Working Paper

Differential Inclusions and Target Problems

Marc Quincampoix

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IIASA

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FOREWORD

The author studies where and how solutions associated to a differential inclusion can or cannot enter a given target. For this purpose, he associates partitions of the target boundary with the dynamics of the system.

He qualitatively describes the behaviour of these solutions in terms of viability and invariance kernels of sets. These kernels determine points such that there exist (respectively all) solutions starting at these point remain in a given set of constraints.

He also studies the sets which are reached in finite time by viable solutions to the system.

Finally, he provides some applications to control systems with one target and he generalizes the concept of semipermeable barrier.

Alexander B. Kurzhanski
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Contents

Differential inclusions and target problems

Marc Quincampoix

1 Introduction

We consider an open set C and a system whose evolution is described by the following differential inclusion[1]:

$$x'(t) \in F(x(t))$$

We assume, throughout this paper, that the set valued map F has nonempty values[2].

A first question is:

On what part of the boundary can the state of the system reach the target C?

For that purpose, we define three areas C^e, C^b, C^i of ∂C such that:

- no solution can enter C crossing C^e
- if a solution enters C, then this solution crosses C^i
- if a solution remains in ∂C, then this solution belongs to C^b.

[1]This allows to incorporate a lack of exact knowledge of dynamics or to represent a control system with state-dependent control map $U(x)$:

$$\begin{cases} x'(t) = f(x(t), u(t)) \\ u(t) \in U(x(t)) \end{cases}$$

through a differential inclusion, by setting: $F(x) := f(x, U(x))$.

[2]If it is not the case, we can study the differential inclusion in the interior of the domain of F.

For answering the above question, we prove a new result about the tangent cone to an intersection. Thanks to this, the three sets C^e, C^b, C^i form a partition of the boundary of C.

Another question is: From what initial conditions can the system reach C?

Let us consider a differential inclusion with constraints:

$$x'(t) \in F(x(t)), \quad x(t) \in K \tag{1}$$

where K is a given closed set (we shall use $K := X \backslash C$).

We need some definitions and properties concerning differential inclusions with constraints (see [5], [6]):

We shall say that a solution $x(\cdot)$ of the differential inclusion (1) is *viable* in K if and only if:

$$\forall\, t \geq 0,\ x(t) \in K$$

The solution $x(\cdot)$ is *locally viable* in K if and only if:

$$\exists\, T > 0, \text{ such that } \forall\, t \leq T,\ x(t) \in K$$

A set K has the *viability property* if and only if for any point x_0 of K, there exists at least one solution to (1) starting at this point which is viable in K.

A set K has the *invariance property* if and only if for any point x_0 of K, all solutions to (1) starting at this point are viable in K.

A closed set K is a viability domain if and only if:

$$\forall\, x \in K,\ F(x) \cap T_K(x) \neq \emptyset$$

The set K is an invariance domain if and only if:

$$\forall\, x \in K,\ F(x) \subset T_K(x)$$

where $T_K(x)$ denotes the contingent cone[3] to K at x.

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[3]Recall that:

$$T_K(x) := \{\, v \in X \mid \liminf_{h \longrightarrow 0+} d(x + hv, K)/h = 0 \,\}.$$

If K is closed, if F is an upper semi-continuous[4] set-valued map with nonempty closed convex compact values and linear growth[5] , then, thanks to Haddad's viability theorem (see [15], [5]), K is a viability domain if and only if the viability property holds for K.

When K is not a viability domain, the question arises to find closed subsets in which it is possible to solve (1) in K. With these assumptions it is possible to define the *viability kernel*:

Definition 1.1 *The viability kernel of a closed set K is the largest closed viability domain contained in K.*

We have some examples of computation of viability kernels in [7] and [14].

In a similar way, thanks to the invariance theorem (see [5]), if K is closed and if F is lipschitzean[6] with compact convex values and linear growth, then K is an invariance domain if and only if the invariance property holds for K.

Under these assumptions it is possible to define (see [5]) the *invariance kernel*:

Definition 1.2 *The invariance kernel of a closed set K is the largest closed invariance domain contained in K.*

In this paper, we prove some properties of the boundaries of viability and invariance kernels. In fact, under adequate assumptions, the boundaries of $\mathrm{Viab}_F(K)$ and $\mathrm{Inv}_F(K)$ are viability domains.

[4]Let us recall that a set valued map F is upper semi continuous at x_0 if and only if:

$$\forall\, \varepsilon > 0,\ \exists\, \alpha > 0,\ F(x_0 + \alpha B) \subset F(x_0) + \varepsilon B$$

[5]We say that a map F has a linear growth if there exists $c > 0$ such that:

$$\forall\, x \in X,\ F(x) \subset c(1 + \|x\|)B$$

[6]Let us recall (see [5]), [6] that a set valued F is lipschitzean if and only if there exists a positive real k such that:

$$\forall\, (x, y) \in X \times X,\ \ F(x) \subset F(y) + k\|x - y\|B.$$

We shall also try to answer another question: is it possible to find a solution to the differential inclusion with constraints which reaches a given point? To investigate this question, we shall introduce and study kernels for $-F$ which yields backward trajectories.

