Working Paper

Weak Asymptotic Stability of Trajectories of Controlled Systems

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IIASA

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Foreword

The author presents a theory of weak asymptotic stability for controlled systems which is further specified for periodic systems. The approach is based on set-valued calculus and the theory of Lyapunov exponents.

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Weak Asymptotic Stability of Trajectories of Controlled Systems

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Introduction

Let us consider the controlled system

$$\dot{x}(t) = f(x(t), u(t)), \; u(t) \in U \subset R^n, \tag{1}$$

where $f : R^n \times U \to R^n$ is a continuous function differentiable with respect to $x \in R^n$. Consider a trajectory $\hat{x}(t)$, $t \in [0, \infty[$ of controlled system (1) corresponding to a control $\hat{u}(t)$, $t \in [0, \infty[$. The trajectory $\hat{x}(\cdot)$ is said to be weakly asymptotically stable if for all $\epsilon > 0$ there exists $\delta > 0$ such that for any x, $|x - \hat{x}(0)| < \delta$ one can find a trajectory $x(\cdot)$ of controlled system (1) satisfying

$$x(0) = x, \;\; |x(t) - \hat{x}(t)| < \epsilon, \; t \in [0, \infty[, \;\; \lim_{t \to \infty} |x(t) - \hat{x}(t)| = 0.$$

This definition is a natural generalization of the notion of asymptotic stability introduced by Lyapunov [1] for solutions to an ordinary differential equation. The investigation of weak asymptotic stability is of great interest in the regulator design theory [2, 3]. The aim of this paper is to derive sufficient conditions for weak asymptotic stability of the trajectory $\hat{x}(\cdot)$. We consider two cases. In the first case $\hat{x}(\cdot)$ is a constant trajectory, i.e. an equilibrium point, and in the second case $\hat{x}(\cdot)$ is a periodic trajectory. To derive sufficient conditions of weak asymptotic stability we use the following approach. First of all we investigate the "first approximation" of the system (1), i.e. the linear controlled system

$$\dot{x}(t) = C(t)x(t) + w(t), \; w(t) \in K(t), \tag{2}$$

where $C(t) = \nabla_x f(\hat{x}(t), \hat{u}(t))$, $K = \text{cone co}[f(\hat{x}, U) - f(\hat{x}(t), \hat{u}(t))]$. For controlled system (2) we obtain necessary and sufficient conditions of the weak asymptotic stability of the zero solution. Then the weak asymptotic stability of the trajectory $\hat{x}(\cdot)$ is derived from the weak asymptotic stability of zero solution to (2).

Weak asymptotic stability of an equilibrium point of a differential inclusion with convex valued right-hand side has been investigated [3]. The controlled system (1) is a differential inclusion with a parametrized right-hand side. In this case we do not need a convexity assumption. Moreover we can consider the case of periodic trajectory.

The outline of the paper is as follows. We devote the first section to some background results from stability theory. We also prove some auxiliary propositions in this section. Section 2 provides an investigation of relationship between controlled systems (1) and (2). Weak asymptotic stability of an equilibrium point of (1) is studied in section 3. We consider the case of periodic trajectory in section 4.

1 Background notes

We shall use the following notations. If $x, y \in R^n$, $A \subset R^n$, then $|x|$ is a norm of the vector x, $\langle x, y\rangle$ is the inner product of the vectors x, y, cl$A$ is closure of A, co$A$ is the convex hull of A, int$A$ is the interior of A, bd$A$ is its boundary, cone$A$=cl$\cup_{\alpha>0}\alpha A$ is a cone spanned by the set A, $d(x, A)$ is a distance between x and A. We denote by K^* the polar cone of a cone $K \subset R^n$, the closed convex cone defined by

$$K^* = \{x^* | \forall x \in K,\ \langle x^*, x\rangle \geq 0\}.$$

If C is $n \times n$ matrix we denote by C^* a transposed matrix. The standard simplex in R^n is denoted by

$$\Gamma^k = \{\bar{\gamma} = (\gamma_1, \ldots, \gamma_k) \in R^k | \gamma_i \geq 0,\ \sum_{i=1}^{k} \gamma_i = 1\}.$$

A unit ball in R^n centered at the origin is denoted by B_n.

Now, we recall some background results from the stability theory [1, 4].

Let $f : R \to R$ be a continuous function. The Lyapunov exponent of the function f is defined by

$$\chi[f(\cdot)] = -\limsup_{t\to\infty} \frac{1}{t} \ln |f(t)|.$$

The Lyapunov exponents possess the following properties

1. $\chi[(f + \phi)(\cdot)] \geq \min\{\chi[f(\cdot)], \chi[\phi(\cdot)]\}$,
2. $\chi[(f\phi)(\cdot)] \geq \chi[f(\cdot)] + \chi[\phi(\cdot)]$,
3. $\chi[(f + 1/f)(\cdot)] \leq 0$,

If $f : R \to R^n$ is a vector function, then the Lyapunov exponent is defined as the minimal value of the Lyapunov exponents of the components $\chi[f^i(\cdot)]$.

Let us consider the linear differential equation

$$\dot{x}(t) = C(t)x(t), \tag{1}$$

where $n \times n$ matrix $C(t)$ has measurable bounded components. Lyapunov proved that the exponent is finite for any nonzero solution of (1). Moreover, the set of all possible numbers that are Lyapunov exponents of some nonzero solution of (1) is finite, with cardinality less than or equal to n. Lyapunov exponents of nonzero solutions to a linear differential equation with constant matrix C coincide with the real parts of the eigenvalues of C taken with the opposite sign.

A fundamental system of solutions of (1) $x_1(\cdot), \ldots, x_n$ is said to be normal if for all $\alpha_1, \ldots, \alpha_n \in R$

$$\chi[(\sum_{i=1}^{n} \alpha_i x_i)(\cdot)] = \max\{\chi[x_i(\cdot)] \mid i : \alpha_i \neq 0\}.$$

Lyapunov proved that a normal system of solutions always exists. Lyapunov exponents $\lambda_1, \ldots, \lambda_n$ of a normal system of solutions (there may be equal quantities among them) are called the Lyapunov spectrum of (1).

