Working Paper

Observability of Parabolic Systems with Interior Observations of Discrete Type

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WP-92-45
July 1992

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Foreword

The paper deals with the comparison of the quality of continuous-in-time and discrete-in-time observations for a distributed parameter system of a parabolic type. A direct method of constructing the discrete observations that preserve the property of continuous observability is given, based on the "sensitivity" points of an associated system with observations that are continuous in time.

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Key words. observability, distributed parameter systems, sensors, discrete observations.

AMS(MOS) subject classifications. primary 93B07; secondary 93C20.

1. Introduction, Statement of Problem.

We consider the following homogeneous problem for the parabolic equation:

$$\frac{\partial u(x,t)}{\partial t} = A(t)u(\cdot,t)\,, \tag{1.1}$$

$$t \in T = (0,\theta)\,,\ x \in \Omega \subset R^n\,,\ \ n \leq 3,$$

$$Q = \Omega \times T\,,\ \Sigma = \partial\Omega \times T,$$

$$u(\xi,t) = 0,\ \ \xi \in \partial\Omega,\ \ t \in T, \qquad u(x,0) = u_0(x),\ \ u_0(\cdot) \in L^2(\Omega)$$

with an unknown initial condition $u_0(x)$.

In the above Ω is a simply-connected open bounded domain with sufficiently smooth boundary $\partial\Omega$ ($\partial\Omega \in C^2$) and an operator $A(t)$ satisfies the condition of uniform parabolicity, namely,

$$A(t) \;=\; \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x,t)\frac{\partial}{\partial x_j} + a_i(x,t)) - \sum_{i=1}^{n} b_i(x,t)\frac{\partial}{\partial x_i} - a(x,t),$$

$$v_1 \sum_{i=1}^{n} \xi_i^2 \;\leq\; \sum_{i,j=1}^{n} a_{ij}\,(x,t)\,\xi_i\xi_j \;\leq\; v_2 \sum_{i=1}^{n} \xi_i^2 \quad \text{for a.e. } \ \xi_i \in R,$$

$$v_1 = \text{const } > 0,\ v_2 = \text{const } > 0.$$

Assuming that the coefficients in (1.1) are sufficiently regular, we will treat the solution of the initial-boundary value problem (1.1) as a generalized one [9, 10] from the Sobolev space $H_0^{2,1}(Q)$, with [15]

$$H_0^{2,1}(Q) = \{\varphi \mid \quad \varphi, \ \frac{\partial \varphi}{\partial t}, \ \frac{\partial \varphi}{\partial x_i}, \ \frac{\partial^2 \varphi}{\partial x_i \, \partial x_j} \in L^2(Q)\}.$$

In physical situations the space of observations is finite-dimensional. Consequently, we assume that all the measurement data on the actual state of the system (1.1) are scalar values and, furthermore, they are available at the discrete times

$$t_1, \ \ldots, \ t_k, \ldots \ \in (0, \theta).$$

We set, therefore,

$$y_k \ = \ \mathbf{G}(t_k) u(\cdot, \cdot), \quad k = 1, \ldots, \tag{1.2}$$

where $y_k, \quad k = 1, \ldots$ are data and the operator $\mathbf{G}(t_k)$ describes the structure of observations.

Below we shall consider a measurement data of two types.

We begin with the discrete-time pointwise observations when the measurements are taken at some spatial points of the domain Ω.

Let $\{x^1, \ldots, x^k, \ldots\}$ be a sequence of spatial points of the domain Ω. Then we set

$$y_k \ = \ u(x^k, t_k), \quad k = 1, \ldots . \tag{1.3}$$

The following type of observations describes the spatially-pointwise time-averaged *(but available at discrete times)* observations [12]:

$$y_k = \frac{1}{\tau_k} \int_{t_k - \tau_k}^{t_k} u\,(x^k, t) dt, \quad k = 1, \ldots . \tag{1.4}$$

when the measurement data are quantities of the solution $u(x, t)$, taken at the spatial points $x^k, k = 1, \ldots$ and time-averaged over the intervals

$$[t_k - \tau_k, t_k] \in T, \ \ k = 1, \ldots,$$

with $\tau_k, \ \ k = 1, \ldots$ being given.

To ensure the meaning of the values in the right-hand sides of (1.3), (1.4) we require a proper regularity of solutions to the mixed problem (1.1).

If the dimension n of the spatial variable x is equal to 1, due to embedding theorems [9, 10, 15] we have

$$H^1(\Omega) \ \subset \ C(\bar{\Omega})$$

and the observations (1.4) are well-defined.