In the last section, results concerning the boundaries of the kernels of a differential inclusion will be used to study the following control system with one target C:

$$x'(t) = f(x(t), u(t)) \quad u(t) \in U(x(t))$$

We shall generalize the concept of *semi permeable barrier* (introduced by Isaacs in [17] for differential games). Recall that a barrier allows to separe the areas from which it is possible to reach C and the areas from which it is not possible. (see also [10], [9], [11]). Recall that a C^1-surface is semi permeable when it is satisfying an equation such that: $\max_u f(x,u).n \leq 0$ or $\min_u f(x,u).n \geq 0$ (where n is the normal vector of the surface). It means that the solutions of (4) are able to cross the surface in "only one direction". In fact, we prove that the solutions of a control system can cross the boundaries of viability and invariance kernels only from the exterior of the kernel to the interior the kernel. *In this sense* the boundaries of invariance and viability kernels are semi permeable.

I am indebted to Halina Frankowska for her help and advice.

2 The target boundary and the dynamics

We study a system whose dynamics are described by the differential inclusion:

$$(2) \qquad x'(t) \in F(x(t))$$

where F is the set valued map, whose values are nonempty convex and compact, from a finite dimensional vector space X into itself. We also consider a set C (the target) which is open nonempty and different from X.

Let us define two closed sets $K := X \backslash C$ and $\widehat{K} := \overline{X \backslash K} = \overline{C}$.

The Haddad's viability theorem [15] provides conditions such that the state never reaches C. Here we study how it is possible to reach C.

We first state our results; their proofs will be given in section **2.4**.

2.1 A geometrical result

We need a result concerning the contingent cone to an intersection of two closed sets.

Definition 2.1 *Let K be a closed set. The Dubovitsky-Milliutin tangent cone is defined by:*

$$D_K(x) := \{ v \in X \mid \exists \alpha > 0 \;\; x +]0, \alpha](v + \alpha B) \subset K \}$$

or equivalently: $D_K(x) = X \backslash T_{\overline{X \backslash K}}(x)$.

Theorem 2.2 *Let K_1 and K_2 be two closed sets of X a normed vector vector space. Then, for any x:*

$$T_{K_1}(x) \cap T_{K_2}(x) \cap D_{K_1 \cup K_2}(x) \subset T_{K_1 \cap K_2}(x)$$

This result allows to characterize the intersection of contingent cones, without assumptions on the regularity of these cones[7].

Corollary 2.3 *Let K_1 and K_2 be two closed sets of x.*
If $x \in K_1 \cap K_2 \cap \mathrm{Int}(K_1 \cup K_2)$, then

$$T_{K_1}(x) \cap T_{K_2}(x) = T_{K_1 \cap K_2}(x).$$

We recall that the same conclusion can be obtained if X is a finite dimensional vector space, when we assume the following transversality condition:

$$C_{K_1}(x) - C_{K_2}(x) = X$$

where $C_K(x)$ denotes the Clarke's cone[8]to K at x.

Remark — These two results allow to express the contingent cone to an intersection in different cases:

[7]It is a pleasure for me to thank Halina Frankowska who suggested to improve corollary 2.3 into theorem 2.2 by using Dubovitsky-Milliutin tangent cones.

[8]Recall the definition of the Clarke's tangent cone (see for instance [6] chapter 4):

$$C_K(x) = \{ v \mid \liminf_{h \longrightarrow 0+,\, y \longrightarrow x,\, y \in K} d(y + hv, K)/h = 0 \}$$

In $\mathbb{R}^2$, we can compute the tangent cone at $(0,0)$ to the intersection of

$$K_1 := \{ (x,y) \mid x \leq 0 \} \text{ and } K_2 := \{ (x,y) \mid x \geq 0 \}$$

thanks to Corollary 2.3, but not from the transversality condition. It is the contrary in the case of

$$K_1 := \{ (x,y) \mid x = y \} \text{ and } K_2 := \{ (x,y) \mid x = -y \} \quad \square$$

We can deduce the very useful corollary:

Corollary 2.4 *Let x belong to X.*

$$T_K(x) \cap T_{\hat{K}}(x) = T_{\partial K}(x)$$

and

$$D_K(x) = T_K(x) \backslash T_{\partial K}(x).$$

This Corollary was be used in the study of the qualitative behaviour of replicator systems in the simplex (see [12]), or to study the fluctuations of solutions around the boundary of a given set (see [18]). Corollary 2.3 can be generalized to compute the contingent cone to an intersection of a finite number of closed sets:

Corollary 2.5 *Let $K_1, K_2, \ldots K_p$ be p closed subsets of a metric vector space X and let x belong to $\bigcap_{i=1}^{i=p} K_i$. If there exists an open set $\mathcal{O}$ which contains x such that:*

$$\forall j \leq p,\ \mathcal{O} \subset K_j \cup \left(\bigcap_{i=1}^{i=j-1} K_i \right)$$

then

$$T_{\bigcap_{i=1}^{p} K_i}(x) = \bigcap_{i=1}^{p} T_{K_i}(x)$$

2.2 First partition of the target boundary

Let us introduce three subsets of the boundary which are depending on the dynamic of the system.

$$\begin{cases} K^i := \{ x \in \partial K \ / \ F(x) \subset D_K(x) \} \\ K^e := \{ x \in \partial K \ / \ F(x) \subset X \backslash T_K(x) \} \\ K^b := \{ x \in \partial K \ / \ F(x) \cap T_{\partial K}(x) \neq \emptyset \} \end{cases}$$

Proposition 2.6 *If $F : X \mapsto X$ is an upper semi continuous set valued map with nonempty convex compact values, if K is closed nonempty then, $(K^e,\ K^i,\ K^b)$ form a partition of the boundary ∂K (in the sense that $\partial K = K^i \cup K^b \cup K^e$ and these three sets are disjoint).*

If x_0 belongs to K^i, then all solutions starting at x_0 enter $\mathrm{Int}(K)$ *and stay in the interior on time interval* $]0,T[$ *(with $T > 0$).*

If x_0 belongs to K^e, then all solutions starting at x_0 enter $\mathrm{Int}(X\backslash K)$ *and stay outside K on* $]0,T[$ *(with $T > 0$).*

If x_0 belongs[9] to $\mathrm{Int}_{\partial K}K^b$*, then there exists a solution starting at x_0 which stays on the boundary ∂K on* $]0,T[$ *(with $T > 0$).*

Remark — We can notice that, when ∂K is a C^1 surface, the subset $K^b \cup K^i$ is often called the *boundary usable part* and K^e the *boundary non usable part.* □

For a set A, we set $\widehat{A} := \overline{X\backslash A}$ and we denote by $\mathrm{Int}(A)$ its interior. We can introduce the same type of partition for the closed set $\overline{C} = \widehat{K}$, i.e., the three sets $\overline{C}^i = \widehat{K}^i$, $\overline{C}^b = \widehat{K}^b$, $\overline{C}^e = \widehat{K}^e$ which form a partition of $\partial\widehat{K}$. A natural question arises: how can we compare $\widehat{K}^i$, $\widehat{K}^b$, $\widehat{K}^e$ and $(K^e,\ K^i,\ K^b)$?

Proposition 2.7 *Let K be a closed set and F a set valued map with nonempty convex values.*

$$K^i = \widehat{K}^e,\quad K^e \subset \widehat{K}^i,\quad K^b \subset \widehat{K}^b$$

Equalities hold true if and only if $\overline{\mathrm{Int}(K)} = K$.

The first statement and the inclusions are obvious. If the equalities $K^e = \widehat{K}^i$, $K^b = \widehat{K}^b$ hold, then necessarily $\partial K = \partial\widehat{K}$ i.e. $\overline{\mathrm{Int}(K)} = K$. It is easy to show that this condition is sufficient (in this case $\widehat{\widehat{K}} = K$).

We can improve this partition to have a more precise one.

2.3 Second partition of the target boundary

Let us consider the following differential inclusion:

$$y'(t) \in -F(y(t)) \tag{3}$$

[9]Here, we denote by $\mathrm{Int}_{\partial K}K^b$ the interior of K^b in the space ∂K.

We can regard the solutions of (3) as solutions of (1) but in the reverse direction (i.e. if $x(\cdot)$ is a solution of (1) on $[0,T]$, then $y(t) := x(T-t)$ is solution of (3) on $[0,T]$). In this way, we get the backward trajectories of (1).

We introduce the subsets:

$$\begin{cases} K^{i-} := \{\, x \in \partial K \;/\; -F(x) \subset D_K(x) \,\} \\ K^{e-} := \{\, x \in \partial K \;/\; -F(x) \subset X \backslash T_K(x) \,\} \\ K^{b-} := \{\, x \in \partial K \;/\; -F(x) \cap T_{\partial K}(x) \neq \emptyset \,\} \end{cases}$$

These three sets also form a partition of ∂K. Consequently, these two partitions yield a new partition of the boundary made of nine subsets.

We can describe the qualitative behaviour of solution, as in the previous section, in the following proposition in which we shall denote by $\mathrm{Int}(K^b)$ the interior of K^b in the space ∂K.