Let $\lambda_1, \ldots, \lambda_n$ be the Lyapunov spectrum of (1). Then the value $S = \lambda_1 + \cdots + \lambda_n$ does not exceed $\chi[\xi(\cdot)]$ where

$$\xi(t) = \exp \int_0^t \mathrm{tr}\, C(s) ds.$$

From this fact we obtain the following consequence. If $z_1(\cdot), \ldots, z_n(\cdot)$ is a fundamental system of solutions of (1), $\nu_1, \ldots, \nu_n$ are corresponding Lyapunov exponents, and $\nu_1 + \cdots + \nu_n = \chi[\xi(\cdot)]$, then the system is normal. Equation (1) is called regular if $S = -\chi[(1/\xi)(\cdot)]$. In this case, obviously,

$$S = \chi[\xi(\cdot)] = -\chi[(1/\xi)(\cdot)].$$

As a consequense we derive that the limit

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \mathrm{tr}\, C(s) dt$$

exists. All linear differential equations with constant or periodic coefficients are regular.

Let us consider along with (1) the adjoint equation

$$\dot{x}^*(t) = -C^*(t) x^*(t), \tag{2}$$

where $C^*(t)$ represents a transposed matrix. An important property of regular equations was established by Perron (see [4], e.g.). If $\lambda_1 \leq \ldots \leq \lambda_n$ is the Lyapunov spectrum of (1) and $\mu_1 \geq \ldots \geq \mu_n$ is the Lyapunov spectrum of (2), then equation (1) is regular if and only if $\lambda_i + \mu_i = 0, i = 1, \ldots, n$.

We denote by $\Lambda_\tau(\delta)$ the subspace consisting of all points $x_0 \in R^n$ such that a solution of (1) with the initial condition $x(\tau) = x_0$ has a Lyapunov exponent greater than $-\delta$ and by $\Lambda_\tau^+(\delta)$ the subspace consisting of all points $x_0^* \in R^n$ such that a solution of (2) with the initial condition $x(\tau) = x_0$ has a Lyapunov exponent greater than or equal to δ.

Lemma 1.1. If the equation (1) is regular then

$$\Lambda_\tau^\perp(\delta) = \Lambda_\tau^+(\delta).$$

Proof. We first establish the inclusion

$$\Lambda_\tau^+(\delta) \subset \Lambda_\tau^\perp(\delta). \tag{3}$$

Assume that $x_0 \in \Lambda_\tau(\delta), x_0^* \in \Lambda_\tau^+(\delta)$ and that $x(\cdot), x^*(\cdot)$ are solutions of equations (1) and (2), respectively. Then

$$\langle x(t), x^*(t) \rangle = \langle x_0, x_0^* \rangle + \int_\tau^t \frac{d}{ds} \langle x(s), x^*(s) \rangle ds =$$

$$\langle x_0, x_0^* \rangle + \int_\tau^t (\langle C(s) x(s), x^*(s) \rangle + \langle x(s), -C^*(s) x^*(s) \rangle) ds = \langle x_0, x_0^* \rangle.$$

Taking into account properties of the Lyapunov exponents, we obtain $\chi[\langle x, x^* \rangle(\cdot)] > 0$. Thus, $\lim_{t \to \infty} \langle x(t), x^*(t) \rangle = 0$ and $\langle x_0, x_0^* \rangle = 0$. The inclusion (3) is proved.

To prove the equality we consider matrices $\Phi(t, \tau)$ and $\Phi^+(t, \tau)$ of the fundamental solutions of equations (1) and (2). Assume that their columns form normal systems of solutions. The subspace $\Lambda_\tau(\delta)$ is spanned by column vectors of the matrix $\Phi(\tau, \tau)$ which correspond to solutions that have Lyapunov exponents greater than $-\delta$, and the subspace

$\Lambda_\tau^+(\delta)$ is spanned by column vectors of the matrix $\Phi^+(\tau,\tau)$, which correspond to solutions that have Lyapunov exponents greater than or equal to δ. Let $\dim\Lambda_\tau(\delta) = k$. Since equation (1) is regular,the Perron theorem implies that the Lyapunov spectra $\lambda_1 \leq \ldots \leq \lambda_n$ and $\mu_1 \geq \ldots \geq \mu_n$ of (1) and (2) satisfy the equalities $\lambda_i + \mu_i = 0, i = 1,\ldots,n$. Thus $\dim\Lambda_\tau^+(\delta) = n - k$. If we combine this with (3), we reach $\Lambda_\tau^\perp(\delta) = \Lambda_\tau^+(\delta)$ and the end of the proof.

We recall that a linear transformation $x = L(t)y$, where $n \times n$ matrix $L(t)$ smoothly depends upon $t \in R$ is said to be Lyapunov transformation if

$$\sup_{t\in R}(|L(t)| + |L^{-1}(t)| + |\dot{L}(t)|) < \infty.$$

Nonautonomus linear differential equation which could be transformed to an autonomus one with the help of a Lyapunov transformation supplies a very important example of regular linear differential equation. Lyapunov proved that if $C(t)$ is $T/2$-periodic matrix than there exists a real T-periodic Lyapunov transformation $x = L(t)y$ leading equation (1) to the equation

$$\dot{y} = Cy$$

with a constant matrix C.

We conclude this section with a generalization of the Perron positive matrix theorem.

Theorem 1.2. Let $K \subset R^n$ be a nonzero convex closed cone which does not contain a line and let $C : R^n \to R^n$ be linear operator. If $Cx \in K$ for all $x \in K$ then there exists eigenvector of the operator C contained in the cone K and corresponding to nonnegative eigevalue.

Proof. A) Suppose that $Cx \in \mathrm{int}K$ for all $x \neq 0$. Consider the set

$$\Omega = \{\omega \in R \mid \exists x \in K \cap \mathrm{bd}B_n \ : \ Cx - \omega x \in K\}.$$

Since sufficiently small $\omega > 0$ belongs to Ω, we conclude that $\Omega \neq \emptyset$. Let us prove that the set Ω is upper bounded. If this is not the case then there exist sequences $\omega_k \to \infty$ and $x_k \to x$ such that

$$x_k \in K \cap \mathrm{bd}B_n,$$
$$\omega_k^{-1}Cx_k - x_k \in K.$$

Taking the limit we obtain that $-x \in K$. This contradicts the inclusion $x \in K$ because the cone K does not contain a line.