For the case of $n = 2,3$ applying the embedding theorems (namely, $H^2(\Omega) \subset C(\bar{\Omega})$, [9, 10, 15]) again ensures the meaning for all the values in (1.4).

For sake of simplicity, when working with the observations of type (1.3), (1.4) we shall assume below that any solution to the system (1.1) is continuous function in t and x, when $t > 0$ (see [3, 9, 10]).

Definition 1.1.([6,4], see [14, 1, 2]) The system (1.1), (1.2) is ***observable***, if a final (or initial) state of the system can be uniquely determined from the observations $y_1, y_2, \ldots$.

We remark here that ***observability*** indicates, in fact, the existence of some one-to-one operator.

However, the infinite - dimensional nature of distributed - parameter systems generates various definitions of observability that are determined by the topological structure of the problem (1.1) - (1.2).

Denote by

$$\mathbf{y} = (y_1, \ldots, y_k, \ldots)$$

the sequence of all the measurement data. Then we can rewrite the equation (1.2) in the form

$$\mathbf{y} = \mathbf{G}u(\cdot,\cdot), \tag{1.5}$$

where

$$\mathbf{G}u(\cdot,\cdot) = (G(t_1)u(\cdot,\cdot), \ldots G(t_k)u(\cdot,\cdot), \ldots).$$

Let us assume that the space of outputs $\mathbf{y}$ for the problem of (1.1), (1.5). is a Banach space and denote the latter by B_d .

Definition 1.2.(see [14, 1, 2]) The system described by (1.1) and (1.2)((1.5)) is said to be ***continuously observable*** at final time $t = \theta$, if

$$\exists \gamma_d > 0 \text{ such that } \| \mathbf{G}u(\cdot,\cdot) \|_{B_d} \geq \gamma_d \| u(\cdot,\theta) \|_{L^2(\Omega)} \tag{1.6}$$

for any solution $u(x,t)$ of the system (1.1).

It is clear that Def.1.2 implies observability.

Below we shall consider the case when $B_d = l^\infty$, so that

$$\| \mathbf{y} \|_{l^\infty} = \sup_{k=1,\ldots} | y_i | \ .$$

We remark also that all the results below may be extended to the more "narrow" space $B_d = l^2$, where the Hilbert space l^2 has its standard meaning.

The case of stationary discrete-time observations of type (1.3) (when $x^k = \bar{x},\ k = 1, \dots$) in infinite and finite time horizon has been considered in [4, 6] on the basis of the theory of harmonic analysis. For the stationary parabolic system the authors have derived the necessary and sufficient conditions for observability (in the sense of Definition 1.1).

One of the problems that naturally arises here is the comparison of the quality of continuous-time and discrete-time observations. We propose a method of constructing the pairs $\{x^k, t_k\}_{k=1}^{\infty}$ for observations that ensure an approximation (optimal in the sense that will be specified below) of the continuous-in-time measurements by the discrete ones.

Consequently, the latter enables us to derive (on the basis of results for an associated system under continuous-in-time observations) sufficient condition for continuous observability (in finite time horizon) for the nonstationary system (1.1) in the case of discrete observations (1.3), (1.4). It, in turn, supplies us with the observability property at final time.

In Section 3 we illustrate the main result for an example of one dimensional heat equation under stationary observations.

2. Interrelation Between Continuous-in-Time and Discrete Observations.

Let $\bar{x}(t) \in \Omega,\ t \in T$ denote a spatial piecewise continuous trajectory in the domain Ω.

Assume that the measurement data are defined by a moving sensor (it is stationary, if $\bar{x}(t) \equiv \bar{x}$, as well), so as

$$y(t) = u(\bar{x}(t), t),\ t \in [\varepsilon, \theta] = T_\varepsilon, \tag{2.1}$$

where a scalar function $y(t),\ t \in T$ is a measurement data and ε is given, $\varepsilon > 0$.

The system described by (1.1) and (2.1) is said [14, 1, 2] to be continuously observable at final time $t = \theta$, if

$$\exists \gamma_c > 0 \text{ such that } \| u(\bar{x}(\cdot), \cdot) \|_{B_c} \geq \gamma_c \| u(\cdot, \theta) \|_{L^2(\Omega)} \tag{2.2}$$

for any solution $u(\cdot, \cdot)$ of the system (1.1).

