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A Note on the Evolution Property of the Assembly of Viable Solutions to a Differential Inclusion

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Foreword

The paper deals with the description of the bundle of viable trajectories for a differential inclusion with phase constraints. The graph of the right-hand side of the differential inclusion is assumed to be star-shaped and characterizes the reachable set multifunction in terms of set-valued solutions to an evolution equation of special type. The author thus characterizes an important class of nonlinear systems. This paper was written under a cooperation with IIASA and finalized during the author's visit to the SDS Program. Dr. Filippova comes from the Institute of Mathematics and Mechanics in Yekatherinburg, Russia.

A Note on the Evolution Property of the Assembly of Viable Solutions to a Differential Inclusion

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1 Introduction

Consider a differential inclusion

$$\dot{x}(t) \in F(t, x(t)), \quad x(t_0) \in X_0, \quad t_0 \leq t \leq \theta \tag{1.1}$$

with a state constraint

$$x(t) \in Y(t), \quad t_0 \leq t \leq \theta \tag{1.2}$$

A solution $x(t)$ to relations (1.1)-(1.2) is said to be a *viable* trajectory to the differential inclusion. In recent years the viability properties of dynamic systems have become an object of strong interest [1,2]. We should mention however that these investigations are mainly concerned with problems of global viability (or weak invariance [6]) when the phase constraints (1.2) have to be satisfied for all the future instants of time $t \geq t_0$.

On the other hand there is a close relation between viability theory for differential inclusions and the "guaranteed" treatment of uncertain dynamic systems, adaptive control and differential games [7-10]. A "local" viability setting is used for studying observation and estimation problems under incomplete data [11-13]. Results obtained in the latter papers allow to describe the reachable set $X[t]$ to the system of inclusions (1.1)-(1.2) at instant t, which in other words is the t-section of the trajectory bundle that combines all the solutions to a differential inclusion (1.1) that are viable on the interval $[t_0, t]$. It was proven in [12] that the reachable set $X[t]$ satisfies the following evolution equation

$$\lim_{\sigma \to 0} \sigma^{-1} h(X[t+\sigma], \bigcup_{x \in X[t]} (x + \sigma F(t, x)) \cap Y(t+\sigma)) = 0 \tag{1.3}$$

then generalizes the so-called "integral funnel" equation [3,14,15] (here h denotes the Hausdorff distance function). The crucial assumption for the last result was the convexity of the graph of the multifunction $F(t, \cdot)$ for every fixed t. We relax this rather restrictive convexity assumption and consider instead a differential inclusion (1.1) with a *star-shaped graph* of the right-hand side $F(t, \cdot)$. This allows to apply the proposed approach in Section 2 to the following uncertain system [10]

$$\dot{x} \in \mathcal{A}(t)x + P(t), \quad x(t_0) \in X_0, \tag{1.4}$$

$$x(t) \in Y(t), \quad t_0 \leq t \leq \theta$$

that depends bilinearly upon the state vector x and the disturbances $A(t) \in \mathcal{A}(t)$ and $p(t) \in P(t)$. Here the multifunctions $\mathcal{A}(\cdot)$ and $P(\cdot)$ reflect the uncertainties in the system (1.4) (Note that the values $\mathcal{A}(t)$ of $\mathcal{A}(\cdot)$ are subsets of the space of all $n \times n$-matrices). In Section 3 we formulate the main result of this paper (Theorem 3.1) which is the description of the evolution of reachable sets $X[t]$ for a nonlinear differential inclusion (1.1) with a star-shaped graph of $F(t, \cdot)$.

Finally, it should be pointed out that the proposed generalization seems to be rather natural because a family of star-shaped sets is close in many respects to the cone of all convex subsets of the space R^n. For example, under quite general assumptions it is possible to introduce algebraic operations (of summation and multiplication by a scalar) within this class so that the duality relation between Minkowski-Gauge functions and star-shaped sets becomes an algebraic isomorphism somewhat similar to the one known in convex analysis for support functions and closed convex sets [5].