Let $\omega_0 = \sup\Omega > 0$. Since the cone K is closed, there exists a vector $x_0 \in K \cap \mathrm{bd}B_n$ such that $Cx_0 - \omega_0 x_0 \in K$. Let $z_0 = Cx_0 - \omega_0 x_0 \neq 0$. Then

$$C(x_0 + z_0/(2\omega_0)) - \omega_0(x_0 + z_0/(2\omega_0)) = z_0/2 + C(z_0/(2\omega_0)).$$

Since $z_0 \neq 0$, we obtain $Cz_0 \in \mathrm{int}K$. Hence, $Cy_0 - \omega_0 y_0 \in \mathrm{int}K$, where $y_0 = (x_0 + z_0/(2\omega_0))/|x_0 + z_0/(2\omega_0)|$. This implies that $\sup\Omega > \omega_0$. Thus, $z_0 = 0$ and $Cx_0 = \omega_0 x_0$.

B) To reduce the general case to that one considered in part **A**, introduce a contraction of the linear operator C on the subspace $K - K$. This permits us to regard that $\mathrm{int}K \neq \emptyset$. Let $x_0 \in \mathrm{int}K$. Consider the sequence of linear operators $C_k : R^n \to R^n$ defined as $C_k x = k^{-1}\langle x_0, x\rangle x_0$. Observe that $(C + C_k)x = Cx + k^{-1}\langle x_0, x\rangle x_0 \in \mathrm{int}K$ for all $x \neq 0$. By part **A** there exist numbers $\omega_k > 0$ and vectors $x_k \in K \cap \mathrm{bd}B_n$ such that $\omega_k x_k \in Cx_k + k^{-1}\langle x_0, x_k\rangle x_0$. Without loss of generality the sequences $\{\omega_k\}$ and $\{x_k\}$ converge to ω_0 and x_0 respectively. Taking the limit we reach $\omega_0 x_0 \in Cx_0$, $\omega_0 \geq 0$ and the end of the proof.

2 First approximation of nonlinear controlled system

Consider the controlled system

$$\dot{x}(t) = f(x(t), u(t)),\ u(t) \in U \subset R^k,\ t \in [0,T], \tag{1}$$

where $f : R^n \times U \in R^n$ is a continuous function differentiable with respect to $x \in R^n$. Suppose that $|\nabla_x f(x,u)| \le L,\ |f(x,u)| \le L$ for all $(x,u) \in R^n \times U$. Let $\hat{x}(t),\ t \in [0,T]$ be a trajectory of (1) corresponding to a control $\hat{u}(t),\ t \in [0,T]$. Set $C(t) = \nabla_x f(\hat{x}(t), \hat{u}(t))$, $K(t) = \text{cone co}[f(\hat{x}(t), U) - f(\hat{x}(t), \hat{u}(t))]$. We shall consider the linear controlled system

$$\dot{x}(t) = C(t)x(t) + w(t),\ w(t) \in K(t),\ t \in [0,T] \tag{2}$$

as the first approximation of (1) in the neighbourhood of the trajectory $\hat{x}(\cdot)$. We shall prove that the set of trajectories of the linear controlled system (2) can be considered as a tangent cone to the set of trajectories of (1) at $\hat{x}(\cdot)$.

To this end we need the following auxiliary statement.

Lemma 2.1. Let $v \in C(t)x + K(t)$. Then

$$\lim_{\lambda \downarrow 0} \lambda^{-1} d(\dot{\hat{x}}(t) + \lambda v,\ \text{co}f(\hat{x}(t) + \lambda x, U)) = 0.$$

Proof. Let $\eta > 0$. There exist vectors $u_i \in U,\ i = 1, \dots, n+1$ and numbers $\alpha > 0,\ \mu_i \ge 0,\ i = 1, \dots, n+1$ satisfying $\sum_{i=1}^{n+1} \mu_i = 1$ and

$$|v - C(t)x - \alpha \sum_{i=1}^{n+1} \mu_i f(\hat{x}(t), u_i) + \alpha f(\hat{x}(t), \hat{u}(t))| < \eta.$$

If $\lambda\alpha < 1$ we obtain the following inequalities:

$$d(\dot{\hat{x}}(t) + \lambda v,\ \text{co}f(\hat{x}(t) + \lambda x, U)) \le$$

$$d(\dot{\hat{x}}(t) + \lambda(C(t)x + \alpha \sum_{i=1}^{n+1} \mu_i f(\hat{x}(t), u_i) - \alpha f(\hat{x}(t), \hat{u}(t))),\ \text{co}f(\hat{x}(t) + \lambda x, U)) + \lambda\eta \le$$

$$|\dot{\hat{x}}(t) + \lambda(C(t)x + \alpha \sum_{i=1}^{n+1} \mu_i f(\hat{x}(t), u_i) - \alpha f(\hat{x}(t), \hat{u}(t)))-$$

$$f(\hat{x}(t) + \lambda x, \hat{u}(t)) - \lambda\alpha(\sum_{i=1}^{n+1} \mu_i f(\hat{x}(t) + \lambda x, u_i) - f(\hat{x}(t) + \lambda x, \hat{u}(t)))| + \lambda\eta \le$$

$$|f(\hat{x}(t), \hat{u}(t)) + \lambda \nabla_x f(\hat{x}(t), \hat{u}(t))x - f(\hat{x}(t) + \lambda x, \hat{u}(t))| +$$

$$\lambda\alpha \sum_{i=1}^{n+1} \mu_i |f(\hat{x}(t), u_i) - f(\hat{x}(t) + \lambda x, u_i)| +$$

$$\lambda\alpha |f(\hat{x}(t), \hat{u}(t)) - f(\hat{x}(t) + \lambda x, \hat{u}(t))| + \lambda\eta.$$

Since $\eta > 0$ is arbitrary, deviding the above inequality by λ and taking the limit we achieve the result.

