Working Paper

On the Reconstruction of a Parameter of an Elliptic System

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Foreword

The problem considered consists of the reconstruction (restoration) of a parameter of an elliptic system based on the results of measuring its state. To solve this problem the method of dynamical approximation is used. The method was proposed by A.V. Kryazhimskii and the author, and is based on the ideas of the differential games theory and the ill-posed problems theory. The reconstruction algorithm presented here is stable with respect to the errors of measurements and is fairly constructive.

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1 Notation and preliminary discussion of the problem statement.

Let V, H, U be real Hilbert spaces with scalar products $(\cdot,\cdot)_V$, $(\cdot,\cdot)_H$, $(\cdot,\cdot)_U$ and norms $|\cdot|_V$, $|\cdot|_H$, $|\cdot|_U$, corresponding to them. Let V^* and H^* be the spaces dual to V and H respectively. Assume that V is densely and continuously imbedded into H. The spaces H and H^* are identified.

Let an operator $A : V \to V^*$ be given. It depends on a parameter $u \in P \subset U$:

$$A = A[u].$$

Assume that for every $u \in P$ the operator $A[u]$ is linear, continuous and self-conjugate. Denote by $a(u;\ \cdot,\cdot)$ the bilinear for some numbers $\lambda_1 > 0$, $\lambda_2 > 0$ and any $u \in P$

$$\lambda_2 \cdot |w|_V^2 \geq a(u; w, w) \geq \lambda_1 \cdot |w|_V^2, \quad w \in V.$$

Let an element $f \in V^*$ be given.

Consider an elliptic system ([14], p. 52-53)

$$A[u]y = f, \quad y \in V. \tag{1.1}$$

In order to emphasize that y is dependent on u we also write $y = y_u$ when necessary.

The problem is the following.

The parameter u in (1.1) is unknown. It is only known to belong to the convex bounded closed set $P \subset U$. It is necessary to construct an approximation u_h to the parameter u based on the approximation ξ to the solution y of the system (1.1) with error no larger than h:

$$|\xi - y|_V \leq h. \tag{1.2}$$

The approximation u_h should satisfy the condition

$$|u_h - u|_U \to 0 \text{ as } h \to 0.$$

2 Specification of the problem statement.

Let P_* be the set of all elements of P which generate the same solution $y = y_u$ of system (1.1) as the parameter u does

$$P_* = \{v \in P : A[v]y_u = f\}.$$

For an element $v \in P$ we put

$$J(v) = \inf_{u \in P_*} |v - u|_U.$$

The initial problem can now be formulated like this.

Problem 2.1. Construct an operator $D : (0, \infty) \times V \to P$ with a property

$$\sup_{\xi \in \Xi} J(D(h, \xi)) \to 0 \text{ as } h \to 0, \tag{2.1}$$

where Ξ is the set of all elements $\xi \in V$ satisfying (1.2).

An operator D satisfying the property (2.1) will be called a reconstruction algorithm.

Remark 2.1. The algorithm D reconstructs the unknown parameter u, if the set P_* consists of the only element u.

3 The finding of reconstruction algorithm D.

Consider an argument t (imaginary time) which varies in the interval $T = [t_0, \vartheta]$, $-\infty < t_0 < \vartheta < +\infty$. Below it is assumed that the operators $A[u]$ satisfy the following additional condition: if a sequence $\{v_k\}$ weakly converges in $L^2(T; P)$ to v_0 then for all $w \in V$ and $t \in T$ the sequence $\{\int_{t_0}^{t} A[v_k(\tau)] w d\tau\}$ converges in V^* to the element $\{\int_{t_0}^{t} A[v_0(\tau)] w d\tau\}$. This property implies, in particular, the weak compactness of P_*.

Fix $h > 0$ and $\xi \in V$ which satisfy (1.2).

Consider a partition of the interval T by points t_i:

$$t_0 < t_1 < \ldots < t_m = \vartheta, \; m = m(h), \; \Delta = \Delta(t) = \max_i (t_{i=1} - t_i) \le C \cdot h,$$

where $C > 0$ is a fixed number. The function $m = m(h)$ is also assumed to be fixed.

