \in u_z(t,x) - Q(t) \tag{3.4}$$

for any $x^0 \in W[t_0]$.

Lemma 3.3. Assume the inclusion (3.4) is generated by a strategy $u_z(t,x)$ for a given h_+-solution $Z(\cdot)$ of the evolution equation (2.1), (2.2). Then, for almost all values of $t \in T$, in the domain $d(x,Z(t)) > 0$ the following estimate is true along the solutions to (3.4)

$$\frac{d}{dt}(d[t]) = (l^0,u[t]-v[t]) - \frac{\partial}{\partial t}\rho(l^0|Z(t)) \leq (l^0,u[t]-v[t]) +$$

$$+ \rho(-l^0|P(t)) - \rho(-l^0|Q(t))$$

where $u[t] \in P(t)$, $v[t] \in Q(t)$.

Lemma 3.4. For a solution $x^0[t]$ of (3.4) the initial condition $x^0[\tau] \in Z(\tau)$ yields $x^0[t] \in Z(t)$ for any $t \in [\tau,t_1]$.

4. Multivalued Integration

Once the evolution equation (2.1), (2.2) is given it is possible to define the crossection $W[t]$ of the solution tube $W(\cdot)$, $W(t_1)\subset\mathcal{M}$ as a certain *multivalued integral*. As we shall see in the sequel this integral generalizes a whole range of "simpler" multivalued integrals. Let us proceed with constructing the corresponding integral sums. Suppose $\mathbf{M}[t',t'']$ stands for the set of $(n\times n)$-matrix valued functions continuous on $[t',t'']$, and $\mathbf{M}$ for the set of square matrices of dimension n. Introduce a subdivision P_m of the interval $[\tau,t_1]$ as

$$P_m : \tau = \tau_0 < \tau_1 < \cdots < \tau_m = t_1$$

$$\Delta_m = \max\{|\tau_i - \tau_{i-1}|,\ i=0,\ldots,m\}$$

and define integral sums of the following three types:

(1)

$$X_m^{(1)}(P_m,\mathcal{M}) = \mathcal{M}$$

$$X_{i-1}^{(1)} = \cap\{[\int_{\tau_{i-1}}^{\tau_i} (I - \int_{\tau_{i-1}}^{t} M(\xi)d\xi)P(t)dt \tag{4.1}$$

$$+ \int_{\tau_{i-1}}^{\tau_i} M(t)Y(t)dt + (I - \int_{\tau_{i-1}}^{\tau_i} M(\xi)d\xi)X_i^{(1)}]$$

$$\dot{-}\, [-\int_{\tau_{i-1}}^{\tau_i} (I - \int_{\tau_{i-1}}^{t} M(\xi)d\xi)Q(t)dt] \mid M(\cdot)\in\mathbf{M}[\tau_{i-1},\tau_i]\}$$

$$i = m, m-1, \ldots, 1$$

(2)

$$X_m^{(2)}(P_m,\mathcal{M}) = \mathcal{M}$$

$$X_{i-1}^{(2)} = \cap\{-\int_{\tau_{i-1}}^{\tau_i} P(t)dt + MY(\tau_i) + (I-M)X_i^{(2)}] \tag{4.2}$$

$$\dot{-}\, [-\int_{\tau_{i-1}}^{\tau_i} Q(t)dt]\ |M\in\mathbf{M}|\ \}$$

$$i = m, m-1, \ldots, 1$$

(3)

$$X_m^{(3)}(P_m,\mathcal{M}) = \mathcal{M}$$

$$X_{i-1}^{(3)} = [-\int_{\tau_{i-1}}^{\tau_i} P(t)dt + X_i^{(3)} \cap Y(\tau_i)]$$

$$\dot{-} [-\int_{\tau_{i-1}}^{\tau_i} Q(t)dt]$$

Symbol $\dot{-}$ stands for the "geometrical" ("Minkowski") difference, i.e. for sets **A**,**B** given

$$\mathbf{C} = \mathbf{A} \dot{-} \mathbf{B} = \{\mathbf{c}{:}\mathbf{c} + \mathbf{B} \subseteq \mathbf{A}\} .$$

