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Invariance envelopes and invariance kernels for Lipschitzean Differential Inclusions

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FOREWORD

The author investigates a differential inclusion whose solutions have to remain in a given closed set. The invariance kernel is the set of the initial conditions starting at which, all solutions to the differential inclusion remain in this closed set. The invariance envelope is the smallest set which contains the given closed set and which is invariant for the differential inclusion. In this paper, the author studies invariance envelopes and he compares this envelope to invariance kernels. He provides an algorithm which determines the invariance kernel and consequently the invariance envelope.

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Marc Quincampoix

1 Introduction

Let us consider a differential inclusion with constraints:

$$\begin{cases} i) & x'(t) \in F(x(t)) \\ ii) & \forall\, t \geq 0,\ x(t) \in K \end{cases}$$

where F is a set valued map and K a closed subset of a finite dimensional vector space X.

Recall that the *contingent cone* to K at x is the set:

$$T_K(x) := \{\ v \in X \mid \liminf_{h \longrightarrow 0+} d(x+hv, K)/h = 0\ \}$$

Under adequate assumptions, the Invariance Theorem (cf [2]) states that, for all $x_0 \in K$, all solutions to the differential inclusion $i)$ starting at x_0 are *viable* (i.e. satisfy $ii)$), if and only if $F(x) \subset T_K(x)$ for any $x \in K$. In this case, K is called an invariance domain.

Of course, generally, K is not an invariance domain, and we have to solve the inclusion in subsets of K and to determine all the initial conditions such that all solutions starting at these points are viable in K. Let us denote by $\mathrm{Inv}_F(K)$, the *Invariance kernel of* K namely the largest *closed* invariance domain contained in K. This set (possibly empty) exists if F is lipschitzean[1] with nonempty compact values (see for instance [2], chapter 5). In this paper, the set valued map F is assumed to be such. In a similar way we define

[1] Let us recall that the set valued map F is k-lipschitzean if and only if:

$$\forall\,(x,y),\ \ F(x) \subset F(y) + k\|x-y\|B.$$

where B denotes the closed unit ball.

the invariance envelope of K denoted by $Env_F(K)$ the smallest invariance domain which contains K. We compare the invariance envelope to the accessibility map. We prove that this envelope can be related with invariance kernels of the opposite inclusion in the following way:

$$\text{Env}_F(K) = \overline{X \backslash \text{Inv}_{-F}(\widehat{K})}$$

where $\widehat{K} := \overline{X \backslash K}$. We provide a result of viability and semipermeability of the boundary of the invariance envelope and consequently we deduce viability and semipermeability properties for the invariance kernel.

Our aim is to provide a constructive algorithm allowing the computation of the invariance kernel when K is assumed to be only closed. Consequently this algorithm allows to construct the invariance envelope. Let us notice that in [10] (see also [9], [17]) we have even provided an algorithm for the *viability kernel* of a differential inclusion. The viability kernel of a given closed K is the largest closed subset of K such that starting at any point of the viability kernel there exists at least one solution viable in K. In the case of differential equations with Lipschitz right-hand side, there is no difference between viability kernels and invariance kernels.

2 Invariance envelopes

In all this paper, X denotes a finite dimensional vector space F a k-Lipschitzean set valued map with compact convex values from X into itself. Consider the following differential inclusion

$$\text{(1)} \qquad \text{for almost all } t \geq 0, \;\; x'(t) \in F(x(t))$$

Let K be a closed set of X, ∂K its boundary, $\mathrm{Int}(K)$ its interior and $\widehat{K} := \overline{X \backslash K}$. We define by $S(x_0)$ the set of solutions to (1) starting at x_0. We consider also the following differential inclusion which provides same trajectories that (1) but in the reverse sense.

$$\text{(2)} \qquad x'(t) \in -F(x(t))$$

Let us recall the definition of invariance domains (cf [2]):

Definition 2.1 *A set A is called an invariance domain of F if and only if:*

$$\forall\, x \in A, \;\; F(x) \subset T_A(x)$$

The invariance kernel $\mathrm{Inv}_F(K)$ *of a closed set K is the largest closed invariance domain of F contained in K.*

If F is lipschitzean with nonempty compact values then, thanks to Invariance Theorem (see [2], [4]) a closed set K is an invariance domain if and only if starting from any point of K, every solution to (1) is viable in K (i.e. remains in the set K).

