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Hypercube Parallel Processing for Ellipsoidal Estimates in Differential Inclusions

Motoyasu Nagata

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IIASA International Institute for Applied Systems Analysis □ A-2361 Laxenburg □ Austria
Telephone: +43 2236 715210 □ Telex: 079 137 iiasa a □ Telefax: +43 2236 71313

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Foreword

This paper presents hypercube parallel processing for ellipsoidal estimates in differential inclusion. The results are broadly applicable to many problems arising in differential inclusion using parallel computer architecture.

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1 Introduction

When one tries to obtain the attainability set of the differential inclusion using ellipsoidal estimates, it is very important to detect inclusion between approximated ellipsoids. Because an inclusion can be detected between two ellipsoids that are estimates for the attainability set at some time, parameterized including external ellipsoid $E_p^+[t]$ (or parameterized included internal ellipsoid $E_p^-[t]$) becomes unnecessary for computation after this time, thus the computation of the ellipsoidal estimates can be reduced. However, since this detection requires C_2^n combinations for couple ellipsoids, where n stands for number of ellipsoids, the sequential processing has not been developed due to heavy overload by every combination of all ellipsoids. Therefore efficient parallel processing has been employed to avoid overload inherent to the traditional sequential processing. For this detection problem, we present a scheme of the hypercube parallel processing.

Evolution equation of the differential inclusion system has been explored with the aid of a "target cone" to the multi-valued map in Aubin and Cellina [1], Aubin and Ekeland [2], Aubin and Frankowska [3], and alternatively explored with the aid of a "funnel equation" in Kurzhanski and Filippova [9], and Kurzhanski and Nikonov [10]. On the other hand, for the purpose of approximation of the attainability set obtained from the evolution equation of the differential equation, ellipsoidal estimation methods have been developed in Chernousko [6], and Kurzhanski and Valyi [11] [12]. Ellipsoidal technique was studied from the viewpoint of a "funnel equation" in [11], [12]. While it is true that the ellipsoidal technique using "funnel equation" requires computation of the evolution according to the number of parameterized ellipsoids, and therefore traditional sequential processing faces the problem of the overload for the detection problem of inclusion between ellipsoids, better parallel-processing results are to be expected.

In this paper we explore a hypercube parallel processing for ellipsoidal estimates in the differential inclusion. The hypercube is one of the structural topologies that represent connection among processing elements (PEs) of multiprocessors, which are executed in a parallel manner [13]. The n-dimensional hypercube Q_n has 2^n PEs that are connected with n adjacent PEs, respectively. The PE mainly consists of a central processing unit and local memory and interfaces with other PEs. The PE and the connection between PEs can be regarded as the node and edge in the hypercube graph. Currently, the hypercube is the most promising structural topology for architecture of parallel computing due to surprisingly fruitful theoretical results in spite of the simple structure.

Our approach will be delineated. First, we propose definitions about partial ordering of ellipsoids that is represented by a Hasse diagram. Although this partial ordering does not necessarily satisfy inclusion between ellipsoids, the Hasse diagram (as a graph) can become a data structure of ellipsoids that is to be embedded into the hypercube. The quotient set of the Hasse diagram with respect to an equivalent relation is also studied. Second, the relaxed-squashed (RS) embedding of the graph into hypercube is considered. We propose a result about RS embedding of multiple graphs into the hypercube, where these graphs correspond to the quotient sets of the Hasse diagram. Our proposed RS embedding guarantees mapping of any adjacent node in the source graph into adjacent subcubes. Third, the parallel processing for detection of inclusion between ellipsoids is studied. For this problem, we propose a parallel algorithm in the hypercube. We are primarily concerned with studying the parallel detection of inclusion between the ellipsoidal estimates that can lead to effective computation of the evolution equation in the differential inclusion.

2 Problem Statement

We consider the following problem for the nonviable differential inclusions: For parameterized external ellipsoids $E_p^+[t]$ and internal ellipsoids $E_p^-[t]$ that approximate the attainability set at time t of the differential inclusion system

$$\dot{x}(t) \in A(t)x(t) + P(t), t \in T = [t_0, t_1], \tag{1}$$

a scheme is designed of the hypercube parallel processing to detect inclusion between parameterized external (or internal) ellipsoids simultaneously.

Let A be a continuous map from T to $R^{n\times n}$, and let P be a continuous map from T to the space of convex compact subsets in R^n. Here, R^n and $R^{n\times n}$ stand for the n-dimensional space and the space of $n\times n$-matrices, respectively. The initial value satisfies a condition

$$x(t_0)\in X^0, \tag{2}$$

where a given set X^0 is the convex compact subset in R^n. Solutions to (1) and (2) are understood in the Caratheodory sense, i.e., absolutely continuous functions verifying (1) and (2) almost everywhere.

Definition 2.1 [10] The Haussdorff distance $h(P',P'')$ is defined as follows:

$$h(P',P'')=\max\{h_+(P',P''),h_+(P'',P')\},$$

where

$$h_+(P',P'')=\min_{r\geq 0}\{r|P'\subseteq P''+rS\}$$

and S is the unit ball in R^n.