In the above B_c is a Banach space of outputs of the system (1.1), (2.1). In accordance with (1.6) we set $B_c = L^\infty(T_\varepsilon)$ or, when it is possible (namely, when $\bar{x}(t)$ is continuous), $B_c = C(T_\varepsilon)$.

Assume now that the measurement curve $\bar{x}(t)$ at the instants t_i, $i = 1, \ldots$ passes through the points x^1, x^2, $\ldots$, so as

$$\bar{x}(t_k) = x^k, \quad k = 1, \ldots . \tag{2.3}$$

We note easily that

$$\| u(\bar{x}(\cdot), \cdot) \|_{L^\infty(T_\varepsilon)} \leq 1, \tag{2.4}$$

implies

$$| u(x^k, t_k) | \leq 1, \; k = 1, \ldots . \tag{2.5}$$

Combining (2.4) and (2.5) leads to

Lemma 2.1. Let $B_c = L^\infty(T_\varepsilon)$, $B_d = l^\infty$ and (2.3) be fulfilled. If the system (1.1), (1.3) is continuously observable at final time with the constant γ_d, then the system (1.1), (2.1) is also continuously observable at final time and $\gamma_c \geq \gamma_d$.

We proceed now to the method of constructing the measurements of discrete type that enable to preserve both observability and continuous observability at final time of the system (1.1) under observations of continuous-in-time type (2.1) (if the latter exists).

Denote by $U_{1\varepsilon}(\cdot)$ a set of all those solutions to the system (1.1) on the time interval $[\varepsilon, \theta]$ that have a unit norm (in the space $L^2(\Omega)$) at the instant θ, so as

$$U_{1\varepsilon}(\cdot) = \{u(x,t) \mid \; x \in \Omega, \; t \in T_\varepsilon, \; \| u(\cdot, \theta) \|_{L_2(\Omega)} = 1\}.$$

In turn, due to (2.1) the latter generates a set of outputs $Y_{1\varepsilon}(\cdot)$, so as

$$Y_{1\varepsilon}(\cdot) = \{z(\cdot) \mid z(t) = u(\bar{x}(t), t), \; t \in T_\varepsilon, \; u(\cdot, \cdot) \in U_{1\varepsilon}(\cdot)\}.$$

We note next that for any positive number δ we may select in $U_{1\varepsilon}(\cdot)$ a denumerable δ-net $U^\delta_{1\varepsilon}(\cdot)$ in the norm of $C(\bar{\Omega} \times T\varepsilon)$, so as

$$U^\delta_{1\varepsilon}(\cdot) = \{u_j(\cdot, \cdot) \in U_{1\varepsilon}(\cdot), \; j = 1, \ldots\}.$$

The latter means that for any solution $u(\cdot, \cdot)$ of the system (1.1) on the time interval T_ε there exists a number j^*, such that

$$| u(x,t) \; - \; \nu u_{j^*}(x,t) \; | \leq \; \delta\nu, , \;\; \forall x \in \bar{\Omega}, \;\; \forall t \in T_\varepsilon, \;\; \| u(\cdot, \theta) \|_{L^2(\Omega)} = \nu. \tag{2.6}$$

Consequently, we obtain

$$| u(\bar{x}(t), t) \; - \; \nu \, u_{j^*}(\bar{x}(t), t) \; | \leq \; \delta\nu, \quad \forall t \in T_\varepsilon. \tag{2.7}$$

Let us consider now a series (over $k = 1, \ldots$) of optimization problems as follows:

$$| u_k(\bar{x}(t),t) | \rightarrow \sup, \quad t \in T_\varepsilon. \tag{2.8}$$

Since the function $\bar{x}(t)$ might, in general, be piecewise continuous, the solution of the problem in the above not always exists.

However, for any positive value β we may designate by $t_1^*,\ t_2^*, \ldots,\ t_k^*, \ldots$ some sequence of instants of time such that

$$|\ | u_k(\bar{x}(t_k^*),t_k^*) | - \sup_{t\in T_\varepsilon} | u_k(\bar{x}(t),t) |\ | \leq \beta, \quad k = 1,\ldots\ . \tag{2.9}$$

Furthermore, in the case of measurements (1.4), we may select the values of $\tau_k^*,\ k = 1, \ldots$ in such a way that

$$|\ | \frac{1}{\tau_k^*} \int\limits_{t_k^*-\tau_k^*}^{t_k^*} u(\bar{x}(t_k^*),t_k^*)dt \ | \ - \ \sup_{t\in T_\varepsilon} | u_k(\bar{x}(t),t) |\ | \leq \beta,\ t_k^* - \tau_k^* \in T_\varepsilon,\ k = 1,\ldots, . \tag{2.10}$$

We note that some of the instants $\{t_k^*\}_{k=1}^{\infty}$ may coincide, but this can only reduce the number of measurements.