2 Bilinear Uncertain Systems

Let us introduce some notations. Denote R^n to be the Euclidean n-dimensional space with the norm $\|x\| = (x, x)^{1/2}$ for $x \in R^n, S = \{x \in R^n : \|x\| \leq 1\}$. Also denote comp R^n to be the space of all compact subsets of R^n. The Hausdorff distance between the sets $A, B \in$ comp R^n will be denoted by $h(A, B)$ while

$$\rho(\ell|A) = \sup\{(\ell, a)|a \in A\}$$

will stand for the support function of $A \in$ comp R^n. We use the symbol $R^{n\times n}$ for the space of all $n \times n$-matrices. Let conv R^n(conv $\mathrm{R}^{n\times n}$) be the set of convex and compact subsets of R^n ($R^{n\times n}$, respectively). The graph of a multifunction $Z : R^m \to$ comp R^n will be denoted by $grZ = \{\{u, v\} : v \in Z(u)\}$. If a multifunction $Z(u, w)$ depends on two variables the symbol $gr_w Z$ is used for grZ_0 where $Z_0(u) = Z(u, w)$ and w is fixed.

Consider the uncertain system (1.4) where $x \in R^n, \mathcal{A}(t), P(t), Y(t), X_0 \in$ conv R^n for all $t \in [t_0, \theta]$. We assume the set-valued functions $\mathcal{A}(\cdot), P(\cdot)$ to be measurable and the following hypotheses to be fulfilled.

Assumption A. For all $t \in [t_0, \theta], 0 \in P(t); 0 \in X_0$.
Assumption B. There exists an $\epsilon > 0$ such that $\epsilon S \subseteq Y(t)$ for every $t \in [t_0, \theta]$.
Assumption C. The multifunction $Y(\cdot)$ satisfies one of the following conditions:

(i) $grY \in$ conv R^{n+1};

(ii) for every $\ell \in R^n$ the support function $f(\ell, t) = \rho(\ell|Y(t))$ is differentiable in t and its derivative $\partial f/\partial t$ is continuous in (ℓ, t).

Every absolutely continuous function $x(\tau)(t_0 \leq \tau \leq \theta)$ satisfying inclusions

$$\dot{x}(\tau) \in \mathcal{A}(\tau)x(\tau) + P(\tau) \text{for a.e.} \tau \in [t_0, \theta]$$

and

$$x(t_0) \in X_0$$

will be called a trajectory of the differential inclusion that starts at X_0. A trajectory $x(\tau)$ is said to be viable on $[t_0, t]$ if $x(\tau) \in Y(\tau)$ for all $\tau \in [t_0, t]$. Denote by $X(t, t_0, X_0)$ the reachable set of (1.4) at instant t that is emitted by X_0:

$$\begin{aligned} X(t, t_0, X_0) \quad = \quad & \{z \in R^n : \text{there exists a trajectory } x(\tau) \text{ such that} \\ & [t_0, t] \text{ and } x(t_0) \in X_0, x(t) = z\}. \end{aligned}$$

Lemma 2.1 *Let Assumptions A , B , C be true. Then for all $\mu > 0, \tau \in [t_0, \theta]$ and for every trajectory $x(\tau)$ such that $x(t_0) \in X_0$ and $x(t) \in Y(t) + \mu S, (t_0 \leq t \leq \tau)$ there exists a solution $x^*(t)$ to (1.4) that satisfies the inequality*

$$\|x(t) - x^*(t)\| \leq C\mu, \quad t_0 \leq t \leq \theta \tag{2.1}$$

where constant C does not depend on $\mu, x(\cdot), \tau$.