Consider a finite set of trajectories of the linear controlled system (2) $x_i(\cdot)$, $i = 1,\ldots,k$. Let $\bar{\gamma} = (\gamma_1,\ldots,\gamma_k) \in \Gamma^k$. We set

$$x(t,\bar{\gamma}) = \sum_{i=1}^{k} \gamma_i x_i(t).$$

Theorem 2.2. For any $\epsilon > 0$ there exists a number $\lambda_0 > 0$ such that for all $\lambda \in]0,\lambda_0[$ and $\bar{\gamma} \in \Gamma^k$ there exists a trajectory of the controlled system (1) $x_{\bar{\gamma}}^{\lambda}(\cdot)$ satisfying

$$x_{\bar{\gamma}}^{\lambda}(0) = \hat{x}(0) + \lambda x(0,\bar{\gamma}),$$

$$|\hat{x}(t) + \lambda x(t,\bar{\gamma}) - x_{\bar{\gamma}}^{\lambda}(t)| < \lambda\epsilon,\ t \in [0,T].$$

Proof. Denote

$$\rho(t,\lambda,\bar{\gamma}) = d(\dot{\hat{x}}(t) + \lambda\dot{x}(t,\bar{\gamma}),\ cof(\hat{x}(t) + \lambda x(t,\bar{\gamma}),U)),$$

$$\rho(t,\lambda) = \max_{\bar{\gamma}\in\Gamma^k} \rho(t,\lambda,\bar{\gamma}).$$

Lemma 2.1 implies that $\lim_{\lambda\downarrow 0}\rho(t,\lambda,\bar{\gamma}) = 0$ for all $\bar{\gamma} \in \Gamma^k$. Let $b(t) = \max\{|\dot{x}_i(t) + L|x_i(t)| \mid i = 1,\ldots,k\}$. It is easy to check that

$$|\rho(t,\lambda,\bar{\gamma}) - \rho(t,\lambda,\bar{\gamma}\prime)| \le \lambda b(t)|\bar{\gamma} - \bar{\gamma}\prime|, \tag{3}$$

$$\rho(t,\lambda) \le \lambda b(t). \tag{4}$$

Let us prove that

$$\lim_{\lambda\downarrow 0} \lambda^{-1}\rho(t,\lambda) = 0. \tag{5}$$

Suppose the opposite. Then there exist a number $\beta > 0$ and sequences $\lambda_i \downarrow 0,\ \bar{\gamma}_i \in \Gamma_k$ satisfying $\lambda_i^{-1}\rho(t,\lambda_i,\bar{\gamma}_i) > \epsilon$. Without loss of generality $\bar{\gamma}_i \to \bar{\gamma}_0$. By (3)

$$\epsilon < \lambda_i^{-1}\rho(t,\lambda_i,\bar{\gamma}_i) \le \lambda_i^{-1}\rho(t,\lambda_i,\bar{\gamma}_0) + b(t)|\bar{\gamma}_i - \bar{\gamma}_0|.$$

Taking the limit we obtain a contradiction. By the Lebesgue theorem (4) and (5) imply that

$$\int_0^T \lambda^{-1}\rho(t,\lambda)dt = 0.$$

Let $\lambda_0 > 0$ be such that

$$\int_0^T e^{L(T-s)}\rho(s,\lambda)ds < \epsilon\lambda/3 \ \text{ for all } \lambda \in]0,\lambda_0[.$$

Let $\lambda \in]0,\lambda_0[,\ \bar{\gamma} \in \Gamma^k$. Set $\delta = \epsilon\lambda/3$. Using the Caratheodory theorem and Filippov lemma [5] it is easy to prove that there exist measurable functions $u_i(\cdot,\bar{\gamma}),\ \mu_i(\cdot,\bar{\gamma}),\ i = 1,\ldots,n+1$ satisfying

$$u_i(t,\bar{\gamma}) \in U,\ \mu_i(t,\bar{\gamma}) \ge 0,\ i = 1,\ldots,n+1,\ t \in [0,T],$$

$$\sum_{i=1}^{n+1} \mu_i(t,\bar{\gamma}) = 1,\ t \in [0,T],$$

$$|\dot{\hat{x}}(t) + \lambda\dot{x}(t,\bar{\gamma}) - \sum_{i=1}^{n+1} \mu_i(t,\bar{\gamma}) f(\hat{x}(t) + \lambda x(t,\bar{\gamma}), u_i(t,\bar{\gamma}))| \le \rho(t,\lambda) + \frac{\delta L}{e^{LT} - 1}.$$

Consider a solution $y(\cdot,\bar\gamma)$ to the Cauchy problem

$$\dot y(t,\bar\gamma)=\sum_{i=1}^{n+1}\mu_i(t,\bar\gamma)f(y(t,\bar\gamma),u_i(t,\bar\gamma)),$$

$$y(0,\bar\gamma)=\hat x(0)+\lambda x(0,\bar\gamma).$$

Obviously,

$$\frac{d}{dt}|\hat x(t)+\lambda x(t,\bar\gamma)-y(t,\bar\gamma)|\le|\dot{\hat x}(t)+\lambda\dot x(t,\bar\gamma)-\dot y(t,\bar\gamma)|\le$$

$$\sum_{i=1}^{n+1}\mu_i(t,\bar\gamma)|f(\hat x(t)+\lambda x(t,\bar\gamma),u_i(t,\bar\gamma))-f(y(t,\bar\gamma),u_i(t,\bar\gamma))|+$$

$$\rho(t,\lambda)+\frac{\delta L}{e^{LT}-1}\le L|\hat x(t)+\lambda x(t,\bar\gamma)-y(t,\bar\gamma)|+\rho(t,\lambda)+\frac{\delta L}{e^{LT}-1}.$$

By the Gronwall inequality

$$|\hat x(t)+\lambda x(t,\bar\gamma)-y(t,\bar\gamma)|\le\int_0^t e^{L(t-s)}\rho(s,\lambda)ds+\delta\le 2\epsilon\lambda/3.$$

Since $|f(x,u)|\le L$ for all $(x,u)\in R^n\times U$ and

$$\dot y(t,\bar\gamma)\in\mathrm{co}\{f(y(t,\bar\gamma),u_i(t,\bar\gamma))\mid i=1,\dots,n+1\}.$$

The Filippov-Wazewski theorem [5] implies that there exists a solution $x^\lambda_{\bar\gamma}(\cdot)$ to the differential inclusion

$$\dot x(t)\in\{f(x(t),u_i(t,\bar\gamma))\mid i=1,\dots,n+1\}$$

satisfying $|x^\lambda_{\bar\gamma}(t)-y(t,\bar\gamma)|<\epsilon\lambda/3$.