Definition 2.2 *We define the* invariance envelope, *denoted by* $\mathrm{Env}_F(K)$, *by the smallest closed invariance domain which contains K*

We can express this set thanks to the *accessibility map*:

Definition 2.3 *We denote by $R_F(x)$ the following set-valued map:*

$$R_F(x) := \{\, y \in X \mid \exists\, x(\cdot) \in S_F(x),\; \exists\, T > 0,\; y = x(T) \,\}$$

And for any set A, $R_F(A) := \bigcup_{x \in A} R_F(A)$.

We can express the relations between the two sets of previous definitions by the following

Proposition 2.4 *If K is a closed set, then*

$$\mathrm{Env}_F(K) = \overline{R_F(K)}$$

Furthermore, if the closed set K satisfies $K = \overline{Int(K)}$, then

$$\mathrm{Env}_F(K) = \overline{\mathrm{Int}(R_F(K))} = \overline{R_F(\mathrm{Int}(K))}$$

We need the following well-known lemma:

Lemma 2.5 *If $\mathcal{O}$ is an open set, then $R_F(\mathcal{O})$ is open too.*

Proof — Consider $y \in R_F(\mathcal{O})$, then there exist $x \in \mathcal{O}$, $x(\cdot) \in S_F(x)$ and a time $T \geq 0$ such that $y = x(T)$. If $y(\cdot) \in S_{-F}(x)$. The concatenation of $\{ x(T-s) \mid s \in [0,T] \}$ and $\{ y(s) \mid s \geq 0 \}$ provide a solution $\tilde{y}(\cdot) \in S_{-F}(y)$ such that $x = \tilde{y}(T) \in \mathcal{O}$. But thanks to Fiippov's Theorem (cf [8]) applied with the backward inclusion (2), we know that there exists a ball $B(y,\delta)$, such that for any $z \in B(y,\delta)$, there exists $z(\cdot) \in S_{-F}(z)$ such that $z(T) \in \mathcal{O}$.
Hence $z \in R_F(z(T))$ and $B(x,\delta) \subset R_F(\mathcal{O})$.

Q.E.D.

Proof of Proposition 2.4 — Let us prove the first equality. It is clear that $R_F(K)$ is contained in any invariant domain containing K in particular in the invariance envelope of K.
Conversely, we shall prove that $\overline{R_F(K)}$ is invariant. If it is not the case, then

$$\exists\, x_0 \in \overline{R_F(K)},\ \exists\, x(\cdot) \in S_F(x_0),\ \exists\, T > 0,\ x(T) \notin \overline{R_F(K)}$$

According Filippov's Theorem, there exists $\delta > 0$ such that for any $y \in B(x_0,\delta)$ there exists $y(\cdot) S_F(y)$ satisfying $y(T) \notin \overline{R_F(K)}$. Consider $y \in B(x_0,\delta) \cap R_F(K)$, then there exist a time T_0, $y_0 \in K$ and $y_0(\cdot) \in S_F(y_0)$ such that $y_0(T_0) = y$. Let us introduce $\tilde{y}(\cdot) \in S_F(y_0)$ the concatenation of $\{ y_0(s) \mid s \in [0,T_0] \}$ and $\{ y(s) \mid s \geq 0 \}$. Then $\tilde{y}(T+T_0) \notin R_F(K)$ a contradiction.