Consider a control system on T (this system will be called a model) which is described by relations:

$$\dot{z}(t) \;=\; A[v_h(t)]\xi \;-\; f, \quad z(t_0) \le t \le \vartheta. \tag{3.1}$$

Here $v_h(\cdot) : T \longrightarrow P$ is a piece-wise constant control formed according to the rule:

$$v_h(t) \;=\; v^*, \quad t_1 \le t < t_{i+1}, \;\; i = 0, \ldots, m-1, \tag{3.2}$$

where v_i^* is an element of P which corresponds to the minimal value of the functional $\Phi(v)$ on the set P,

$$\Phi(v) \;=\; -2 \cdot a(v; \xi, z(t_i) - \xi) \;+\; a(h) \cdot |v|_U^2,$$

$a(h)$ is a non-negative function on $(0, \infty)$ with a property:

$$a(t) \;\longrightarrow 0 \;\text{ and }\; h/a(h) \longrightarrow 0 \text{ as } h \longrightarrow 0$$

(according to the terminology of the ill-posed problems theory [12, 13] $a(h)$ can be called a regularization parameter). Note that there always exist the minimizing elements v_i^*.

Now D can be defined as the rule which to every pair (h, ξ) puts into correspondence the element u_h:

$$u_h \;=\; \frac{1}{\vartheta - t_0} \cdot \int_T v_h(t)dt \in P. \tag{3.3}$$

Theorem 3.1. *If the set P_* is convex, then the operator D solves the Problem 2.1.*

Proof. To prove the theorem it suffices to show that, whatever the sequences $\{h_k\}$ $(h_k > 0, h_k \to 0)$, $\{\xi_k\}$ $(\xi_k \in V,\ |\xi_k - y|_V \leq h_k)$, the following relation holds

$$J(D(h_k, \xi_k)) \ \to\ 0 \text{ as } k \to \infty.$$

Fix arbitrary sequences $\{h_k\}$ and $\{\xi_k\}$, which satisfy the above-mentioned properties and denote by $v_k = v_{h_k}$, $u_k = D(h_k, \xi_k), z_k(\cdot)$ the solution of the model (3.1)-(3.2),

$$z_k(t) \ = \ \xi_k \ + \int_{t_0}^{t} (A[v_k(\tau)]\xi_k \ - \ f)d\tau, \quad t_0 \leq t \leq \vartheta,$$

$$\Lambda_k(t) \ = \ |y - z_k(t)|^2_{V^*} + a(h_k) \ \cdot \int_{t_0}^{t} |v_k(\tau)|^2_U \, d\tau \ - \ a(h_k) \cdot \int_{t_0}^{t} |u_*|^2_U \, d\tau,$$

u_* is an arbitrary element of P_* whose norm is minimal.

Taking into account the rule, according to which the function v_k is formed, one can obtain the following estimate for the functional Λ_k

$$\max_{t_0 \leq t \leq \vartheta} \Lambda_k(t) \ \leq \ \gamma_1 \cdot h_k,$$

where $\gamma > 0$ is some number which depends on the known parameters of the problem, but not on number k. In particular, this estimate implies

$$\max_{t_0 \leq t \leq \vartheta} |y - z_k(t)|^2_{V^*} \ \leq \ \gamma_1 \cdot h_k \ + \ 2 \cdot b \cdot (\vartheta - t_0) \ \cdot a(h_k), \qquad (3.4)$$

$$b \ = \ \max_{u \in P} \ |u|^2_U$$

$$\int_{t_0}^{\vartheta} |v_k(\tau)|_U^2 d\tau \;\leq\; \int_{t_0}^{\vartheta} |u_*|_U^2 d\tau \;+\; \gamma_1 \cdot h_k/a(h_k).$$