Set in the measurement equations (1.3) and (1.4)

$$t_1 = t_1^*,\ t_2 = t_2^*, \ldots,\ t_k = t_k^* \ldots ; \quad x^1 = \bar{x}(t_1^*),\ x^2 = \bar{x}(t_2^*), \ldots,\ x^k = \bar{x}(t_k^*), \ldots\ , \tag{2.11}$$

$$\tau_1 = \tau_1^*,\ \tau_2 = \tau_2^*, \ldots,\ \tau_k^*, \ldots . \tag{2.11*}$$

Now our aim is to evaluate the value of continuous output of the system (1.1), (2.1) on the basis of an associated (due to (2.11), (2.11*)) discrete output of the system (1.1), (1.3) or (1.4).

Consider first the case of pointwise observations of type (1.3).

Let us take an arbitrary solution $u^*(\cdot,\cdot)$ of the system (1.1) and assume that the following condition is fulfilled:

$$\sup_{k=1,\ldots} | u^*(x^k,t_k) | \leq 1. \tag{2.12}$$

Denote

$$\| u^*(\cdot,\theta) \|_{L^2(\Omega)} = \nu.$$

Let j_* be an index of an element of the net $U_{1\varepsilon}^{\delta}(\cdot)$ that corresponds to $u^*(\cdot,\cdot)$. Then, via (2.7), we come to

$$| u^*(\bar{x}(t),t) - \nu u_{j_*}(\bar{x}(t),t) | \leq \nu\delta, \quad \forall t \in T_\varepsilon. \tag{2.13}$$

Due to (2.8)-(2.11), we obtain next

$$\sup_{t\in T_\varepsilon} | u^*(\bar{x}(t),t) | \leq \nu \sup_{t\in T_\varepsilon} | u_{j_*}(\bar{x}(t),t) | + \nu\delta \ \leq \ \nu \,| u_{j_*}(x^{j_*},t_{j_*}) | + \nu(\delta+\beta). \tag{2.14}$$

Combining (2.14) and (2.12) yields

$$\sup_{t\in T_\epsilon} \mid u^*(\bar{x}(t),t) \mid \leq 1+\nu(2\delta+\beta). \tag{2.15}$$

On the other hand, assuming that the system (1.1), (2.1) is continuously observable, we obtain

$$\nu = \parallel u^*(\cdot,\theta) \parallel_{L^2(\Omega)} \leq \frac{1}{\gamma_c} \sup_{t\in T_\epsilon} \mid u^*(\bar{x}(t),t) \mid . \tag{2.16}$$

Finally, if the parameters δ and β are sufficiently small, so as

$$2\delta+\beta<\gamma_c, \tag{2.17}$$

we obtain the estimate

$$\sup_{t\in T_\epsilon} \mid u^*(\bar{x}(t),t) \mid \leq \frac{1}{1-\gamma_c^{-1}(2\delta+\beta)} . \tag{2.18}$$

Thus, we have proved

Theorem 2.1. Let the system (1.1), (2.1) be continuously observable at final time in the sense (2.2) and let the discrete observations of type (1.3) be constructed along the relations (2.11), (2.17) on the basis of the measurement trajectory $\bar{x}(t)$ from (2.1). Then, if the discrete output (1.3) of some solution $u(\cdot,\cdot)$ to the system (1.1) satisfies the estimate (2.12), the latter has due to (2.1) a continuous-in-time output that satisfies the estimate (2.18).

In the case of observations of type (1.4) we come to the same estimate after slight modification of the formula (2.14) on the basis of the relation (2.10) instead of (2.9) and the condition

$$\sup_{k=1,\dots} \mid \frac{1}{\tau_k} \int_{t_k-\tau_k}^{t_k} u(\bar{x}(t_k),t_k)dt \mid \leq 1 \tag{2.19}$$

substituted for (2.12).

Theorem 2.2. Let the system (1.1), (2.1) be continuously observable at final time in the sense (2.2) and let the discrete observations of type (1.4) be constructed along the relations (2.11), (2.11*), (2.17) on the basis of the measurement trajectory $\bar{x}(t)$ from (2.1). Then, if the discrete output (1.4) of some solution $u(\cdot,\cdot)$ to the system (1.1) satisfies the estimate (2.19), the latter has due to (2.1) a continuous-in-time output that satisfies the estimate (2.18).