Proposition 2.8 *Let K be a closed nonempty set.*

x_0 *is an element of*	*Properties of solutions which start at* x_0
$K^i \cap K^{i-}$	*all solutions enter* K *at* x_0 *no solution come from the exterior of* K
$K^i \cap K^{e-}$	*all solutions enter* K *at* x_0 *no solution come from the interior of* K
$K^i \cap \mathrm{Int}(K^{b-})$	*all solutions enter* K *at* x_0 *there exists at least one trajectory locally viable on the boundary which come into* $\mathrm{Int}(K)$ *at* x_0
$K^e \cap K^{e-}$	*all solutions go outside* K *no solution come from the interior of* K
$K^e \cap K^{i-}$	*all solutions go outside* K *no solution come from the exterior of* K
$K^e \cap \mathrm{Int}(K^{b-})$	*all solutions go outside* K *there exists at least one solution locally viable on the boundary which comes into* $\mathrm{Int}(K)$ *at* x_0
$\mathrm{Int}(K^b) \cap \mathrm{Int}(K^{b-})$	*there exists a solution passing through* x_0 *(i.e.* $\exists\, \tau > 0,\ x(\tau) = x_0$ *)* *and locally viable (for* F and $-F$*) on the boundary*
$\mathrm{Int}(K^b) \cap K^{i-}$	*no solution come from the exterior of* K *there exists a solution locally viable on* ∂K *which comes from the interior of* K
$\mathrm{Int}(K^b) \cap K^{e-}$	*no solution come from the interior of* K *there exists a solution locally viable on* ∂K *which comes from the exterior of* K

2.4 Proofs

Proof of theorem 2.2 — Let v be in $T_{K_1}(x) \cap T_{K_2}(x) \cap D_{K_1 \cup K_2}(x)$. According to the definitions of these sets, there exist sequences h_n^1, h_n^2 of nonnegative reals converging to 0, sequences v_n^1, v_n^2 converging to v and a real α such that:

$$\begin{cases} \forall\, n \quad x + h_n^1 v_n^1 \in K_1, \ \ x + h_n^2 v_n^2 \in K_2 \\ \qquad\ x +]0, \alpha](v + \alpha B) \subset K_1 \cup K_2 \end{cases}$$

Clearly there exists N such that:

$$\forall\, n > N, \ \ x + h_n^i v_n^i \in x + [0, \alpha[(v + \alpha B) \ \text{ for } i = 1, 2$$

Since for all $n > N$, the two points $x + h_n^1 v_n^1$ and $x + h_n^2 v_n^2$ belong to the convex set $x + [0, \alpha](x + \alpha B)$,

$$[x + h_n^1 v_n^1,\ x + h_n^2 v_n^2] \subset x + [0, \alpha](x + \alpha B) \subset K_1 \cup K_2$$

On the other hand:

$$([x + h_n^1 v_n^1,\ x + h_n^2 v_n^2] \cap K_1) \cup ([x + h_n^1 v_n^1,\ x + h_n^2 v_n^2] \cap K_2) = [x + h_n^1 v_n^1,\ x + h_n^2 v_n^2]$$

We cannot form a partition of a connected set made of two nonempty closed sets. Hence the intersection of the two closed sets in the left-hand side of the above equality is nonempty. Consequently, there exists λ_n in $[0, 1]$ such that:

$$\lambda_n(x + h_n^1 v_n^1) + (1 - \lambda_n)(x + h_n^2 v_n^2) \in\ K_1 \cap K_2$$

(if one of these sets is empty, we obtain the same conclusion by setting $\lambda_n := 0$ or 1). By setting:

$$\begin{cases} h_n := \lambda_n h_n^1 + (1 - \lambda_n) h_n^2 \\ v_n := (\lambda_n h_n^1 v_n^1 + (1 - \lambda_n) h_n^2 v_n^2)/h_n \end{cases}$$

We see that $v_n \longrightarrow v$, $h_n \longrightarrow 0$ and $x_n + h_n v_n \in\ K_1 \cap K_2$. The proof is completed. □

Corollaries 2.3 and 2.4 are obvious consequences of theorem 2.2 if we notice that:

- $D_{K_1 \cup K_2}(x) = X$ by assumption of Corollary 2.3
- $T_{K_1 \cap K_2}(x) \subset T_{K_1}(x) \cap T_{K_2}(x)$
- $D_{K \cup \hat{K}}(x) = X$ trivially.

Proof of corollary 2.5 — Thanks to corollary 2.3, we have:

$$T_{K_p}(x) \cap T_{\bigcap_{i=1}^{i=p-1} K_i}(x) = T_{\bigcap_{i=1}^{p} K_i}(x)$$

But we have (with the assumption in the case $j = p - 1$):

$$\mathcal{O} \subset (K_{p-1}) \cup (\bigcap_{i=1}^{i=p-2} K_i)$$

Hence according to corollary 2.3:

$$T_{K_{p-1}}(x) \cap T_{\bigcap_{i=1}^{i=p-2} K_i}(x) = T_{\bigcap_{i=1}^{p-1} K_i}(x)$$

An obvious induction argument allows to complete the proof. □

Proof of proposition 2.6 — Thanks to corollary 2.4, we can divide the space X into three sets $D_K(x)$, $T_{\partial K}(x)$, $X \backslash T_K(x)$. That provides the partition of the boundary. In fact, we observe that $K^i \cap K^e = \emptyset$ and if $F(x) \cap T_K(x) \neq \emptyset$ and $F(x) \cap T_{\hat{K}}(x) \neq \emptyset$, then using that values of F are connected, we deduce that $x \in K^b$. Now, is easy to characterize each area (see [5]). □