Let us prove the second equality. According to Lemma 2.5, $\overline{R_F(\text{Int}(K))} \subset \overline{\text{Int}(R_F(K))}$, and obviously $\overline{\text{Int}(R_F(K))} \subset \text{Env}_F(K)$.
Conversely, consider $y \in R_F(K)$, we shall prove that $y \in \overline{R_F(\text{Int}(K))}$. Consder $\varepsilon > 0$, there exist $x_\epsilon \in K$, $x_\epsilon(\cdot) \in S_F(x_\epsilon)$ and $T_\epsilon \geq 0$ such that $x_\epsilon(T_\epsilon) \in B(x, 2\varepsilon)$. Since $x_\epsilon \in K = \overline{\text{Int}(K)}$, according to Filippov's Theorem, there exist $\delta_\varepsilon > 0$ and $y_\epsilon \in B(x_\epsilon, \delta_\varepsilon) \cap \text{Int}(K)$, such that there exists $y_\varepsilon(\cdot) \in S_F(y_\varepsilon)$ satisfying $y_\epsilon(T_\varepsilon) \in B(x, \varepsilon)$. Hence for all $\varepsilon > 0$, $B(x, \varepsilon) \cap R_F(\text{Int}(K)) \neq \emptyset$, consequently $\overline{R_F(K)} \subset \overline{R_F(\text{Int}(K))}$.

Q.E.D.

Now, we state our main result for invariance envelopes:

Theorem 2.6 *If K is a closed nonemptyset such that $\overline{\text{Int}(K)} = K$, if we denote by $\widehat{K} := \overline{X \backslash K}$ then*

- $\text{Env}_F(K) = \overline{X \backslash \text{Inv}_{-F}(\widehat{K})}$
- *The set $\overline{X \backslash \text{Env}_F(K)}$ is an invariant domain for (2)*
- *The set $\partial \text{Env}_F(K)$ is locally viable for (2) and is a semipermeable set (cf [16] or [15]).*

Proof — Since these two sets contain K, it is enough to prove the equality for the elements outside of K. Consider $x_0 \notin \text{Inv}_{-F}(\overline{X \backslash K})$ (and $x_0 \notin K$). Then, $\exists\ x(\cdot) \in S_{-F}(x_0)$, $\exists\ T \geq 0$, such that $y_0 := x(T) \notin \overline{X \backslash K}$ i.e $y_0 \in \text{Int}(K)$. Consider $y(\cdot) \in S_F(x_0)$.

$$(3) \qquad \tilde{y}(s) := \begin{cases} x(T-s) & \text{si} \quad 0 \leq s \leq T \\ y(s-T) & \text{si} \quad s \geq T \end{cases}$$

then $\tilde{y}(\cdot) \in S_F(y_0)$ and $\tilde{y}(T) = x_0 \in R_F(y_0) \subset R_F(K)$. Then the open set $X \backslash \text{Inv}_{-F}(\overline{X \backslash K})$ is contained in $R_F(K)$, and by proposition 2.4 in $\text{Env}_F(K)$.

Conversely, consider $y \in \text{Int}(R_F(K)) \backslash K$. Thanks to Lemma 2.5 and Proposition 2.4, $\text{Int}(R_F(K)) = R_F(\text{Int}(K))$. Hence

$$\exists\ x \in \text{Int}(K),\ \exists\ x(\cdot) \in S_F(x),\ \exists\ T,\ y = x(T) \in X \backslash K$$

Consider $y(\cdot) \in S_{-F}(y)$. We can again define $\tilde{y}(\cdot)$ by formula (3) and we obtain a solution to (2) starting at y which is not viable in $\overline{X \backslash K}$ (because $\tilde{y}(T) = x \in \operatorname{Int}(K)$). Hence $y \notin \operatorname{Inv}_F(\overline{X \backslash K})$ and consequently $\overline{\operatorname{Int}(R_F(K))} \subset \overline{X \backslash \operatorname{Inv}_{-F}(\overline{X \backslash K})}$.