The convergence of the integral sums with $m \to \infty$ to a value that does not depend on the subdivision P_m is ensured by the following assumption:

Assumption 4.1. There exists a function $\beta(t)$, continuous in t, $t \in T$ and such that $\beta(t)>0$ for $t \in [t_0,t_1)$ and that for any subdivision P_m of the interval T the following inclusion is true

$$X_i^{(3)}(P_m,\mathcal{M}) \cap Y(\tau_i) \dot{-} \beta(\tau_i)S \neq \phi$$

$$(i = 0,1,\cdots,m)$$

Theorem 4.1. Under Assumption 4.1 the limit

$$J(\tau,t_1,\mathcal{M}) = \lim_{\substack{m \to \infty \\ \Delta_m \to 0}} X_0^{(i)}(P_m,\mathcal{M})$$

depends neither on the sequence of subdivisions P_m ***nor on the index*** $i = 1,2,3$. ***The following equality is true***

$$W[\tau] = J(\tau,t_1,\mathcal{M}) .$$

The definition of the integral $J(\tau,t_1,\mathcal{M})$ is therefore correct.

We will now follow several particular cases starting from the simplest one.

5. Attainability Domains for Control Systems

Assume $Q(t) = \{0\}$, $Y(t) \equiv \mathbf{R}^n$. Then $W[\tau]$ is the attainability domain for system

$$\dot{x} = u, \quad u \in P(\tau), \quad x(t_1) = \mathcal{M}$$

written in backward time from t_1 to t_0 and evolving from set $\mathcal{M}$. A funnel equation for differential inclusions in the absence of state constraints was studied in [8,9] in terms of the *Hausdorff distance* $h(Z_1,Z_2)$, where

$$Z_1,Z_2 \in \text{comp } \mathbf{R}^n , \quad h(Z_1,Z_2) = \max \{h_+(Z_1,Z_2), h_+(Z_2,Z_1)\}$$

The funnel equation for $W[\tau]$ is as follows

$$\lim_{\sigma \to +0} \sigma^{-1} h(Z(t-\sigma), Z(t) - \sigma P(t)) = 0 \tag{5.1}$$

$$Z(t_1) = \mathcal{M} .$$

Lemma 5.1. Under conditions $Q(t) \equiv \{0\}$, $Y(t) \equiv \mathbf{R}^n$, the set $W[t]$, $W[t_1] = \mathcal{M}$ is the unique solution to equation (5.1) and also the unique maximal solution to equation

$$\lim_{\sigma \to 0} \sigma^{-1} h_+(Z(t-\sigma), Z(t) - \sigma P(t)) = 0$$

$$Z(t_1) = \mathcal{M} .$$

It may be represented as a "multivalued Lebesque integral" ("Aumann's integral")

$$W[t] = J(t,t_1,\mathcal{M}) = \mathcal{M} + \int_{t_1}^{t} P(\xi) d\xi$$

6. "Viability" Tubes.

Consider the particular case $Q(t) \equiv \{0\}$ (a system with state constraints in the absence of uncertainty). Then $W[\tau]$ is the set of states from each of which there exists a "viable" trajectory (relative to constraint (1.3)) that ends in $\mathcal{M}$. In other words, for each $x^0 \in W[\tau]$ there exists a control $u[t]$ restricted by (1.2) that generates a trajectory $x[t] \in Y(t)$ for all $t \in T$ and such that $x[t_1] \in \mathcal{M}$. ($W[\tau]$ is also the attainability domain for system (1.1)-(1.4), $Q(t) \equiv \{0\}$ in backward time.)