Proof. Suppose that

$$\dot{x}(t) = A(t)x(t) + p(t),$$
$$x(t_0) = x_0, \quad t_0 \leq t \leq \theta$$

for some $A(\cdot) \in \mathcal{A}(\cdot), p(\cdot) \in P(\cdot)$ and $x_0 \in X_0$ and

$$x(t) \in Y(t) + \mu S, \quad t_0 \leq t \leq \tau. \tag{2.2}$$

Denote $p^*(t) = \epsilon(\mu + \epsilon)^{-1}p(t), x_0^* = \epsilon(\mu + \epsilon)^{-1}x_0$. Under Assumptions A-C we have

$$x_0^* \in X_0, \quad p^*(t) \in P(t) \quad (t_0 \leq t \leq \theta).$$

Let $x^*(t)$ be

$$x^*(t) = \epsilon(\mu + \epsilon)^{-1}x(t), \quad t_0 \leq t \leq \theta. \tag{2.3}$$

Then

$$\dot{x}^*(t) = A(t)x^*(t) + p^*(t)$$
$$x^*(t_0) = x_0^*, \quad t_0 \leq t \leq \theta$$

Hence we can conclude that $x^*(\cdot)$ is a solution to the uncertain bilinear system (1.4).

The following inclusion follows from Assumption B:

$$\epsilon\mu(\mu + \epsilon)^{-1}S \subseteq \mu(\mu + \epsilon)^{-1}Y(t), \quad t_0 \leq t \leq \theta.$$

From (2.2)-(2.3) we obtain

$$x^*(t) \in \epsilon(\mu + \epsilon)^{-1}(Y(t) + \mu S) = \epsilon(\mu + \epsilon)^{-1}Y(t) + \epsilon\mu(\mu + \epsilon)^{-1}S.$$

Then for every $t \in [t_0, \tau]$

$$\begin{aligned} x^*(t) \;&+\; \epsilon\mu(\mu + \epsilon)^{-1}S \subseteq \epsilon(\mu + \epsilon)^{-1}Y(t) + \mu(\mu + \epsilon)^{-1}Y(t) \\ &+\; \epsilon\mu(\mu + \epsilon)^{-1}S \subseteq Y(t) + \epsilon\mu(\mu + \epsilon)^{-1}S. \end{aligned}$$

(We use here the convexity of the set $Y(t)$.) Hence we have

$$x^*(t) \in Y(t), \qquad t_0 \leq t \leq \tau.$$

It means that $x^*(\tau) \in X(\tau; t_0, X_0)$. Now let us estimate the difference

$$\|x(t) - x^*(t)\| = \|x(t) - \epsilon(\mu + \epsilon)^{-1}x(t)\| = \mu(\mu + \epsilon)^{-1}\|x(t)\| \leq \mu\epsilon^{-1}K, \quad t_0 \leq t \leq \theta$$

(Here $K > 0$ does not depend on the choice of $x(\cdot)$). From the last relations we obtain the inequality (2.1) (for $C = K\epsilon^{-1}$). The lemma is proved.

Denote $X_\mu(\cdot; \tau, t_0, X_0)$ to be the set of all viable trajectories to a bilinear system (1.4) (with respect to a perturbed constraint $Y_\mu(t) = Y(t) + \mu S$) and let

$$X_\mu[\tau] = X_\mu(\tau; t_0, X_0) = X_\mu(\tau; \tau, t_0, X_0).$$

The following result is a direct consequence of Lemma 2.1.

Lemma 2.2 *Suppose that Assumptions A-C are fulfilled. Then the multivalued functions* $X_\mu(\cdot;\tau,t_0,X_0)$ *and* $X_\mu[\tau]$ *are Lipschitz-continuous in* $\mu > 0$ *at point* $\mu = +0$ *(in spaces* $C^n[t_0,\theta]$ *and* R^n *respectively).*

Denote $\mathcal{M} \circ X = \{z \in R^n : z = Mx, M \in \mathcal{M}, x \in X\}$ for $\mathcal{M} \in \text{conv } \mathrm{R}^{n\times n}, \mathrm{X} \in \text{comp } \mathrm{R}^n$.