The second statement is easily deduced from the first one. Let us prove the last one. Consider $x_0 \in \partial \operatorname{Env}_F(K)$. Assume for a moment that there exist $x_0(\cdot) \in S_{-F}(x_0)$ and $T > 0$ such that $x_0(T) \in Int(\operatorname{Env}_F(K))$. Fix $\alpha > 0$, according to Filippov's theorem, there exists $x \notin \operatorname{Env}_F(K)$ and $x \in B(0, \alpha)$ such that $x(T) \in Int(\operatorname{Env}_F(K))$. But there exists $y \in K$, $y(\cdot) \in S_F(Y)$ and a time $\tau > 0$ such that $y(\tau) = x(T)$. Let introduce $z(\cdot) \in S_F(y)$ the concatenation of $\{ y(s) \,|\, 0 \leq s \leq \tau \}$ and $\{ x(T - s) \,|\, 0 \leq s \leq T \}$.

We obtain a solution to (1) starting at $y \in K$ such that $\tilde{y}(T + \tau) \notin \operatorname{Env}_F(K)$ a contradiction. Hence every solution starting at a point of the boundary is viable on the boundary of the invariance envelope.

Q.E.D.

Our results can be used to study target problems. In fact, the invariance envelope $\operatorname{Env}_{-F}(C)$ of a closed set C is exactely the set of point starting from which there exists at least one trajectory reaching C in finite time. This is the *possible victory domain* for target problems. Furthermore this set has the crucial property of viability and semipermeability of its boundary, it is the barrier of our target problem.

3 Algorithm for invariance kernels

We impose the following assumptions on the set-valued map F from X into itself:

$$(4) \qquad \begin{cases} F \text{ is a } k \text{ -lipschitzean set valued map with nonempty} \\ \text{compact values, satisfying the following boundedness} \\ \quad \text{condition } M := \sup_{x \in K} \sup_{y \in F(x)} \|y\| < \infty \end{cases}$$

Let us notice that the boundedness condition is automatically satisfied when K is compact.

Let us consider the following subset of the boundary of K

$$(5) \qquad K^a := \{ x \in \partial K \,|\, F(x) \cap (X \backslash T_K(x)) \neq \emptyset \}$$

When $K = \overline{\mathrm{Int}(K)}$, where $\overline{A}$ denotes the closure of a set A and $\mathrm{Int}(A)$ its interior, then (see [16])

$$K^a := \{ x \in \partial K \mid F(x) \cap D_{\widehat{K}}(x) \neq \emptyset \}$$

Where $\widehat{K} = \overline{X \backslash K}$ and $D_K(x)$ denotes the *Dubovitsky-Miliutin* tangent cone to K at x defined by:

$$D_K(x) := \{ v \in X \mid \exists\, \alpha > 0,\ x +]0, \alpha[(v + \alpha B) \subset K \}.$$

This enables us to express the Invariance Theorem in the following way:

Proposition 3.1 *A nonempty closed set K is an invariance domain for F if and only if the set K^a is empty.*

Furthermore, if $K^a \neq \emptyset$ then, $\mathrm{Inv}_F(K) \cap K^a = \emptyset$.

The second statement holds because the invariance kernel is a closed set.

There is a "natural" algorithm (see [2]) defined by the following subsequence:

$$K_0 := K,\ K_1 := \overline{K \backslash K^a},\ \ldots\ , K_{n+1} := \overline{K_n \backslash K_n^a}. \tag{6}$$

In some particular cases, this sequence may converge, but, generally, it is not the case. In fact, it is easy to notice that this sequence is constant ($= K$) as soon as:

$$K = \overline{\mathrm{Int}(K)} \tag{7}$$

The idea of our algorithm is to subtract to K not only K^a, but an open neighbourhood of K^a. In fact, since $\mathrm{Inv}_F(K)$ is closed, for any $x_0 \in K^a$, there exists a real $\varepsilon_{x_0}^0 > 0$ such that:

$$\mathrm{Inv}_F(K) \cap B(x_0, \varepsilon_{x_0}^0) = \emptyset,$$

where $B(x_0, \varepsilon_{x_0}^0)$ is the *closed* ball of center x_0 and radius $\varepsilon_{x_0}^0$ and $\overset{\circ}{B}(x_0, \varepsilon_{x_0}^0)$, the open one. A sequence of closed subsets of K can be defined in the following way:

$$
(8) \qquad \begin{cases} K_0 := K \\ K_1 := K_0 \setminus \bigcup_{x_0 \in K_0^a} \overset{\circ}{B}(x_0, \varepsilon_{x_0}^0) \\ \text{where } \ B(x_0, \varepsilon_{x_0}^0) \cap \mathrm{Inv}_F(K) = \emptyset \\ \dots \\ K_{n+1} := K_n \setminus \bigcup_{x_0 \in K_n^a} \overset{\circ}{B}(x_0, \varepsilon_{x_0}^n) \\ \text{where } \ B(x_0, \varepsilon_{x_0}^n) \cap \mathrm{Inv}_F(K) = \emptyset \\ \dots \end{cases}
$$

Of course, such sequence depends on the choice of $\varepsilon_{x_0}^n$. Also, since we do not know in advance the set $\mathrm{Inv}_F(K)$, we have to find a procedure which allows to determine $\varepsilon_{x_0}^n$ from the knowledge of K_n and F for all $n \geq 0$. Below, we suggest a particular choice of $\varepsilon_{x_0}^n$ which leads to the invariance kernel. Thanks to the very definition of the sequence K_n and Proposition 3.1, we have

Proposition 3.2 *Consider a sequence of closed subsets K_n, $n \geq 0$ satisfying (8). Set $K_\infty := \bigcap_{n>1} K_n$. Then,*

$$
\begin{array}{l} \mathrm{Inv}_F(K) \subset K_\infty \subset \ldots \subset K_{n+1} \subset K_n \subset \ldots \subset K_1 \subset K \\ \text{and} \\ \mathrm{Inv}_F(K) = \mathrm{Inv}_F(K_i) \ \text{ for } \ i \geq 1 \end{array}
$$

Below, for each n and for each $x_0 \in K_n^a$, we compute numbers $\varepsilon_{x_0}^n$ depending only on K_n and F. Let us introduce the Hausdorff semidistance between two closed set A and B (see [1] for instance).

$$
\delta(A,B) := \sup_{a \in A} d(a,B) = \sup_{x \in X} (d(x,B) - d(x,A)) = \sup_{a \in A} \inf_{b \in B} d(a,b)
$$

Then $\delta(A,B) = 0$ if and only if $A \subset B$. Let us notice that if A is compact and B closed there exists $u \in A$ such that $\delta(A,B) = d(u,B)$.

Proposition 3.3 *Let $x_0 \in K^a$ and $\varepsilon := \delta(F(x_0), T_K(x_0))$. Consider $u_0 \in F(x_0)$, such that $\delta(F(x_0), T_K(x_0)) = d(u_0, T_K(x_0))$. Define*

$$
(9) \qquad t_{max} := \sup\{\, t > 0 \mid (x_0 +]0,t](u_0 + \frac{\varepsilon}{2} B)) \cap K = \emptyset \,\}
$$

and set

$$
(10) \qquad t_{x_0} := \min\{\, t_{max}, \ \tfrac{\varepsilon}{4kM}, \ \tfrac{ln 2}{k} \,\}, \quad \varepsilon_{x_0}^0 := \frac{\varepsilon t_{x_0}}{8 e^{k t_{x_0}}}
$$

Then,

$$\mathrm{Inv}_F(K) \cap B(x_0, \varepsilon^0_{x_0}) = \emptyset$$

and furthermore,

$$\forall\, y_0 \in B(x_0, \varepsilon^0_{x_0}),\ \exists\, y(\cdot) \in S(y_0),\ d(y(t_{x_0}), K) \geq \frac{\varepsilon t_{x_0}}{8}$$

In order to prove this proposition, we need two results concerning the distance of a solution starting at $x_0 \in K^a$ from the set K. Let us denote by $S(x_0)$ the set of solutions to (1) starting at x_0 and defined on $[0, \infty[$.