3 Boundaries of invariant and viability kernels

In this section, X denotes a finite dimensional vector space. Our purpose is to describe the boundary of the set of initial conditions of (1) from which it is possible to reach the target C. This set is the complement of the viability kernel of $K = X \backslash C$ associated with (1). We shall, now, characterize the boundary of these two kernels:

Theorem 3.1 *Let $F : X \mapsto X$ be a lipchitzean set valued map with nonempty convex compact values and with linear growth, and K be a closed nonempty set.*

If x_0 belongs to $\partial \mathrm{Viab}_F(K) \backslash \partial K$, then there exists a solution viable in K, starting at x_0 which stays in the boundary of $\mathrm{Viab}_F(K)$ as long as it does not cross ∂K. Furthermore, every viable *solution starting at x_0 has the same behaviour.*

Proof — We prove that there exists a viable solution starting at x_0 which stays on the boundary of the viability kernel until it reaches ∂K.

In fact, let be $x(\cdot)$ a viable solution starting at x_0 which enters the interior of $\mathrm{Viab}_F(K)$ (i.e. $\exists\, T > 0$ such that $x(T) \in \mathrm{Int}(\mathrm{Viab}_F(K))$ and $x([0,T]) \subset \mathrm{Int}(K) \cap \mathrm{Viab}_F(K)$. According to Filippov's Theorem[10], there exists $l > 0$

[10] If F is lipchitzean with nonempty values, $y(\cdot) \in S_T(y_0)$ (set of solutions of (1) starting at y_0). There exists $l > 0$ such that:

$$d_{W^{1,1}(0,T)}(y, S_T(x_0)) \leq l \|x_0 - y_0\|$$

such that, for all y in K, there exists a solution $y(\cdot)$ starting at y_0 such that:

$$\forall\, t \leq T,\ y(t) \in\ x(t) + l\|y_0 - x_0\|B$$

Hence, it is possible to find $\alpha > 0$ such that, for all y_0 in $(x_0+\alpha B)\backslash\mathrm{Viab}_F(K)$, we have $y([0,T]) \subset K$ and $y(T) \in \mathrm{Viab}_F(K)$. But there exists a viable solution $\tilde{y}(\cdot)$ starting at $y(T)$ ($\in \mathrm{Viab}_F(K)$).

Let us define a new trajectory $\bar{y}(\cdot)$:

$$\bar{y}(s) := \begin{cases} y(s) & \text{if } \ s \leq T \\ \tilde{y}(s) & \text{if } \ s \geq T \end{cases}$$

Then $\bar{y}(\cdot)$ is a solution to (1) viable in K. We have shown that $\mathrm{Viab}_F(K) \cup \{\ \bar{y}(t),\ t \geq 0\ \}$ (which contains strictly $\mathrm{Viab}_F(K)$) is a viability domain; this is a contradiction. □

Corollary 3.2 *If assumptions of theorem 3.1 hold true and if* $\mathrm{Viab}_F(K) \subset \mathrm{Int}(K)$*, then* $\partial\mathrm{Viab}_F(K)$ *is a viability domain and the set* $\overline{X\backslash\mathrm{Viab}_F(K)}$ *is an invariance domain*

Proof — If we notice that $\partial\mathrm{Viab}_F(K) \cap \partial K = \emptyset$ then, thanks to the theorem 3.1, $\partial\mathrm{Viab}_F(K)$ is a viability domain. Let consider a solution $x(\cdot)$ starting at $x_0 \in \overline{X\backslash\mathrm{Viab}_F(K)}$, if a solution reaches $\partial\mathrm{Viab}_F(K)$, thanks to theorem 3.1, it can not enter in the interior of the viability kernel. □

Remark and example — If $\mathrm{Viab}_F(K) \subset \mathrm{Int}(K)$, the sets $\partial\mathrm{Viab}_F(K)$ and $\partial\mathrm{Inv}_F(K)$ are viability domains; but generally, they are not invariance domains. We can notice that in the following simple examples in the two dimensional space $\mathbf{R}^2$.