Theorems 2.1, 2.2 provide us with the estimate

$$\parallel u(\bar{x}(\cdot),\cdot) \parallel_{B_c} \leq \frac{1}{1-\gamma_c^{-1}(2\delta+\beta)} \parallel \mathbf{G}u(\cdot,\cdot) \parallel_{l^\infty} \tag{2.20}$$

under condition of continuous observability of the system (1.1), (2.1).

Consequently, combining (2.20), (2.2) and (2.17) yields the following estimate

$$\| \mathbf{G}u(\cdot,\cdot) \|_{l^\infty} \geq \gamma_c \left(1 - \gamma_c^{-1}(2\delta + \beta)\right) \| u(\cdot,\theta) \|_{L^2(\Omega)} \tag{2.21}$$

for any solution $u(\cdot,\cdot)$ of the system (1.1).

Theorem 2.3. Let the discrete observations of type (1.3) or (1.4) be constructed along the relations (2.11)-(2.11*), (2.17) on the basis of the measurement trajectory $\bar{x}(t)$ from (2.1). Then the transition from the continuous-in-time observations of type (2.1) to the observations of discrete type (1.3) or (1.4) preserves the property of continuous observability at final time with the constant

$$\gamma_d = \gamma_d(\beta,\delta) = \gamma_c \left(1 - \gamma_c^{-1}(2\delta + \beta)\right). \tag{2.22}$$

Remark 2.1. For given constant γ_d^* the procedure (2.8)-(2.11*) of constructing the observations of discrete type on the basis of continuous-in-time ones may be considered in some sense as optimal. Indeed, the omitting even one of the measurement instants might increase the value of γ_d^*. However, we remark that the selection of the measurement points (2.11), (2.11*) is non-unique.

Remark 2.2. The sequence $\{t_k^*\}_{k=1}^{\infty}$ in (2.11) admits, in general, the instant $t = \varepsilon$ as a limit point. In the case when the value of γ_c does not depend upon the length of the interval of observations (see [5]) the procedure (2.8)-(2.11) can be modified for constructing a sequence of measurements of discrete type on the time interval T_ε that ensures (2.22) and contains the minimal in time (the first) measurement instant.

3. Continuous Observability under the Stationary Discrete-Time Observations.

Theorem 2.3 allows us to construct the measurements of discrete type (1.3), (1.4) in finite time horizon that make the system (1.1), (1.2) be continuously observable at final time.

In this section we apply the latter for the case of stationary pointwise observations.

Let us consider the one-dimensional heat equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 < x < 1, \quad t \in T, \tag{3.1}$$

$$u(t,0) = u(t,1) = 0, \quad u(x,0) = u_0(x)$$

under stationary continuous-in-time observations

$$y(t) = u(\bar{x}, t), \;\; t \in T_\varepsilon. \tag{3.2}$$

In the above

$$\bar{x}(t) \equiv \bar{x}, \quad t \in T_\varepsilon.$$

The system (3.1), (3.2) was studied by various authors (see [14, 1, 2]) and the sufficient and necessary conditions for both observability and continuous observability at final time were established. Below we assume that the latter are fulfilled, so as $\bar{x}$ is an irrational number of special type.

Our aim here to construct a discrete pointwise observations of type (1.3), namely,

$$y_k \;=\; u(\bar{x}, t_k), \;\; k = 1, \ldots \tag{3.3}$$

that make the system (3.1), (3.3) be continuously observable at $t = \theta$.

Hence, we have to specify the sequence of measurement instants $\{t_k\}_{k=1}^{\infty}$.

It is well-known that the eigenvalues and the (orthonormalized) eigenfunctions for problem (3.1) are as follows

$$\lambda_k = -(\pi k)^2, \omega_k(x) = \sqrt{2}\,\mathrm{Sin}\;\pi k x, \quad k = 1, 2, \ldots .$$

Expanding the solution of system (3.1), (3.2) in a series of exponentials, we obtain the following description for the set $U_{1\varepsilon}(\cdot)$:

$$U_{1\varepsilon}(\cdot) \;=\; \{u(\cdot,\cdot) \mid \; u(x,t) =$$

$$= \sqrt{2}\sum_{k=1}^{\infty} e^{-(\pi k)^2 t}\, u_{0k}\omega_k(x), \quad x \in \Omega, \;\; t \in T_\varepsilon; \;\; \sum_{k=1}^{\infty} e^{-2(\pi k)^2 \theta}\, u_{0k}^2 \;=1\}, \tag{3.4}$$

where

$$u_{0k} = \sqrt{2}\int_0^1 u(x,0)\,\mathrm{Sin}\;\pi k x\; dx.$$

For any given $\delta > 0$ we shall select in the latter a δ-net $U_{1\varepsilon}^{\delta}(\cdot)$ as follows.