From Lemmas 2.1-2.2 one can prove the following theorem:

Theorem 2.1 *Let Assumptions A, B, C be true. Then the multivalued function* $X[t] = X(t,t_0,X_0)$ *is the solution to the following evolution equation*

$$\lim_{\sigma\to 0} \sigma^{-1} h(X[t+\sigma], ((E+\sigma\mathcal{A}(t)) \circ X[t] + \sigma P(t)) \cap Y(t+\sigma)) = 0 \qquad \textit{for a.e. } t \in [t_0, t] \quad (2.4)$$

with initial condition $X[t_0] = X_0$.

The following example demonstrates that under our assumptions the reachable sets $X[t]$ need not be convex.

Example 1. Consider a differential inclusion in R^2

$$\begin{cases} \dot{x}_1 \in [-1,1] \cdot x_2, \quad 0 \le t \le 1, \\ \dot{x}_2 = 0, \quad X_0 = \{x \in R^2 : x_1 = 0, |x_2| \le 1/2\}, \end{cases}$$

$$Y(t) = \{x \in R^2 : |x_1| \le 1, |x_2| \le 1/2\}.$$

Then $X(1,0,X_0) = X[1] = X^1 \cup X^2$ where $X^1 = \{x \in R^2 : |x_1| \le x_2 \le 1/2\}$, $X^2 = \{x \in R^2 : |x_1| \le -x_2 \le 1/2\}$. Obviously the set $X[1]$ is not convex.

Definition. A set $Z \subseteq R^n$ will be called star-shaped (with a center at 0) if $0 \in Z$ and $\lambda Z \subseteq Z$ for all $\lambda \in (0,1]$.

Proposition. Assume X_0 to be star-shaped. Then for every $t \in [t_0,\theta]$ the reachable set $X(t,t_0,X_0)$ of the system (1.4) is a compact star-shaped subset of R^n.

3 The Main Result

Now consider a nonlinear differential inclusion (1.1) where $F(t,x)$ is a multifunction measurable in t and Lipschitz continuous in $x(F : [t_0,\theta]xR^n \to$ conv R^n. Denote $x[t] = x(t;t_0,x_0)$ to be the Caratheodory-type solution to (1.1) that starts at $x[t_0] = x_0 \in X_0$. We further require all the solutions $\{x(t;t_0,x_0) : x_0 \in X_0\}$ to be extendable until the instant θ [4]. As before, the symbol $X[t] = X(t;t_0,X_0)$ stands for the reachable set (at instant t) to a differential inclusion (1.1) with phase constraint (1.2).

Assumption D.

(i) For all $t \in [t_0,\theta]$ we have $0 \in F(t,0)$ and gr_tF is a star-shaped subset of R^{2n};

(ii) the set $X \subseteq R^n$ is star-shaped.

Theorem 3.1 *Under Assumptions B, C, D the multifunction* $X[t] = X(t,t_0,X_0)$ *is the solution to the following evolution equation*

$$\lim_{\sigma\to 0} \sigma^{-1} h(X[t+\sigma], \quad \bigcup_{x\in X[t]} (x+\sigma F(t,x)) \cap Y(t+\sigma)) = 0$$

for a.e. $t \in [t_0,\theta]$ *that starts at* $X_0 : X[t_0] = X_0$.

Example 2. Let $F(t,x)$ be of the form

$$F(t,x) = G(t,x)U + P(t)$$

where the $n \times n$-matrix function $G(t,x)$ is measurable in t, Lipschitz continuous and positively homogeneous in x; $U \in$ conv R^n. A function $P : [t_0, \theta] \to$ conv R^n is assumed to be measurable. We suppose also that for all $t \in [t_0, \theta], 0 \in P(t)$. One can easily verify that Assumption D holds in this case.

The proof of Theorem 3.1 is based on the ideas of paper [12] and follows from the next two results.