Choose a subsequent $\{v_k\}$ of the sequence $\{v_{k_j}\}$, which provides the upper limit $\overline{\lim}_{k\to\infty} J(u_k)$ and weakly converges in $L^2 = L^2(T;P)$ to some element $v_* \in L^2$ (v_k and $u_k = D(h_k, \xi_k)$ are connected by (3.3)). From the above-mentioned properties of operators A and estimate (3.4) it follows for all $t \in T$ that

$$z_{k_j}(t) \;\rightarrow\; y \text{ in } V^*,$$

$$z_{k_j}(t) \;\rightarrow\; z_*(t) \;=\; y \;+\; \int_{t_0}^{t} (A[v_*(\tau)]y \;-\; f)d\tau \;\text{ in }\; V^*.$$

Hence, the uniqueness of the limit gives an equality

$$\int_{t_0}^{t} (A[v_*(\tau)]y \;-\; f)d\tau \;=\; 0, \quad t \in T,$$

which implies that $v_*(t) \in P_*$ for almost all $t \in T$ and

$$\int_{t_0}^{\vartheta} |u_*|_U^2 d\tau \;\leq\; \int_{t_0}^{\vartheta} |v_*(\tau)|_U^2 d\tau.$$

Thus,

$$\int_{t_0}^{\vartheta} |v_{k_j}(\tau)|_U^2 \, d\tau \;=\; |v_{k_j}|_{L^2}^2 \;\rightarrow\; |v_*|_{L^2}^2 \;=\; |u_*|_{L^2}^2,$$

and so,

$$v_{k_j} \rightarrow v_* \text{ in } L^2,$$

$$u_{k_j} = \frac{1}{\vartheta - t_0} \cdot \int_T v_{k_j}(\tau)d\tau \rightarrow \bar{v}_* = \frac{1}{\vartheta - t_0} \cdot \int_T v_*(\tau)d\tau \text{ in } U.$$

Taking into account the convexity of the set P_* we obtain

$$\bar{v}_* \in P_*$$

(more than that, it is easy to see that $|\bar{v}_*|_U = |u_*|_U$). Consequently, $J(u_{k_j}) \rightarrow 0$. The theorem is prooved.

Remark 3.1. If the set P_* is not convex, then in the general case one can guarantee the convergence

$$\min_{u \in U_*} \int_T |v_h(t) - u(t)|_U^2 dt \rightarrow 0,$$

where $U_* = \{u \in L^2(T;P) : u(t) \in P_*, \text{ for almost all } t \in T\}$.

Remark 3.2. The properties of the algorithm D can be characterized by the following estimates:

$$\sup_{u \in P_*} \sup_{t \in T} \left(\int_{t_0}^{t} |v_h(\tau)|_U^2 d\tau - \int_{t_0}^{t} |u|_U^2 d\tau\right) = \leq \gamma_2 \cdot h/a(h),$$

$$\sup_{u \in P_*} \sup_{t \in T} \left|\int_{t_0}^{t} A[v_h(\tau)]y d\tau - \int_{t_0}^{t} A[u]y d\tau\right|_{v^*} \leq \gamma_3 \cdot \sqrt{h + a(h)},$$

where $\gamma_2 > 0$ and $\gamma_3 > 0$ are constants dependent only on the known parameters of the initial problem, but are not dependent on h. In particular, if

$$|\int_{t_0}^{\vartheta} A[v_h(\tau)]yd\tau - \int_{t_0}^{\vartheta} A[u]yd\tau]_{V^*} \geq \mu \cdot |v_h(\cdot) - u|_{L^2},$$

where $\mu > 0$ is some constant (probably, dependent of y), then

$$\mu \cdot |v_h - u|_U \leq \sqrt{\vartheta - t_0} \cdot \gamma_3 \cdot \sqrt{h + a(h)}.$$

Remark 3.3. When dealing with concrete problems of restoration of coefficients in elliptic systems, one can often take for U the space of function defined on the corresponding domain Ω, where the space variables of the elliptic operator take values, which can be imbedded into the space of continuous functions on Ω. Then Theorem 3.1 implies the uniform convergence of the coefficients being restored.