Denote by

$$U_{1\varepsilon}(m,\cdot) \;=\; \{u(\cdot,\cdot) \mid \; u(x,t) =$$

$$= \sqrt{2}\sum_{k=1}^{m} e^{-(\pi k)^2 t}\, u_{0k}\omega_k(x), \quad x \in \Omega, \;\; t \in T_\varepsilon; \;\; \sum_{k=1}^{m} e^{-2(\pi k)^2 \theta}\, u_{0k}^2 \;=1\} \tag{3.5}$$

a sequence of finite dimensional subsets in $L_2(\Omega), \;\; m = 1, \ldots$.

It is clear that

$$U_{1\varepsilon}(m,\cdot) \;\subset\; U_{1\varepsilon}(m+1,\cdot), \;\; m = 1, \ldots,$$

and

$$U_{1\varepsilon}(\cdot) = \mathrm{cl}\,(\bigcap_{m=1,\dots} U_{1\varepsilon}(m,\cdot)), \tag{3.6}$$

where "*cl*" stands for the closure in the norm of $C(\bar{\Omega} \times T_\varepsilon)$.

Therefore, to find a δ-net in $U_{1\varepsilon}(\cdot)$, it is sufficient to find such a net in each of the sets $U_{1\varepsilon}(m,\cdot)$. To do the latter we recall now for the maximum principle [3, 9] for solutions of the system (1.1), namely,

$$M \max_{x\in\bar{\Omega}} \mid u(x,t') \mid \geq \max_{x\in\bar{\Omega}} \mid u(x,t'') \mid, \; t'' \geq t' \geq \varepsilon > 0, \;\; M = \text{const}. \tag{3.7}$$

We remark that $M = 1$ for the system (3.1).

The estimate (3.7) allows us to reduce the problem of constructing a net for each of the sets (3.5) to the same problem for the cross-sections of the latters at the only instant $t = \varepsilon$.

Applying the procedure (2.8)-(2.11) for each of the sets $U_{1\varepsilon}(m,\cdot)$ yields a finite number of measurements instants for each $m = 1,\dots$:

$$\varepsilon < t_1^m < \dots < t_k^m < \dots < t_{K_m}^m < \theta. \tag{3.8}$$

Finally, we can defined the observations of discrete type as

$$y_k^m = u(\bar{x}, t_k^m), \quad k = 1,\dots,K_m; \; m = 1,\dots \; . \tag{3.9}$$

The following assertion immediately follows from Theorem 2.3.

Theorem 3.1. Let the system (3.1), (3.2) be continuously observable at final time in the sense of (2.2). Then the system (3.1), (3.9) is also continuously observable in the sense of relation (1.6) and the estimate (2.22) holds.

Denote by $U(m,\cdot)$ the set of all solutions to the system (3.1) that are generated by initial conditions from the subspace of $L^2(\Omega)$ spanned by the first m eigenfunctions. Then we come to

Corollary 3.1. Let the system (3.1), (3.2) be continuously observable at final time in the sense of (2.2). Then, for any $m = 1,\dots$ a finite number of discrete measurements, namely,

$$y_k^m = u(\bar{x}, t_k^m), \quad k = 1,\dots,K_m \,, \tag{3.10}$$

is able to ensure the estimate (1.6), (2.22) (or the continuous observability at final time) for the set $U(m,\cdot)$ of solutions to the system (3.1).

4. Continuous Observability under Discrete Measurements: The General Case.

We note first that the problem of existence and constructing the measurement trajectories (both continuous and piecewise continuous) that are able to make the system (1.1), (2.1) be continuously observable at final time have been discussed in [8, 5].

The scheme described in the previous section for constructing the discrete type observations can also be applied for the general case. We note only that the constructing the δ-net, that is a crucial point in Theorem 2.3, can be achieved, for example, by applying the Galerkin's method.

Theorem 4.1. Let the conditions of Theorem 2.1(2.2) be fulfilled. Then the system (1.1), (1.3)((1.4)) be continuously observable at final time and the estimate (2.22) holds.

Corollary 4.1. Let the conditions of Theorem 2.1(2.2) be fulfilled. Then the system (1.1), (1.3)((1.4)) be observable.

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