Thus, for any $\bar\gamma\in\Gamma^k$ and $\lambda\in]0,\lambda_0[$ a trajectory of controlled system (1) $x^\lambda_{\bar\gamma}(\cdot)$ satisfying

$$|\hat x(t)+\lambda x(t,\bar\gamma)-x^\lambda_{\bar\gamma}(t)|\le\lambda\epsilon,\ t\in[0,T]$$

is found. The theorem is proved.

3 Weak asymptotic stability of an equilibrium of a controlled system

Let us consider the controlled system

$$\dot x(t)=f(x(t),u(t)),\ u(t)\in U\subset R^k. \tag{1}$$

We assume that the function $f:R^n\times U\to R^n$ satisfies all conditions stated in the previous section. Suppose that there exists a vector $u_0\in U$ such that $f(0,u_0)=0$. This implies that the point $x=0$ is an equilibrium point of system (1). To derive sufficient conditions of weak asymptotic stability of the equilibrium consider the linear controlled system

$$\dot x(t)=Cx(t)+w(t),\ w(t)\in K, \tag{2}$$

where $C=\nabla_x f(0,u_0)$, $K=\text{cone co}f(0,U)$. Let us introduce along with (2) two linear differential equations

$$\dot{x}(t) = Cx(t) \tag{3}$$

$$\dot{x}^*(t) = -C^*x^*(t) \tag{4}$$

We denote by **P** the set consisting of all points $x \in R^n$ such that there exists a trajectory of (2) that has a positive Lyapunov exponent starting at x. The set **Q** consists of all points $x^* \in R^n$ such that there exists a solution to (4) that has a nonegative Lyapunov exponent starting at x^* and contained in K^* for all $t \geq 0$. Let Λ be the subspace consisting of all points $x \in R^n$ such that a solution $x(\cdot)$ to the equation (3) with the initial condition $x(0) = x$ has a positive Lyapunov exponent, and let Λ^+ be the subspace consisting of all points $x^* \in R^n$ such that a solution $x^*(\cdot)$ of (4) with the initial condition $x^*(0) = x^*$ has a nonnegative Lyapunov exponent. Denote by P_T the set consisting of all points $x \in R^n$ such that there exists a trajectory $x(\cdot)$ of (2) satisfying $x(0) = x,\ x(T) = 0$. Let Q_T be the set consisting of all points $x^* \in R^n$ such that there exists a solution $x^*(\cdot)$ to (4) satisfying $x^*(0) = x^*,\ x^*(t) \in K^*$ for all $t \in [0, T]$.

Obviously, the sets **P**, **Q**, P_T, Q_T are convex cones and

$$P_{T_1} \subset P_{T_2},\ \ Q_{T_1} \supset Q_{T_2} \ \text{ when } T_1 < T_2.$$

We set

$$P = \bigcup_{T>0} P_T,\ \ Q = \bigcap_{T>0} Q_T.$$

Now, we prove some properties of the above cones.

Lemma 3.1 The equalities $Q_T = -P_T^*,\ Q = -P^*$ hold true.

Proof. Let $q \in Q_T,\ p \in P_T$. By the definition there exist a solution $x^*(\cdot)$ to differential equation (4) with $x^*(0) = q$ satisfying $x^*(t) \in K^*$ for all $t \in [0, T]$ and a trajectory $x(\cdot)$ of (2) satisfying $x(0) = p$ and $x(T) = 0$. Observe that

$$\langle q, p\rangle = \langle x^*(0), x(0)\rangle - \langle x^*(T), x(T)\rangle =$$

$$-\int_0^T \frac{d}{dt}\langle x^*(t), x(t)\rangle dt = -\int_0^T (\langle \dot{x}^*(t), x(t)\rangle + \langle x^*(t), \dot{x}(t)\rangle)dt =$$

$$-\int_0^T (\langle -C^*x^*(t), x(t)\rangle + \langle x^*(t), Cx(t) + w(t)\rangle)dt = -\int_0^T \langle x^*(t), w(t)\rangle dt \leq 0,$$

where $w(t) \in K$ is a control corresponding to the trajectory $x(\cdot)$. Thus, $Q_T \subset -P_T^*$.

Let $q \in -P_T^*$. This implies that the control $w(t) \equiv 0,\ t \in [0, T]$ solves the following optimal control problem

$$\langle x(0), q\rangle \to \sup$$

$$\dot{x}(t) = Cx(t) + w(t),\ \ w(t) \in K\ \ t \in [0, T],$$

$$x(T) = 0.$$

The Pontryagin maximum principle [6] is equivalent to the inclusion $q \in Q_T$. Thus, $Q_T = -P_T^*$. Now, the second equality follows immediately from the definition of the cones Q and P.

Theorem 3.2. The equalities **Q**$= Q \cap \Lambda^+ = -$**P*** hold true.

Proof. The equality **Q**$= Q \cap \Lambda^*$ is an obvious consequence of the definition. To prove the inclusion $\mathbf{Q} \subset -\mathbf{P}^*$ consider any $q \in \mathbf{Q}$. There exists a solution $x^*(\cdot)$ to the equation (4) satisfying $x^*(0) = q$, $x^*(t) \in K^*$, $t \in [0, \infty[$ and $\chi[x^*(\cdot)] \leq 0$. Let $p \in \mathbf{P}$ and let $x(\cdot)$ be a trajectory of controlled system (2) satisfying $\chi[x(\cdot)] > 0$. Then $\chi[\langle x^*, x\rangle(\cdot)] > 0$. Taking the limit in the inequality (see the proof of lemma 3.1)

$$\langle x^*(t), x(t)\rangle = \langle q, p\rangle + \int_0^t \frac{d}{dt}\langle x^*(s), x(s)\rangle ds \geq \langle q, p\rangle$$

we conclude that $\mathbf{Q} \subset -\mathbf{P}^*$.