We consider a constant set valued map and two closed sets:

$$\begin{cases} F := \{\ 1\ \} \times [0,1] \\ K_1 = \{\ (x,y) \in\ \mathbf{R}^2 \mid x > 0,\ y \leq 1/x\ \} \cup \mathbf{R}^- \times \mathbf{R} \\ K_2 := \{\ (x,y) \in\ \mathbf{R}^2 \mid x > 0,\ y \geq -1/x\ \} \cup \mathbf{R}^- \times \mathbf{R} \end{cases}$$

Then, it is easy to check that:

(see [6] chapter 10).

$$\begin{cases} \mathrm{Viab}_F(K_1) = \boldsymbol{R} \times \boldsymbol{R}^- \ \ \mathrm{Inv}_F(K_1) = \emptyset \\ \mathrm{Inv}_F(K_2) = \boldsymbol{R} \times \boldsymbol{R}^+ \\ \mathrm{Viab}_F(K_2) = \\ \{ (x,y) \in \boldsymbol{R}^2 \mid x \geq 1, \ y \geq -1/x \} \cup \{ (x,y) \in \boldsymbol{R}^2 \mid x \leq 1, \ y \geq x-2 \} \end{cases}$$

Here, the boundaries of $\mathrm{Viab}_F(K_1)$ and $\mathrm{Inv}_F(K_2)$ are viability domains but they are not invariance domains. We can see that all solutions starting at a point of $\{ (x,y) \mid x \leq 1, \ y = x-2 \}$ stay in this set until they reach the boundary of $\mathrm{Viab}_F(K_2)$ (at $(+1,-1)$). □

We are proving now a dual result:

Proposition 3.3 *Let F be a lipchitzean set valued map with nonempty convex compact values, and K a closed compact nonempty set.*

If $\mathrm{Inv}_F(K) \subset \mathrm{Int}(K)$, *then the boundary $\partial \mathrm{Inv}_F(K)$ is a viability domain.*

Proof of proposition 3.3 — We shall show a more precise result: $\overline{X \backslash \mathrm{Inv}_F(K)}$ is a viability domain (so that, if $x_0 \in \partial \mathrm{Inv}_F(K)$ there will exist a solution viable in $\overline{X \backslash \mathrm{Inv}_F(K)}$ which necessarily is also viable $\mathrm{Inv}_F(K)$, hence it is viable in the boundary). For doing this, it is sufficient to show that there exists a solution starting at any point of $\overline{X \backslash \mathrm{Inv}_F(K)}$ which never crosses $\mathrm{Inv}_F(K)$ so that: $X \backslash \mathrm{Inv}_F(K) \subset \mathrm{Viab}_F(\overline{X \backslash \mathrm{Inv}_F(K)})$.

Since K is a compact set, there exists a nonnegative number α such that: $\mathrm{Inv}_F(K) + 2\alpha B \subset \mathrm{Int}(K)$. We shall need the following

Lemma 3.4 *Under the assumptions of proposition 3.3, let $x(\cdot)$ be a solution to (1) and $\alpha > 0$. Then there exists $\tau > 0$ such that for all $t \geq t' \geq 0$:*

$$x[t',t] \subset K, \ \|x(t) - x(t')\| \geq \alpha \quad \Longrightarrow \quad t - t' \geq \tau$$

Proof of the Lemma 3.4 — Let us define $M := \sup_{x \in K} \|F(x)\|$ and $\tau := \alpha / M$. As K is compact and F lipschitzean with compact values, M is different from infinity[11]. Since $x(\cdot)$ is absolutely continuous, we have:

$\alpha \leq \|x(t) - x(t')\| = \| \int_{t'}^{t} x'(s) ds \| \leq M|t - t'|$

Hence $|t - t'| \geq \tau$ The proof of the Lemma is completed. □

Let $x \in K \backslash \mathrm{Inv}_F(K)$; let us build a solution starting at x viable in $\overline{K \backslash \mathrm{Inv}_F(K)}$. We know that there exists at least one solution $x(\cdot)$ which goes

[11] **this is even true if we assume that F is upper semi continuous with compact values.**

outside K (i.e. $\exists\ \tau_1 > 0,\ x(\tau_1) \in\ X\backslash K$). If this solution stays outside $\mathrm{Inv}_F(K)$ then the proof is achieved. Otherwise, there exists a time $T_1 > \tau_1$ such that $x(T_1) \in\ (\mathrm{Inv}_F(K) + \alpha B)\backslash \mathrm{Inv}_F(K)$. According to the lemma 3.4, because $\|x(T_1) - x(\tau_1)\| \geq \alpha$, we have $|T_1 - \tau_1| \geq \tau$. But starting at $x(T_1)$ there exists a solution $\tilde{x}(\cdot)$ which goes outside K ($\exists\ \tau_2,\ \tilde{x}(\tau_2) \notin K$). In a similar way than in the proof of theorem 3.1, we obtain a solution to (1) starting at x_0 (again denoted $x(\cdot)$) such that $\tilde{x}(\tau_2) = x(T_1 + \tau_2)$. If this solution stays outside $\mathrm{Inv}_F(K)$, then the proof is achieved, otherwise:

$$\exists\, T_2 > T_1,\ x(T_2) \in\ (\mathrm{Inv}_F(K) + \alpha B)\backslash \mathrm{Inv}_F(K) \ \text{ with }\ T_2 - T_1 \geq \alpha.$$

If there is a finite number of T_i the proof is clearly achieved. If there is an infinite number of T_i this sequence converges to ∞ because $T_{n+1} - T_n \geq \alpha$. We have obtained a solution of (1) viable in $\overline{X\backslash \mathrm{Inv}_F(K)}$ □

When K is not compact but only closed, it is possible to prove a similar result:

Proposition 3.5 *Let K be a closed set and F a set-valued map satisfying the assumptions of proposition 3.3.*

If $\mathrm{Viab}_F(K) \subset \mathrm{Int}(K)$, then $\overline{X\backslash \mathrm{Inv}_F(K)}$ and $\partial \mathrm{Inv}_F(K)$ are viability domains.