Lemma 3.1 *Let the hypotheses of Theorem 3.1 be true. Then for every $t \in [t_0, \theta]$ the reachable set $x(t; t_0, X_0)$ is a compact star-shaped subset of R^n.*

Lemma 3.2 *Under Assumptions B-D the multivalued map $X_\mu(\cdot; \tau, t_0, X_0)$ satisfies the Lipschitz condition with respect to $\mu > 0$ (from the right) at point $\mu = +0$, namely*

$$X_\mu(\cdot; \tau, t_0, X_0) \subseteq X(\cdot; \tau, t_0, X_0) + C\mu S(\cdot),$$

where $S(\cdot) = \{x(\cdot) \in C^n[t_0, \theta] : \|x(\cdot)\| \le 1\}$ and $C > 0$ does not depend on $\{\tau, \mu\}$.

4 The Uniqueness of the Solution to the Funnel Equation

Let us denote $\mathcal{Z}[t_0, \theta]$ to be the set of all multivalued functions $Z(\cdot) : [t_0, \theta] \to$ comp R^n such that $Z(t_0) = X_0$ and

$$\sigma^{-1} h(Z(\tau + \sigma), \bigcup_{x \in Z(\tau)} (x + \sigma F(\tau, x)) \cap Y(\tau + \sigma)) \to 0 \quad (\sigma \to 0+) \tag{4.1}$$

uniformly with respect to $\tau \in [t_0, \theta]$.

Under Assumptions A-D we have

$$X[\cdot] = X(\cdot; t_0, X_0) \in \mathcal{Z}[t_0, \theta]$$

Let us begin however with the comon case when we don't require these assumptions to be fulfilled.

Consider some properties of the maps $Z(\cdot) \in \mathcal{Z}[t_0, \theta]$.

Lemma 4.1 *Assume that the multivalued function $Y(\cdot)$ satisfies the Lipschitz condition (with constant $k > 0$):*

$$h(Y(t_1), Y(t_2)) \le k(t_1 - t_2), \quad t_0 \le t_1, t_2 \le \theta.$$

Then for every $Z(\cdot) \in \mathcal{Z}[t_0, \theta]$ the following inclusion is true

$$Z(\tau) \subseteq X[\tau] = X(\tau; t_0, X_0), \quad t_0 \le \tau \le \theta \tag{4.2}$$

Proof. Let τ be an arbitrary instant, $\tau \in [t_0, \theta]$, and $z \in Z(\tau)$. Consider the subdivision $\{t_i; i = 1, \ldots, N\}$ of the interval $[t_0, \tau]$ with uniform step $\sigma_N = (\tau - t_0)/N$:

$$t_i = t_0 + i\sigma_N, \quad (i = 1, \ldots, N), \quad t_N = \tau.$$

Let

$$o(\sigma; Z) = \sup_{t_0 \le t \le \theta} h(Z(t + \sigma), \bigcup_{x \in Z(t)} (x + \sigma F(t, x)) \cap Y(t + \sigma)). \tag{4.3}$$

From the definition of $Z(\cdot)$ we obtain

$$\sigma^{-1}o(\sigma; Z) \to 0 \quad (\sigma \to t+0).$$

It is clearly possible to find a finite sequence of vectors $\{z_i, f_i\}_{i=0,1,\dots,N}$. such that

$$\begin{aligned} z_i &\in Z(t_i), \quad f_i \in F(t_i, z_i), \\ z_N &= z, \quad z_0 \in X_0, \quad z_i + \sigma_N f_i \in Y(t_{i+1}), \\ z_i &= z_{i-1} + \sigma_N f_{i-1} + \ell_i, \|\ell_i\| \le o(\sigma_N; Z), \\ & *i = 1, \dots, N-1. \end{aligned}$$

Consider the piecewise linear interpolation $z_{(N)}(\cdot)$:

$$z_{(N)}(t_i) = z_i, \quad z_{(N)}(t) = z_i + (z_{i+1} - z_i)(t - t_i)\sigma_N^{-1}, \quad (t_i \le t \le t_{i+1}, \quad i = 0, 1, \dots, N-1).$$