Obviously, $P + \Lambda \subset \mathbf{P}$. By lemmas 1.1 and 3.1

$$-\mathbf{P}^* \subset -(P + \Lambda)^* = Q \cap \Lambda^{\perp} = Q \cap \Lambda^+ = \mathbf{Q}.$$

Theorem 3.3. The following conditions are equivalent:

1. $\mathbf{P} = R^n$,
2. the matrix C^* has neither eigenvectors corresponding to nonnegative eigenvalues contained in the cone K^* nor proper invariant subspaces contained in the subspace $\Lambda^+ \cap K^* \cap -K^*$.

Proof. The second condition can be derived from the first one by a simple contradiction argument (see the proof of Theorem 4.1 for more details).

Suppose that the second condition holds true. Theorem 3.2 implies that it is sufficient to prove the equality $\mathbf{Q} = \{0\}$. Since $e^{-C^*t}Q \subset Q$ for all $t \geq 0$, we conclude that $e^{-C^*t}(Q \cap -Q) \subset Q \cap -Q$ for all $t \geq 0$. Hence,

$$C^*(Q \cap -Q) \subset Q \cap -Q. \tag{5}$$

Cone $\mathbf{Q}$ does not contain a line. Indeed, if this is not the case, then the inclusions $Q \subset K^*$, $C^*\Lambda^+ \subset \Lambda^+$ and (5) imply that the cone $K^* \cap \Lambda^+$ contains a proper invariant subspace of the matrix C^*. This contradicts the second condition.

Let $\mathbf{Q} \neq \{0\}$. Since $\mathbf{Q} = Q \cap \Lambda^+$, we conclude that for any natural k and any vector $x^* \in \mathbf{Q}$ the inclusion $\exp[k^{-1}C^*]x^* \in \mathbf{Q}$ holds true. By Theorem 1.2 there exist a unit vector x_k^* and a number ω_k such that $k(\omega_k - 1)x_k^* = k(\exp[k^{-1}C^*] - E)x_k^*$. Without loss of generality x_k^* converges to some $x_0^* \in \mathbf{Q}$ and $k(\omega_k - 1)$ converges to ω_0 as k becomes infinite. Taking the limit we obtain $\omega_0 x_0 \in C^*x_0^*$. Since $x_0^* \in \Lambda^+$, we conclude that $\omega_0 \geq 0$. Thus, we reach a contradiction and the end of the proof.

The set of trajectories of the linear controlled system (2) is a convex cone. This, obviously, implies that weak asymptotic stability of zero equilibrium point of the system (2) is equivalent to the following condition:

(H) for any $x_0 \in R^n$ there exists a trajectory of the controlled system (2) with the initial condition $x(0) = x_0$ satisfying

$$\lim_{t \to \infty} x(t) = 0.$$

Proposition 3.4. The conditions $\mathbf{P} = R^n$ and **(H)** are equivalent.

This proposition is an evident consequence of the following result.

Lemma 3.5. Suppose that condition **(H)** holds true. Then there exist numbers $\gamma > 0$

and $a > 0$ such that for any $x_0 \in R^n$ one can find a trajectory of linear controlled system (2) with $x(0) = x_0$ satisfying

$$|x(t)| \leq a|x_0|e^{-\gamma t}, \quad t \in [0, \infty[. \tag{6}$$

Proof. Consider a simplex $\sigma^{n+1} \subset R^n$ containing a unit ball centered at zero. Let x_k, $k = 1, \ldots, n+1$ be its vertices. By condition **(H)** there exist trajectories $x_k(\cdot)$, $k = 1, \ldots, n+1$ of the system (2) with $x_k(0) = x_k$, $k = 1, \ldots, n+1$ which tend to zero as t becomes infinite. There exists $\tau \geq 0$ satisfying $|x_k(\tau)| \leq 1/e$, $k = 1, \ldots, n+1$. Let $y \in \text{bd}B_n$. Then $y = \sum_{k=1}^{n+1} \lambda_k x_k$ for some $\lambda_k \geq 0$, $k = 1, \ldots, n+1$ satisfying $\sum_{k=1}^{n+1} \lambda_k = 1$. Obviously, the trajectory $x(\cdot, y) = \sum_{k=1}^{n+1} \lambda_k x_k(\cdot)$ of the controlled system (2) with $x(0, y) = y$ satisfies $|x(\tau, y)| \leq 1/e$. We define for $y \in R^n$

$$x_y(t) = |y|x(t, y/|y|), \; t \in [0, \tau].$$

Let $x_0 \in R^n$. For $t \geq 0$ we set

$$x(t) = \begin{cases} x_{x_0}(t) & t \in [0, \tau], \\ x_{x(m\tau)}(t - m\tau) & t \in]m\tau, (m+1)\tau]. \end{cases}$$

This trajectory satisfies (6) with $\gamma = 1/\tau$ and

$$a = e \max\{|x_k(t)| \mid t \in [0, \tau], \; k = 1, \ldots, n+1\}.$$

Taking into account Proposition 3.4 and Theorem 3.3 we achieve the following result.

Theorem 3.6. The following conditions are equivalent:

1. The zero equilibrium point of linear controlled system (2) is weakly asymptotically stable,

2. The matrix C^* has neither eigenvectors corresponding to nonnegative eigenvalues contained in the cone K^* nor proper invariant subspaces contained in the subspace $\Lambda^+ \cap K^* \cap -K^*$.

Now, we establish sufficient conditions for weak asymptotic stability of the zero equilibrium point of nonlinear controlled system (1).

Theorem 3.7. Let the zero equilibrium point of linear controlled system (2) be weakly asymptotically stable. Then the zero equilibrium point of nonlinear controlled system (1) is also weakly asymptotically stable.