We then deduce a result which follows from theorem 3.1 and proposition 3.3:

Corollary 3.6 *Let F be a lipchitzean set valued map with nonempty convex compact values, and K a closed compact nonempty set. If $x_0 \in\ \partial \mathrm{Inv}_F(K) \cap \partial \mathrm{Viab}_F(K)$, then all solutions starting at x_0 stay on the boundary of $\mathrm{Viab}_F(K)$, as long as they do not cross ∂K.*

Furthermore if $\partial \mathrm{Inv}_F(K) \cap \partial \mathrm{Viab}_F(K)\ \subset\ \mathrm{Int}(K)$, it is an invariance domain.

4 Backward trajectories for a differential inclusion

In previous sections, we were interested in studying solutions starting at a given point; now, we shall study solutions reaching a given point.

We compare in this section kernels associated to (1) and kernels associated to (3) .

Roughly, the concatenation of solutions of (1) and (3) gives us a solution of the differential inclusion on $]-\infty, +\infty[$.

Let $\mathrm{Viab}_{-F}(K)$ (respectively $\mathrm{Inv}_{-F}(K)$) denote the viability kernel of (3) (respectively the invariance kernel) of K for the set valued map $-F$. Of course all results concerning boundaries of these sets are still available.

Proposition 4.1 *Let F be a lipschitzean set valued map with nonempty convex compact values and linear growth, and K a closed set.*

- *The set* $\mathrm{Inv}_F(K) \cap \mathrm{Viab}_{-F}(K)$ *is an invariance domain for F.*
- *The set* $\mathrm{Inv}_{-F}(K) \cap \mathrm{Viab}_F(K)$ *is an invariance domain for $-F$.*
- *The set* $\mathrm{Viab}_F(K) \cap \mathrm{Viab}_{-F}(K)$ *is a viability domain for F and $-F$.*
- *The set* $\mathrm{Inv}_F(K) \cap \mathrm{Inv}_{-F}(K)$ *is a viability domain for F and $-F$.*

Proof — To prove this, we use a technique similar[12] to the proof of theorem 3.1. Let us prove, for instance, the first result.

Let x_0 belong to $\mathrm{Inv}_F(K) \cap \mathrm{Viab}_{-F}(K)$ and $x(\cdot)$ be a solution to (1). Fix $T > 0$. By setting $y(t) := x(T-t)$ ($t \in [0,T]$), we obtain a solution to (3) such that $y(T) = x_0 \in \mathrm{Viab}_{-F}(K)$. Hence, there exists $\tilde{y}(\cdot)$ a solution to (3) starting at x_0 and viable in K with respect to $-F$. The concatenation of $y([0,T])$ and $\tilde{y}([T,\infty[)$ provides a solution starting at $x(T)$ viable in K (with respect to $-F$). Hence $x(T) \in \mathrm{Viab}_{-F}(K)$; since T is arbitrary the proof follows. □

Proposition 4.2 *If assumptions of proposition 4.1 hold true, then:*

$$\begin{array}{lll} i) & \mathrm{Viab}_F(K) \subset \mathrm{Int}(K) \implies & \mathrm{Viab}_F(K) \subset \mathrm{Inv}_{-F}(K) \\ ii) & \mathrm{Viab}_{-F}(K) \subset \mathrm{Int}(K) \implies & \mathrm{Viab}_{-F}(K) \subset \mathrm{Inv}_F(K) \end{array}$$

[12] Let us recall:

$$\mathrm{Inv}_F(K) \subset \mathrm{Viab}_F(K) \text{ and } \mathrm{Inv}_{-F}(K) \subset \mathrm{Viab}_{-F}(K)$$

It follows by exactly the same arguments that in the previous proof.

Remark — We can now "mix" all these subsets and kernels to prove results of the type:

$$\begin{cases} \mathrm{Viab}_F(K) \cap \mathrm{Viab}_F(\widehat{K}) \subset K^\flat \\ \mathrm{Viab}_F(K) \cap \mathrm{Inv}_{-F}(\widehat{K}) \subset K^\flat \cap \widehat{K}^{i-} \end{cases}$$

and so on ... □

5 An application to control systems with one target: semi permeable barriers.

A question naturally arises: Why is it useful to study the boundary of viability or invariance kernels? We give an example of controlled system with one target.

5.1 Certain and possible victory and defeat domains

We can modelize the controlled system[13]

$$(4) \qquad \begin{cases} x'(t) = f(x(t), u(t)) \\ u(t) \in U(x(t)) \end{cases}$$

through the differential inclusion (1) by setting $F(x) := f(x, U(x))$. We shall assume that F is lipschitzean with convex compact nonempty values[14].