Remark 3.4. In the case when the set P_* does not consist of the one element the algorithm D provides the convergence of the elements u_h to the elements of minimal norm of the set P_*.

Remark 3.5. Let M be a compact subset of P. Assume that for every $u \in M$ the set $P_*(u) = \{v \in P;\ A[v]y_u = f\}$ is convex, and $\Xi(h, u) = \{\xi \in V; |\xi - y_u|_V \leq h\}$. Then

$$\sup_{u \in M} \sup_{\xi \in \Xi(h,u)} \inf_{v \in P_*(u)} |v - u_h|_U \to 0 \text{ as } h \to 0.$$

4 An example.

Consider an elliptic system

$$
\begin{cases}
\frac{\partial}{\partial x_1}(u_1(x_1,x_2)\cdot\frac{\partial y}{\partial x_1}) + 2\cdot\frac{\partial^2 y}{\partial x_1 \partial x_2} + \frac{\partial}{\partial x_2}(u_2(x_1,x_2))\cdot\frac{\partial y}{\partial x_2} = -1 \\
0 < x_1 < 1, \;\; 0 < x_2 < 1 \\
y(0,x_2) = 0 = y(1,x_2), \;\; 0 \le x_2 \le 1, \\
y(x_1,0) = 0 = y(x_1,1), \;\; 0 \le x_1 \le 1.
\end{cases}
$$

The unknown coefficients to be restored are

$$
u_1 = 1 = x_2\cdot(1-x_2), \quad u_2 = 1 + x_1\cdot(1-x_1).
$$

The approximations are taken equal to

$$
\xi(x_1,x_2) = y(x_1,x_2) + h \cdot \frac{\sin kx_1}{k}\cdot\frac{\sin lx_2}{l},
$$

where k and l are some numbers. Take the regularization parameter in the form $a = \sqrt{h}$. Let

$$
U = W_2^1 \times W_2^1 \quad (W_2^1 \text{ is the S.L. Sobolev space }),
$$

$$
P = \{(u_1,u_2) \in U: \;\; 1 \le u_1(x_1,x_2) \le 3, \;\; 1 \le u_2(x_1,x_2) \le 3,
$$

$$
\left|\frac{\partial u_1(x_1,x_2)}{\partial x_1}, \frac{\partial u_1(x_1,x_2)}{\partial x_2}, \frac{\partial u_2(x_1,x_2)}{\partial x_1}, \frac{\partial u_2(x_1,x_2)}{\partial x_2}\right| \le 1\}.
$$

The calculation results are shown in the figures. The numbers 1, 2, 3 denote the results of restoration for $h = 0.5$, $h = 0.2$, $h = 0.02$ respectively. Figure 1 shows the dynamics of discrepancy $r(t) = |u - v_h(t)|_C$. The results of reconstruction of coefficients u_1 and u_2 are depicted in Figure 2 and 3 respectively.