Lemma 3.4 *Let x_0 belong to K^a. If there exist α, $\bar{t} > 0$ and $u_0 \in F(x_0)$ such that*

$$(x_0 +]0, \bar{t}\,](u_0 + \alpha B)) \cap K = \emptyset$$

then,

$$\forall\, t \in [0, \min(\bar{t}, \tfrac{\alpha}{2kM}, \tfrac{ln 2}{k})],\ \ \exists\, x(\cdot) \in S(x_0),\ \ d(x(t), K) \geq t\alpha/2$$

Proof — For any $t \in]0, \bar{t}[$, we have:

$$K \cap (x_0 + t(u_0 + \alpha B)) = \emptyset \tag{11}$$

Hence $x_0 + tu_0 \notin K + \alpha t B$ and therefore $d(x_0 + tu_0, K) \geq \alpha t$
Thanks to Filippov's Theorem (see Corollary 5.3.2 in [2]) there exists $x(\cdot) \in S(x_0)$ such that $x(0) = x_0$ and $x'(0) = u_0$ which satisfies

$$\|x(t) - x_0 - tu_0\| \leq \frac{\|u_0\|}{k}(e^{kt} - 1 - kt) \leq \frac{\|u_0\|}{2k} k^2 t^2 e^{kt} \leq \frac{1}{2} k t^2 M e^{kt}$$

Because $u_0 \in F(x_0) \subset MB$. Furthermore, for any $t \leq \bar{t}$,

$$t\alpha \leq d(x_0 + tu_0, K) \leq d(x(t), x_0 + tu_0) + d(x(t), K)$$

Hence $d(x(t), K) \geq \frac{\alpha t}{2}$ as soon as $\frac{1}{2} k t^2 M e^{kt} \leq \frac{\alpha t}{2}$. If $t \leq \frac{ln 2}{k}$, it is enough to have $t \leq \frac{\alpha}{2kM}$.

Q.E.D.

If a solution $x(\cdot)$ behaves as in the claim of Lemma 3.4, it is the case for at least one solution of $S(y_0)$, for any y_0 near x_0.

Lemma 3.5 *Let x_0 belong to K and $T > 0$. If there exists $x(\cdot) \in S(x_0)$ such that $d(x(T), K) \geq \alpha T/2$, then*

$$\forall\, y_0 \in B(x_0, \tfrac{\alpha T}{4e^{kT}}),\quad \exists\, y(\cdot) \in S(y_0),\quad d(y(T), K) \geq \tfrac{\alpha T}{4}$$

Proof — According to Filippov's Theorem ([8] or [2] corollary 5.3.3), there exist at least one solution $y(\cdot) \in S(y_0)$ such that:

$$\forall\, t \geq 0,\quad \|x(t) - y(t)\| \leq \|x_0 - y_0\| e^{kt}$$

But

$$d(y(T), K) + \|x(T) - y(T)\| \geq d(x(T), K) \geq \frac{\alpha T}{2}$$

Hence $d(y(T), K) \geq \frac{T\alpha}{4}$ as soon as $\|x_0 - y_0\| e^{kT} \leq \frac{\alpha T}{4}$. For this, it is enough to have $\|x_0 - y_0\| \leq \frac{\alpha T}{4} e^{-kT}$.

Q.E.D.

Thanks to lemmas 3.4 and 3.5, we shall determinate a radius $\varepsilon^0_{x_0}$ such that $\mathrm{Inv}_F(K) \cap B(x_0, \varepsilon^0_{x_0}) = \emptyset$, and consequently, we shall define the first step of our algorithm:

Proof of proposition 3.3 — Let us consider $x_0 \in K^a$, then $\varepsilon := \delta(F(x_0), T_K(x_0)) = d(u_0, T_K(x_0)) > 0$ hence

$$T_K(x_0) \cap]0, +\infty[(u_0 + \frac{\varepsilon}{2} B) = \emptyset$$

Since $F(x_0)$ is compact, by the very definition of the contingent cone, we can find a positive t satisfying:

$$K \cap (x_0 +]0, t](u_0 + \frac{\varepsilon}{2} B)) = \emptyset \tag{12}$$

We have defined t_{max}, the largest t (possibly equal to $+\infty$) satisfying (12). Thanks to lemma 3.4, we know that:

$$\forall\, 0 \leq t \leq t_{x_0} := \min(t_{max}, \tfrac{\varepsilon}{2kM}, \tfrac{ln2}{k}),\quad \exists\, x(\cdot) \in S(x_0),\quad d(x(t), K) \geq \tfrac{t\varepsilon}{4}$$

From lemma 3.5, we deduce that

$$\mathrm{Inv}_F(K) \cap B(x_0, \frac{\varepsilon t_{x_0}}{8e^{kt_{x_0}}}) = \emptyset$$

This is ending the proof of proposition 3.3.