Proof. Let $\epsilon > 0$. Consider a simplex $\sigma^{n+1} \subset R^n$ containing the origin as its interior point. Let $x_1, \ldots, x_{n+1}$ be vertices of the simplex. By lemma 3.5 there exist numbers $\gamma > 0$, $a > 0$ and trajectories $x_i(\cdot)$ of the linear system (2) with $x_i(0) = x_i$ satisfying

$$|x_i(t)| \leq a|x_i|e^{-\gamma t}, \; i = 1, \ldots, n+1, \; t \in [0, \infty[.$$

There exists a number $T > 0$ such that $x_i(T) \in \frac{1}{4}\sigma^{n+1}$, $i = 1, \ldots, n+1$. By Theorem 2.2 there exists a number $\lambda_0 < \epsilon/(2a \max\{|x_i| \mid i = 1, \ldots, n+1\})$ such that for all $\bar{\gamma} \in \Gamma^{n+1}$ and $\lambda \in]0, \lambda_0[$ one can find a trajectory $x_{\bar{\gamma}}^{\lambda}(\cdot)$ of controlled system (1) satisfying

$$x_{\bar{\gamma}}^{\lambda}(0) = \lambda x(0, \bar{\gamma}),$$

$$x_{\bar{\gamma}}^{\lambda}(t) \in \lambda x(t, \bar{\gamma}) + \frac{\lambda}{4}\sigma^{n+1},\ t \in [0, T], \tag{7}$$

where $x(t, \bar{\gamma}) = \sum_{i=1}^{n+1} \gamma_i x_i(t)$, $\bar{\gamma} = (\gamma_1, \dots, \gamma_{n+1})$. Let $\delta > 0$ be such that $\delta B_n \subset \frac{\lambda_0}{2}\sigma^{n+1}$. Now, for any $x \in \delta B_n$ we shall define a trajectory $x(\cdot)$ of the controlled system (1) satisfying

$$x(0) = x,\ \ |x(t)| < \epsilon,\ t \in [0, \infty[,\ \lim_{t \to \infty} x(t) = 0.$$

Let $\lambda_1 = \lambda_0/2$. There exists $\bar{\gamma}_1 \in \Gamma^{n+1}$ such that $\lambda_1 x(0, \bar{\gamma}_1) = x$. We set

$$x(t) = x_{\bar{\gamma}_1}^{\lambda_1}(t),\ t \in [0, T].$$

The inclusion (7) implies that

$$x(t) \in \epsilon B_n,\ t \in [0, T],$$

$$x(T) \in \frac{\lambda_1}{2}\sigma^{n+1}.$$

We define the trajectory $x(\cdot)$ by induction. Let λ_{k-1} and $x(t)$ for $t \in [0, (k-1)T]$ be determined. Then we set $\lambda_k = \lambda_{k-1}/2$. There exists $\bar{\gamma}_k \in \Gamma^{n+1}$ such that $\lambda_k x(0, \bar{\gamma}_k) = x((k-1)T)$. We set

$$x(t) = x_{\bar{\gamma}_k}^{\lambda_k}(t - (k-1)T),\ t \in [(k-1)T, kT].$$

The inclusion (7) implies that

$$x(t) \in \frac{\epsilon}{2^{k-1}} B_n,\ t \in [(k-1)T, kT],$$

$$x(kT) \in \frac{\lambda_k}{2}\sigma^{n+1}.$$

Thus, the equilibrium point of (1) is weakly asymptotically stable.

4 Weak asymptotic stability of the periodic trajectory of a controlled system

Consider the controlled system

$$\dot{x}(t) = f(x(t), u(t)),\ u(t) \in U \subset R^k. \tag{1}$$

We suppose that the function $f : R^n \times U \to R^n$ satisfies all conditions stated in section 2. Let $\hat{x}(t)$, $t \in [0, T/2]$ be a periodic trajectory of the system (1) and let $\hat{u}(t)$, $t \in [0, T/2]$ be a corresponding control. To obtain sufficient conditions of weak asymptotic stability of $\hat{x}(\cdot)$ consider the first approximation of the system (1) in the neighbourhood of the trajectory $\hat{x}(\cdot)$, i.e. the linear controlled system

$$\dot{x}(t) = C(t)x(t) + w(t),\ w(t) \in M(t) \tag{2}$$

where $C(t) = \nabla_x f(\hat{x}(t), \hat{u}(t))$, $M(t) = \text{cone co}[f(\hat{x}(t), U) - f(\hat{x}(t), \hat{u}(t))]$.

With the help of T-periodic Lyapunov transformation one can transform the system (2) to the system

$$\dot{x}(t) = Cx(t) + w(t),\ w(t) \in K(t), \tag{3}$$

where C is a constant matrix and $K(t)$ is a closed convex cone satisfying $K(t+T) = K(t)$ for all t.

As in the previous section we shall consider along with the system (3) two linear differential equations

$$\dot{x}(t) = Cx(t), \tag{4}$$

$$\dot{x}^*(t) = -C^*x^*(t). \tag{5}$$

We denote by $\mathbf{P}_\tau$ the set consisting of all points $x \in R^n$ such that there exists at least one trajectory $x(\cdot)$ of the system (3) with $x(\tau) = x$ satisfying $\chi[x(\cdot)] > 0$. The set $\mathbf{Q}_\tau$ consists of all points $x^* \in R^n$ such that there exists a solution $x^*(\cdot)$ to (4) with $x^*(\tau) = x^*$ satistying $\chi[x^*(\cdot)] \geq 0$, $x^*(t) \in K^*(t)$ for almost all $t \geq \tau$. Let Λ be the subspace consisting of all points $x \in R^n$ such that solution to the equation (4) with the initial condition $x(0) = x$ has positive Lyapunov exponent, and let Λ^+ be the subspace consisting of all points $x^* \in R^n$ such that a solution to the equation (5) with the initial condition $x^*(0) = x^*$ has nonnegative Lyapunov exponent. Denote by $P_\tau^{\tau\prime}$ the set consisting of all points $x \in R^n$ such that there exists a trajectory $x(\cdot)$ of (3) satisfying $x(\tau) = x$, $x(\tau\prime) = 0$. Let $Q_\tau^{\tau\prime}$ be the set consisting of all points $x^* \in R^n$ such that there exists a solution $x^*(\cdot)$ to the equation (5) with $x^*(\tau) = x^*$ satisfying $x^*(t) \in K^*(t)$ for almost all $t \in [\tau, \tau\prime]$.