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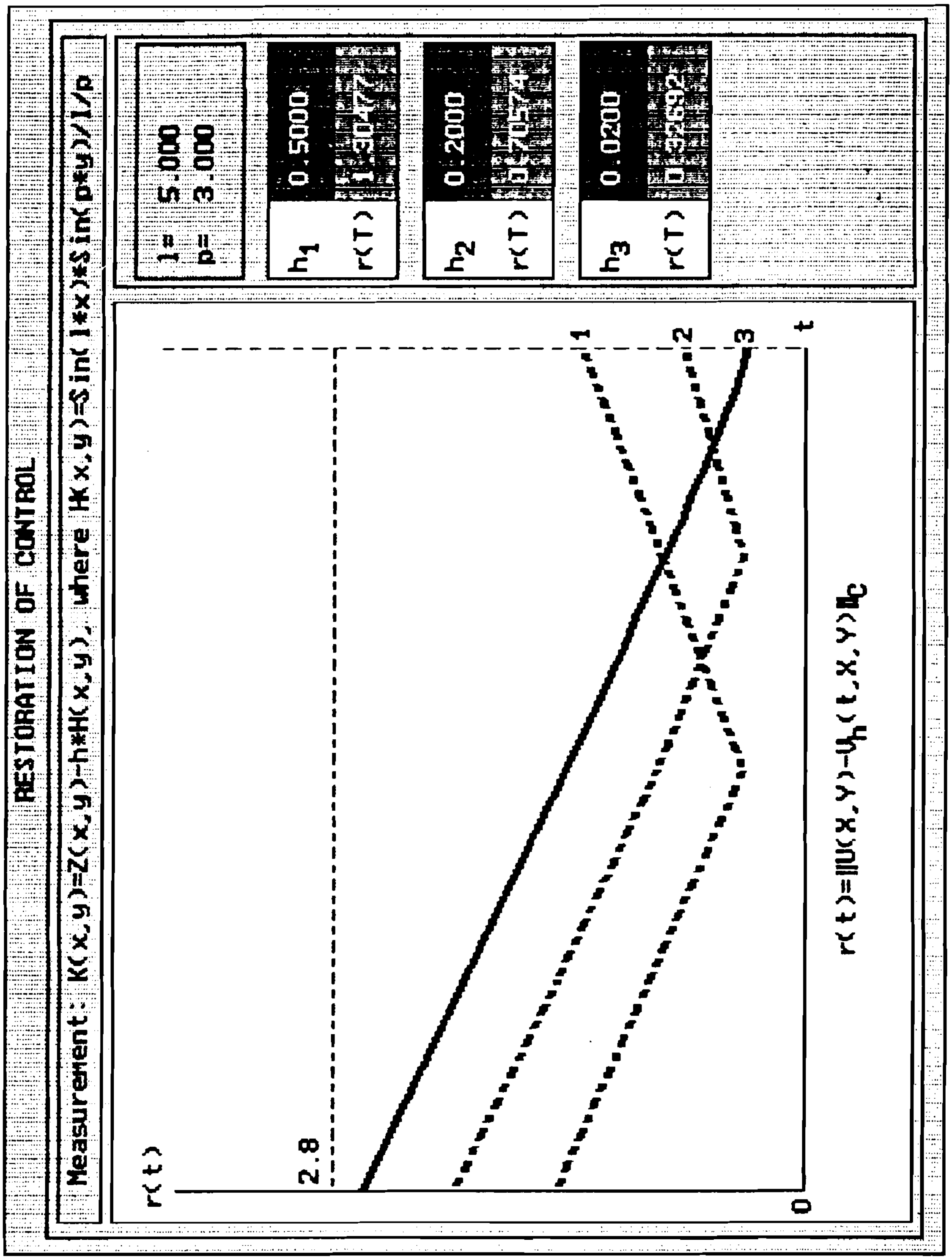
RESTORATION OF CONTROL
Measurement: K(x,y)=Z(x,y)-h*H(x,y), where H(x,y)=Sin(l*x)*Sin(p*y)/l/p
l= 5.000
p= 3.000
h1 0.5000
r(T) 1.30477
h2 0.2000
r(T) 0.70574
h3 0.0200
r(T) 0.32692
r(t)
2.8
0
t
1
2
3
r(t)=||U(x,y)-U_h(t,x,y)||_C