Q.E.D.

Now, we have defined for each $x_0 \in K^a$, a positive number $\varepsilon^0_{x_0}$ and consequently the set K_1 by using (8). Clearly, K_1 is a closed subset of K. This and the induction argument allow us to define a decreasing sequence of closed sets.
Set $K_\infty := \bigcap_{n\geq 1} K_n$.

Remark — If the set K is convex, then, in Proposition 3.3, we can take $t_{max} = \infty$. □

4 Convergence of the algorithm

In the previous section, we have shown that algorithms defined by formula (8) lead to the inclusion $\mathrm{Inv}_F(K) \subset \bigcap_{n>1} K_n$. Thanks to proposition 3.3, we have choosen numbers $\varepsilon^n_{x_0}$ satisfying requirements of (8), namely $B(x_0, \varepsilon^n_{x_0}) \cap \mathrm{Inv}_F(K) = \emptyset$ for all $x_0 \in K^a_n$. Now we check that our algorithm converges to the invariance kernel, i.e. that $\mathrm{Inv}_F(K) = \bigcap_{n>1} K_n$.

Theorem 4.1 *Let K be a closed set and K_∞ be defined as in section 3. Then,*

$$\mathrm{Inv}_F(K) = K_\infty$$

Proof — By proposition 3.2 and the choice of $\varepsilon^n_{x_0}$, $\mathrm{Inv}_F(K) \subset K_\infty$. Let us assume, for a moment, that K_∞ is not an invariance domain, namely $K^a_\infty \neq \emptyset$. Pick x in K^a_∞ and set $\bar{\varepsilon} := \delta(F(x), T_{K_\infty}(x)) > 0$. Let us consider $u \in F(x)$ such that $d(u, T_{K_\infty}(x)) = \bar{\varepsilon}$. Let us define the following *finite* number:

$$\bar{t}_{max} := \sup\{\ t \in\ [0,1] \mid (x+]0,t](u + \frac{\bar{\varepsilon}}{2}B)) \cap K_\infty = \emptyset\ \} > 0$$

We shall state, thanks to a technical lemma that:

$$(13) \qquad \begin{cases} \exists\, N > 0, \text{ such that } \forall\, n > N, \\ \exists\, x_n \in K_n \cap (x + [0, \frac{1}{2}\bar{t}_{max}](u + (\bar{\varepsilon}/2)B)) \\ \text{satisfying } K_n \cap (x_n +]0, \frac{1}{2}\bar{t}_{max}](u + \frac{\bar{\varepsilon}}{2}B)) = \emptyset \end{cases}$$

For this aim, we need the following result, we have proved in [10]:

Lemma 4.2 *Let C be a convex closed cone and H be a compact subset of X. If C does not contain any whole line, then there exists $y \in H$ such that:*

$$(y + C) \cap H = \{ y \}$$

Since $x \in K_\infty^a$, $u \neq 0$, and the convex closed cone $C := \mathbb{R}_+(u + \frac{\bar{\varepsilon}}{2}B)$ does not contain any whole line .By setting $H := K_n \cap (x + [0, \bar{t}_{max}](u + \frac{\bar{\varepsilon}}{2}B))$, we can assert, thanks to Lemma 4.2:

$$\exists\, x_n \in H, \text{ such that } (x_n + C) \cap H = \{ x_n \}$$

On the other hand, by the very definition of K_∞ and the choice of x, the bounded sequence $(x_n)_n$ converges to x. Hence for all n large enough,

$$x_n \in x + [0, \frac{1}{2}\bar{t}_{max}](u + \frac{\bar{\varepsilon}}{2}B).$$

Thus,

$$\begin{aligned} &K_n \cap (x_n + [0, \tfrac{1}{2}\bar{t}_{max}](u + \tfrac{\bar{\varepsilon}}{2}B)) \subset \\ &K_n \cap (x + [0, \bar{t}_{max}](u + \tfrac{\bar{\varepsilon}}{2}B)) \cap (x_n + C) = (x_n + C) \cap H = \{ x_n \}. \end{aligned}$$