The evolution equation (2.1) here forms to be

$$\lim_{\sigma \to +0} \sigma^{-1} h_+(Z(t-\sigma), (Z(t) \cap Y(t)) - \sigma P(t)) = 0 . \tag{6.1}$$

Theorem 6.1. Under condition $Q = \{0\}$ the multivalued function $Z = W[t]$ is the only maximal solution to equation (6.1), (2.2) with respect to the partial order $\leq$. This solution may be presented in the form of a multivalued convolution integral

$$W[\tau] = J(\tau, t_1, \mathcal{M}) \tag{6.2}$$

$$= \cap \{ \int_{\tau}^{t_1} [S(t)P(t) - \dot{S}(t)Y(t)]dt + S(t_1)\mathcal{M} \,|\, M(\cdot) \in \mathbf{M}[\tau, t_1] \}$$

where $S(t)$ and $M(t)$ are connected through the equation

$$\frac{dS(t)}{dt} = - M(t),\ S(\tau) = I,\ M(\cdot) \in \mathbf{M}[\tau, t_1] .$$

The intersection (6.2) is taken over all matrix functions $M(\cdot) \in \mathbf{M}[\tau, t_1]$. The integral (6.2), introduced in [4] is a "multivalued" version of the convolution integral described in [12].

Another version of the funnel equation for $W[t]$, as given in [4], is the following:

$$\lim_{\sigma \to 0} \sigma^{-1} h(Z(t-\sigma), (Z(t) - \sigma P(t)) \cap Y(t)) = 0 . \tag{6.3}$$

Here we use the Hausdorff distance and the equation is true if either the function $Y(t)$ has a convex graph, where

$$\text{graph } Y(t) = \{t, x : x \in Y(t),\ t \in T\}$$

or if the support function $\rho(\ell | Y(t)) = f(t, \ell)$ is continuously differentiable in t, ℓ. The transition to the Hausdorff semidistance in the context of equation (6.1) allows to drop the additional requirements on $Y(t)$ but yields no unicity of solution. The latter is regained, however, if we consider the maximal ($\leq$) solution to (6.1).

7. Solution Sets for Game-theoretic Control Synthesis.

Assume $Y(t) \equiv \mathbf{R}^n$ so that there is no state constraint. Then the initial problem transforms to one of synthesizing a control strategy in a differential game with fixed time with terminal cost being the distance to set $\mathcal{M}$. A guaranteed solution strategy $u(t,x)$ should ensure $d(x[t_1],\mathcal{M}) = 0$ so that $x[t_1] \in \mathcal{M}$.

The funnel equation for the set $W[t]$ that would generate a solution strategy of type (3.2) is now as follows

$$\lim_{\sigma \to +0} \sigma^{-1} h_+(Z(t-\sigma) - \sigma Q(t),\, Z(t) - \sigma P(t)) = 0 . \tag{7.1}$$

The formulae (4.1)-(4.3) for the integral sums (with $Y(t) \equiv \mathbf{R}^n$) now coincide with the "alternated sums" introduced in [7].

Theorem 7.1. With $Y(t) \equiv \mathbf{R}^n$ the multivalued map $W[\tau]$ is the unique maximal ($\leq$) solution to equation (7.1), (2.2). Under Assumption 4.1 the function $W[\tau]$ is continuous and satifies (7.1) for all $t \in T$. It may be presented through the "alternated integral" of L.S. Pontriagin [7], as

$$W[\tau] = J(\tau,t_1,\mathcal{M}) = \int_{\mathcal{M},t_1}^{\tau} (P(t)dt \dot{-} Q(t)dt) . \tag{7.2}$$

Assuming that $P(t)$, $Q(t)$ are of "similar type" (i.e.,$0 \in P(t)$, $0 \in Q(t)$ and $P(t) = \alpha Q(t)$, $\alpha > 0$), the Hausdorff semidistance h_+ in (7.1) may be substituted for the distance h and the integral (7.2) transforms into

$$J(\tau,t_1,\mathcal{M}) = \int_{\mathcal{M},t_1}^{\tau} (P(t) \dot{-} Q(t))dt$$

We then arrive at the "regular case" for the respective differential game [1].

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