Figure 1

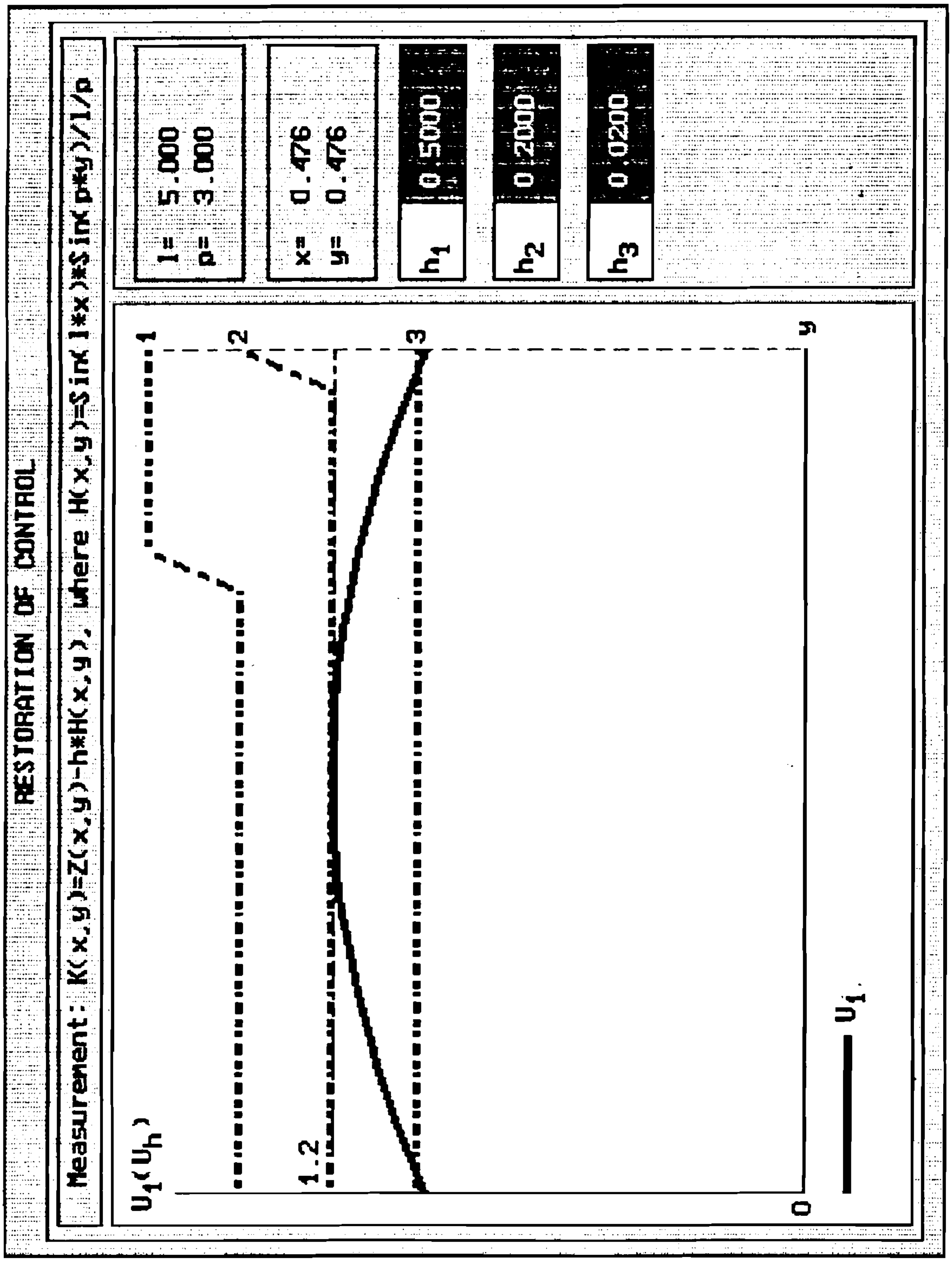
RESTORATION OF CONTROL
Measurement: K(x,y)=Z(x,y)-h*H(x,y), where H(x,y)=Sin(l*x)*Sin(p*y)/l/p
l= 5.000
p= 3.000
x= 0.476
y= 0.476
h1 0.5000
h2 0.2000
h3 0.0200
U1(Uh)
1.2
0
y
1
2
3
Ui