Our problem is to drive in finite time the state x inside a given open set C starting at a point outside of C. This has a precise mathematical sense by using the viability and invariance kernels of $K := \overline{X \backslash C}$. Let us introduce some definition of victory and defeat domains (see[3]).

Definition 5.1 *We define*

- *the domain of* certain defeat *by the set* $\mathrm{Inv}_F(X \backslash C)$

[13] Results of this paper can be easily extended to the non autonomous case i.e. $x'(t) = f(t, x(t), u(t)) \quad u(t) \in U(t, x(t))$ (see [24])

[14] In particular, it is satisfied if f is lipschitzean affine with respect to the control and U is a lipschitzean set valued map with nonempty convex compact values.

- *the domain of* possible defeat *by the set* $\mathrm{Viab}_F(X\backslash C)$
- *the domain of* certain victory *by the set* $\overline{K\backslash\mathrm{Viab}_F(X\backslash C)}$
- *the domain of* possible victory *by the set* $\overline{K\backslash\mathrm{Inv}_F(X\backslash C)}$.

Let us make more precise the qualitative behaviour of solutions in these domains.

Proposition 5.2 *If* $x_0 \in \mathrm{Inv}_F(X\backslash C)$ *then, no solution to (4) starting at* x_0*, can reach* C *(*certain defeat*).*

If $x_0 \in \mathrm{Viab}_F(X\backslash C)$*, there exist solutions of (4) starting at* x_0*, which never reach* C *(*possible defeat*)*

If $x_0 \in X\backslash\mathrm{Viab}_F(X\backslash C)$ *then, all solutions of (4), starting at* x_0*, reach* C *in finite time (*certain victory*).*

If $x_0 \in K\backslash\mathrm{Inv}_F(X\backslash C)$*, there exist solutions to (4) starting at* x_0*, which reach* C *in finite time (*possible victory*).*

Proof — It is the obvious consequence of definitions 1.2 and 1.1. □

5.2 Semi permeable barrier

We shall define some subsets of the boundaries of these victory and defeat domains.

Definition 5.3 *The* barrier *is the set:*

$$\partial\mathrm{Inv}_F(X\backslash C)\backslash\partial C$$

The strict barrier *is the set:*

$$\partial\mathrm{Viab}_F(X\backslash C)\backslash\partial C$$

We can notice that the barrier is contained in the intersection of the certain defeat domain an the possible victory domain. The strict barrier is contained in the intersection of the possible defeat domain and the certain victory domain. We can translate the results of section 3 and so we have a qualitative description of the behaviour of solution on the barriers.

Proposition 5.4 *The strict barrier is a* local viability domain[15]. *Furthermore, all solution starting at any state x_0 of the strict barrier which are viable in $X\backslash C$, remain in this set until it reaches $\overline{C}$ (and there exists such solution).*

The barrier is a viability domain[16] *as soon as:*

$$\overline{C} \cap \mathrm{Viab}_F(X\backslash C) = \emptyset.$$

Proof — The first result is a consequence of theorem 3.1, and the second one is a consequence of proposition 3.5. □

This generalizes the concept of semi-permeable barriers (see [9], [10]). Recall that a C^1-surface is semi permeable when it is satisfying an equation such that: $\max_u f(x,u).n \leq 0$ or $\min_u f(x,u).n \geq 0$ (where n is the normal vector of the surface). It means that the solutions of (4) are able to cross the surface in *only one direction*. Let us make this idea more precise by using the partition of section 2.

Definition 5.5 *Let A be a closed set. A subset B of ∂A, is* semi permeable *for A if and only, for any point x_0 of B, any solution $x(\cdot)$ starting at x_0 is locally viable in A.*

Remark — Let us notice that an obvious consequence of this definition is $B \cap A^c = \emptyset$. □

Thanks to proposition 5.4, we can state the following

Proposition 5.6 *The strict barrier is semi permeable for* $\mathrm{Viab}_F(X\backslash C)$.

In fact, thanks to proposition 5.4, we know that a solution of (4) cannot cross the strict barrier if it comes from the exterior of the viability kernel $\mathrm{Viab}_F(X\backslash C)$, but the converse is possible (i.e it could exist solutions coming from the interior of the kernel which cross the strict barrier).
We can notice that, thanks to corollary 3.6, the intersection of the strict barrier and the barrier is *a local invariance domain*[17] (if it is nonempty).

[15] A set K is a local viability domain if and only if, starting at any point of K, there exists at least one solution locally viable in K.

[16] When we assume that $X\backslash C$ is compact, according to proposition 3.3, we have the same conclusion if:

$$\mathrm{Inv}_F(X\backslash C) \cap \overline{C} = \emptyset.$$

[17] A set K is a local invariance domain if and only if all solutions starting at any point of K are locally viable in K.

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