This is proving (13) and clearly $x_n \in \partial K_n$. For n large enough, as F is lipschitzean, we have $F(x) \subset F(x_n) + \frac{\bar{\varepsilon}}{4}B$, hence there exists $u_n \in F(x_n) \cap (u + \frac{\bar{\varepsilon}}{4})$ such that:

$$\forall\, t \in]0, \tfrac{1}{2}\bar{t}_{max}], \quad (x_n + t(u_n + \tfrac{\bar{\varepsilon}}{4}B)) \cap (K_n + \tfrac{t\bar{\varepsilon}}{4}B) = \emptyset.$$

Consequently, for any $t < \frac{1}{2}\bar{t}_{max}$,

$$\delta(x_n + t(F(x_n) + (\bar{\varepsilon}/4)B), K_n) \geq \bar{\varepsilon}t/4.$$

Thus, since $F(x_n)$ is compact, $\delta(F(x_n), T_{K_n}(x_n)) \geq \bar{\varepsilon}/2$, and since $0 \in T_{K_n}(x_n)$, we have also

$\delta(F(x_n), T_{K_n}(x_n)) \leq M$. Let us denote by $\bar{t} := \min\{ \frac{\bar{t}_{max}}{2}, \frac{\bar{\varepsilon}}{2kM}, \frac{ln2}{k} \}$. If $t^n_{x_n}$ is defined by (10) for the set K_n, then

$$\frac{\bar{t}}{2} \leq \min\{ \frac{1}{2}\bar{t}_{max}, \frac{\bar{\varepsilon}}{4kM}, \frac{ln2}{k} \} \leq t^n_{x_n} \leq \frac{1}{2k}.$$

Since the function $\sigma \mapsto \sigma e^{-k\sigma}$ is increasing for $\sigma \in [0, \frac{1}{k}]$, we can assert thanks to the definition of $\varepsilon^n_{x_n}$ (see (10) in proposition 3.3):

$$\frac{\bar{\varepsilon}\bar{t}}{32} e^{-k\frac{\bar{t}}{2}} \leq \varepsilon^n_{x_n}$$

By the very definition of K_{n+1}:

$$K_{n+1} \cap \overset{\circ}{B}(x_n, \frac{\bar{\varepsilon}\bar{t}}{32} e^{-k\frac{\bar{t}}{2}}) = \emptyset$$

Let us notice that $\frac{\bar{\varepsilon}\bar{t}}{32} e^{-k\frac{\bar{t}}{2}}$ does not depend on n. Consequently, since x belongs to K_{n+1}, the two following contradictory statements would hold:

$$\begin{cases} i) & \|x_n - x\| \geq \frac{\bar{\varepsilon}\bar{t}}{32} e^{-k\frac{\bar{t}}{2}} \\ ii) & \lim_{n \longrightarrow \infty} x_n = x \end{cases}$$

Q.E.D.

Let us notice that in [10], we provided a modified algorithm in the convex case, because for viability kernels[2] it is enough to check the tangent condition in extremal points. For the invariance property, this is not, in general, the case. But the same idea may be used if the set-valued map has the following "linear" property:

$$\begin{cases} \forall (x, y) \in K \times K, \ \ \forall \lambda \in [0, 1] \\ F(\lambda x + (1 - \lambda)y) = \lambda F(x) + (1 - \lambda)F(y) \end{cases}$$

In this case, we could modify the algorithm for conxex sets (see [10]).

[2]If K is convex and F is convex (i.e. its graph is convex).

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