Figure 2

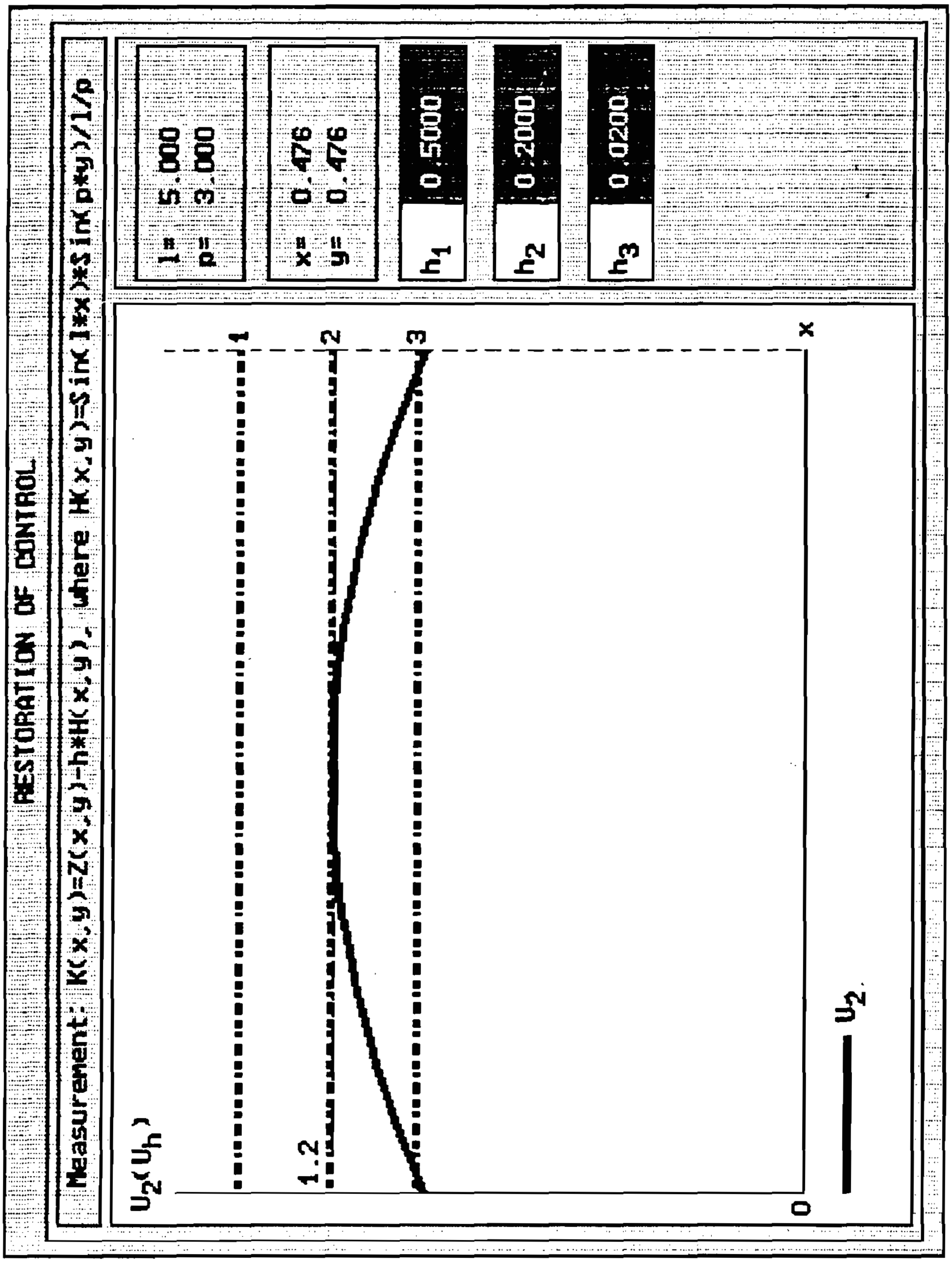
RESTORATION OF CONTROL
Measurement: K(x,y)=Z(x,y)-h*H(x,y), where H(x,y)=Sin(l*x)*Sin(p*y)/l/p
l= 5.000
p= 3.000
x= 0.476
y= 0.476
h1 0.5000
h2 0.2000
h3 0.0200
U2(Uh)
1.2
1
2
3
0
x
U2

Figure 3