Then for every $t \in [t_i, t_{i+1}]$ $(i = 0, 1, \dots, N-1)$:

$$\begin{aligned} z_{(N)}(t_i) &= z_i \in Y(t_i) + \ell_i \subseteq Y(t) + (k\sigma_N + o(\sigma_N; Z))S, \\ z_{(N)}(t_{i+1}) &= z_{i+1} \in Y(t_{i+1}) + \ell_{i+1} \subseteq Y(t) + (k\sigma_N + o(\sigma_N; Z))S, \end{aligned}$$

Hence

$$z_{(N)}(t) \in Y(t) + (k\sigma_N + o(\sigma_N; Z))S, \quad t_0 \le t \le \tau \tag{4.4}$$

(as the set $Y(t)$ is convex). It is not difficult to prove that the sequence $\{z_{(N)}(\cdot)\}(N \to \infty)$ has a limit point $x_*(\cdot)$ in the space $C^n[t_0, \tau]$ and that the function $x_*(\cdot)$ is a solution to the differential inclusion.

$$\dot{x}_* \in F(t, x_*), \qquad t_0 \le t \le \tau,$$
$$x_*(t_0) \in X_0, \quad x_*(\tau) = z.$$

From (4.4) we have

$$x_*(t) \in Y(t), \quad t_0 \le t \le \tau.$$

Therefore, $x_*(\cdot) \in X(\cdot; \tau, t_0, X_0)$ and $x_*(\tau) = z \in X[\tau]$. The lemma is thus proved.

Corollary 4.1 *Under assumptions of Lemma 4.1 the following relations are true*

- *(i)* $z(t) \subseteq Y(t)$ *for every* $t \in [t_0, \theta]$,
- *(ii)* $z(t+\sigma) \subseteq z(t) + \zeta\sigma S, \quad t_0 \le \tau \le \tau + \sigma \le \theta$

where $\zeta > 0$.

Example 4.1. Consider the following system in R^2:

$$\begin{cases} \dot{x} = x_1 x_2^2 - x_1 \\ \dot{x}_2 = x_1^2 x_2 - x_2 \end{cases} \quad 0 \le t \le \theta$$

with set

$$X_0 = \{x = (x_1, x_2) : x_2 = 1, |x_1| \le 1\}$$

and the state restriction,

$$Y = \{x = (x_1, x_2) : |x_1| \le 2, 1 \le x_2 \le 2\}.$$

For every $\tau \in (0, \theta]$ we have

$$X[\tau] = X(\tau; t_0, X_0) = \{x^{(1)}\} \cup \{x^{(2)}\},$$

where

$$x^{(1)} = (x_1^{(1)}, x_2^{(1)}) = (1, 1)$$
$$x^{(2)} = (x_1^{(2)}, x_2^{(2)}) = (-1, 1)$$

Obviously, $\mathcal{Z}[t_0, \theta] = \cup\{Z_i(\cdot) | i = 1, 2, 3\}$, where $Z_1(\cdot) = X(\cdot; t_0, X_0)$,

$$Z_i(t) = \begin{cases} X_0, & t = t_0 = 0 \\ \{x^{(i)}\}, & 0, t < \theta \end{cases} \quad (i = 2, 3) \quad .$$

It should be pointed out that in this example both viable trajectories $x^{(1)}(t), x^{(2)}(t)$ lie on the boundary of set Y. The next result will show that for the "interior" trajectory $x_*(t)$ the above-mentioned situation $x_*(t) \notin Z(t)$ will be impossible.

Denote for every $\tau \in [t_0, \theta]$

$$X_{\text{int}}[\tau] = X_{\text{int}}(\tau; t_0, X_0) = \{z \in R^n : \exists x(\cdot) \in X(\cdot; \tau, t_0, X_0) x(\tau) = z, \\ x(t) \in \text{int}Y(t), \forall t \in [t_0, \tau]\}.$$

Lemma 4.2 *Let Assumption B be fulfilled. Then for every $\tau \in [t_0, \theta]$*

$$X_{\text{int}}[\tau] \subseteq Z(\tau)$$

where $Z(\cdot)$ is an arbitrary multifunction from the class $\mathcal{Z}[t_0, \theta]$.