Obviously, the sets $\mathbf{P}_\tau$, Q_τ, $P_\tau^{\tau\prime}$, $Q_\tau^{\tau\prime}$ are convex cones and

$$P_\tau^{\tau_1\prime} \subset P_\tau^{\tau_2\prime},\ Q_\tau^{\tau_1\prime} \subset Q_\tau^{\tau_2\prime} \text{ for all } \tau_1\prime < \tau_2\prime.$$

We set

$$P_\tau = \bigcup_{\tau\prime > \tau} P_\tau^{\tau\prime},\ Q_\tau = \bigcap_{\tau\prime > \tau} Q_\tau^{\tau\prime}.$$

As in Section 3 one can prove that

$$Q_\tau = -P_\tau^*,$$

$$\mathbf{Q}_\tau = Q_\tau \cap \Lambda^+ = -\mathbf{P}_\tau^*.$$

Moreover observe that

$$e^{-C^*t}Q_\tau \subset Q_{\tau+t},$$

$$Q_{\tau+T} = Q_\tau,\ \mathbf{Q}_{\tau+T} = \mathbf{Q}_\tau,\ \mathbf{P}_{\tau+T} = \mathbf{P}_\tau.$$

Theorem 4.1. The following conditions are equivalent:

1. $\mathbf{P}_0 = R^n$,
2. the matrix e^{-C^*T} has neither eigenvector $\mathbf{l}$ nor nontrivial invariant subspace $\mathbf{L}$ satisfying the inclusions
$$\mathbf{l} \in \Lambda^+,\ e^{-C^*t}\mathbf{l} \in K^*,\ t \in [0, T],$$
$$\mathbf{L} \subset \Lambda^+,\ e^{-C^*t}\mathbf{L} \subset K^*,\ t \in [0, T]$$
respectively.

Proof. Suppose that the condition 1 holds true. Assume that there exists an eigenvector or a nontrivial subspace of the matrix e^{-C^*T} satisfying corresponding inclusions. Then there exists a nontrivial solution $x^*(\cdot)$ to the differential equation (5) satisfying $x^*(t) \in K^*(t)$, $t \geq 0$ and $\chi[x^*(\cdot)] \geq 0$. Let $x \in R^n$. Since $\mathbf{P}_0 = R^n$, there exists a trajectory $x(\cdot)$ of the controlled system (3) with $x(0) = x$ and $\chi[x(\cdot)] > 0$. Obviously,

$$\langle x(t), x^*(t)\rangle = \langle x(0), x^*(0)\rangle + \int_0^t \frac{d}{ds}\langle x(s), x^*(s)\rangle ds =$$

$$\langle x(0), x^*(0)\rangle + \int_0^t (\langle Cx(s), x^*(s)\rangle + \langle u(s), x^*(s)\rangle + \langle x(s), -C^*x^*(s)\rangle)ds \geq$$
$$\langle x(0), x^*(0)\rangle = \langle x, x^*(0)\rangle.$$

Since the function $\langle x(t), x^*(t)\rangle$ has a positive Lyapunov exponent, we get

$$\lim_{t\to\infty} \langle x(t), x^*(t)\rangle = 0.$$

This implies that $\langle x, x^*(0)\rangle \leq 0$ for all $x \in R^n$. Thus, $x^*(0) = 0$. This contradicts the nontriviality of $x^*(\cdot)$.

Now, suppose that condition 2 is verified. It is enough to prove that $Q_0 \cap \Lambda^+ = \{0\}$. Assume that $Q_0 \cap \Lambda^+ \neq \{0\}$. We claim that the cone $Q_0 \cap \Lambda^+$ does not contain a line. If this is not the case, then $\mathbf{L} = Q_0 \cap -Q_0 \cap \Lambda^+ \neq \{0\}$. Observe that

$$e^{-C^*t}\mathbf{L} \subset Q_t \cap -Q_t \cap \Lambda^+, \; t \in [0, T].$$

By periodicity of Q_t

$$e^{-C^*T}\mathbf{L} \subset Q_0 \cap -Q_0 \cap \Lambda^+ = \mathbf{L}.$$

This contradicts condition 2. Hence, the cone $Q_0 \cap \Lambda^+$ does not contain a line. Since

$$e^{-C^*T}(Q_0 \cap \Lambda^+) \subset Q_0 \cap \Lambda^+,$$

Theorem 1.2 implies that the matrix e^{-C^*T} has an eigenvector $\mathrm{l} \in Q_0 \cap \Lambda^+$. Obviously, $e^{-C^*t}\mathrm{l} \in Q_t \cap \Lambda^+$ for all $t \in [0, T]$. Thus, we achieve a contradiction and, hence, the result.

With the help of reasoning similar to that provided in the previous section taking into account periodicity and the properties of Lyapunov transformation, one can prove the following results.

Theorem 4.2. The following conditions are equivalent:

1. The zero equilibrium point of the linear controlled system (2) is weakly asymptotically stable,
2. The matrix e^{-C^*T} has neither eigenvector l nor nontrivial invariant subspace $\mathbf{L}$ satisfying the inclusions

$$\mathrm{l} \in \Lambda^+, \; e^{-C^*t}\mathrm{l} \in K^*, \; t \in [0, T],$$

$$\mathbf{L} \subset \Lambda^+, \; e^{-C^*t}\mathbf{L} \subset K^*, \; t \in [0, T]$$

respectively.

Theorem 4.3. Assume that zero equilibrium point of the linear controlled system (2) is weakly asymptotically stable. Then the periodic trajectory $\hat{x}(\cdot)$ of the controlled system (1) is also weakly asymptotically stable.

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