The proof of this lemma is similar to that of Lemma 4.1.

Corollary 4.2 *Under the assumptions of Lemmas 4.1-4.2 the following inclusions are true:*

$$c\ell X_{\text{int}}[t] \subseteq Z(t) \subseteq X[t], \quad t_0 \leq t \leq \theta \quad \textit{for all } Z(\cdot) \in \mathcal{Z}[t_0, \theta].$$

We are now able to formulate the uniqueness theorem.

Theorem 4.1 *Let Assumptions B, C, D be true. Then the multivalued function $X[\tau] = X(\tau; t_0, X_0)$ is the unique solution to the funnel equation (1.3) in the class $\mathcal{Z}[t_0, \theta]$ of all multivalued mappings $Z(\cdot)$ that satisfy this equation uniformly in t.*

Proof. Under the conditions of Theorem 4.1 one can prove the equality

$$c\ell X_{\text{int}}[t] = X[t].$$

Then from Corollary 4.2 we conclude that $X[t] = Z(t)$ for any $Z(\cdot) \in \mathcal{Z}[t_0, \theta]$ and Theorem 4.1 is proved.

References

[1] Aubin, J.-P., A. Cellina, Differential Inclusions. Springer-Verlag, Heidelberg, 1984.

[2] Aubin, J.-P., A survey of viability theory. SIAM J. Control Opt., 28, 4, pp. 749-788, 1990.

[3] Barbashin, E.A. On the theory of generalized dynamic systems. Uchen. Zap. Moscow Univ., Matematika, 2, 135 (in Russian), 1949.

[4] Blagodatskikh, V.I., A.F. Filippov. Differential inclusions and optimal control. Trudy Matem. Inst. Akad. Nauk SSSR, Nauka, Moscow, 169, pp. 194-252 (in Russian), 1985.

[5] Dem'janov, V.F., A.M. Rubinov. Quasidifferential calculus. Optimization Software, Inc. Publications Division, New York, 1986.

[6] Guseinov, H.G., A.I. Subbotin, V.N. Ushakov. Derivatives for multivalued mappings with applications to game-theoretical problems of control. Problems of Control and Inform. Theory, 14, 3, pp. 155-167, 1985.

[7] Krasovski, N.N. The control of a dynamic system. Nauka, Moscow (in Russian), 1985.

[8] Krasovski, N.N., A.I. Subbotin. Positional differential games. Nauka, Moscow (in Russian), 1974.

[9] Kurzhanski, A.B. Control and observation under uncertainty. Nauka, Moscow (in Russian), 1977.

[10] Kurzhanski, A.B. Dynamic control system estimation under uncertainty conditions. Problems of Control and Inform. Theory, 2, 10, 1, pp. 33-42, 1981.

[11] Kurzhanski, A.B. On the analytical description of a set of viable trajectories of a differential system. Dokl. Acad. Nauk SSSR, 287, 5, pp. 1047-1050 (in Russian), 1986.

[12] Kurzhanski, A.B., T.F. Filippova. On the set-valued calculus in problems of viability and control for dynamic processes: the evolution equation. Les Annales de l'Institut Henri Poincaré, Analyse nonlinéaire, pp. 339-363, 1989.

[13] Kurzhanski, A.B., O.I. Nikonov. On the control strategies synthesis problem. Evolution equations and set-valued integration. Dokl. Akad. Nauk SSSR, 311, 4, pp. 788-793 (in Russian), 1990.

[14] Panasyuk, A.I., V.I. Panasyuk. Asymptotic magistral optimization of control systems. Nauka i Tekhnika, Minsk, (in Russian), 1986.

[15] Roxin, E. On the generalized dynamic systems defined by contingent equations. J. Diff. Equations, 1, pp. 188-205, 1965.