Definition 2.2 [11] The attainability set for (1), denoted by $X[t]=X(t,t_0,X^0)$, is the set of values at $t\in T$ of all single-valued trajectories starting from X^0, i.e., the attainability domains for (1). The attainability set $X[t]$ satisfies the funnel equation

$$\lim_{\sigma\to 0}\sigma^{-1}h(X[t+\sigma],(E+\sigma A(t))X[t]+\sigma P(t))=0, \tag{3}$$

where

$$X[t_0]=X^0. \tag{4}$$

Definition 2.3 [12] A solution $X^*[t]$ is defined to be a maximal solution of (3) if for all $t\in T$

$$\{X[t]|X^*[t]\subset X[t]\ and\ X^*[t]\neq X[t]\}=\Phi,$$

where Φ is the null set.

Lemma 2.4 The funnel equation (3) and (4) has a unique maximal solution that is convex, compact and continuous in t. This maximal solution coincides with the attainability set.

The ellipsoidal estimation that externally and internally approximates the attainability set was studied by Kurzhanski and Valyi [11] by assuming a set $P(t)$ as an ellipsoid $E(\overline{p}(t),\overline{P}(t))$.

Definition 2.5 [11] The external ellipsoid $E[t] = E(a^+(t), Q^+(t))$ is a solution to the following funnel equation:

$$\lim_{\sigma \to 0} \sigma^{-1} h(E^+[t+\sigma], E^+[t+\sigma|t]) = 0 \tag{5}$$

$$E^+[t_0] = E(a^+(t_0), Q^+(t_0)).$$

The ellipsoid $E^+[t+\sigma|t] = E(a^+(t+\sigma|t), Q^+(t+\sigma|t))$ is an external estimate of the Minkowski sum of $(E + \sigma A(t))E^+[t]$ and $\sigma P(t)$, defined by

$$a^+(t+\sigma|t) = (E + \sigma A(t))a^+(t) + \sigma\overline{p}(t)$$

$$Q^+(t+\sigma|t) = (trQ_1(t) + trQ_2(t))(Q_1(t)/(trQ_1(t)) + Q_2(t)/(trQ_2(t))),$$

where

$$Q_1(t) = (E + \sigma A(t))Q^+(t)(E + \sigma A(t))^t$$

$$Q_2(t) = \sigma\overline{P}(t).$$

Definition 2.6 [11] The internal ellipsoid $E^-[t] = E(a^-(t), Q^-(t))$ is a solution to the following funnel equation:

$$\lim_{\sigma \to 0} \sigma^{-1} h(E^-[t+\sigma], E[t+\sigma|t]) = 0 \tag{6}$$

$$E^-[t_0] = E(a^-(t_0), Q^-(t_0)).$$

The ellipsoid $E^-[t+\sigma|t] = E(a^-(t+\sigma|t), Q^-(t+\sigma|t))$ is an internal estimate of the Minkowski sum of $(E + \sigma A(t))E^-[t]$ and $\sigma P(t)$, defined by

$$a^-(t+\sigma|t) = (E + \sigma A(t))a^-(t) + \sigma\overline{p}(t)$$

$$[Q^-(t+\sigma|t)]^{\frac{1}{2}} = [(E + \sigma A(t))Q^-(t)(E + \sigma A(t))^T]^{\frac{1}{2}} + \sigma[\overline{P}(t)]^{\frac{1}{2}}.$$

The trajectories of the ellipsoids $E^+[t]$ and $E^-[t]$ are obtained in the non-viable case [11] and the viable case [12]. From the results, we obtain a result for the nonviable differential inclusion.

Lemma 2.7 Given initial conditions of the internal and external ellipsoids such that $E^-[t_0] \subset X[t_0] \subset E^+[t_0]$. Then the attainability set $X[t]$ for the differential inclusion (1) and (2) is obtained from intersection of the external ellipsoids and union of the internal ellipsoids such that

$$X[t] = \bigcap_p E_p^+[t] = \overline{\bigcup_p E_p^-[t]}$$

and

$$E_p^-[t] \subset X[t] \subset E_p^+[t],$$

where $E_p^+[t]$ and $E_p^-[t]$ stand for parameterized trajectories of the external and internal ellipsoids according to various initial ellipsoids denoted by $E_p^+[t_0]$ and $E_p^-[t_0]$, and overline indicates closure of the set.

We can obtain the following lemmas that are directly linked with our problem:

Lemma 2.8 If $E_{p_1}^+[t] \subset E_{p_2}^+[t]$, then $E_{p_1}^+[s] \subset E_{p_2}^+[s]$ for $s \geq t$. For obtaining the attainability set $X[s]$, $E_{p_2}^+[s]$ can be discarded.

Lemma 2.9 If $E_{p_1}^-[t] \subset E_{p_2}^-[t]$, then $E_{p_1}^-[s] \subset E_{p_2}^-[s]$ for $s \geq t$. For obtaining the attainability set $X[s]$, $E_{p_1}^-[s]$ can be discarded.

Remark 2.10 This paper is concerned with parallel processing for Lemma 2.8 and Lemma 2.9. Two initial ellipsoids, $E^+[t_0]$ and $E^-[t_0]$ depend on the pairs $(a^+(t_0), Q^+(t_0))$ and $(a^-(t_0), Q^-(t_0))$, where $a^+(t_0)$ (or $a^-(t_0)$) and $Q^+(t_0)$ (or $Q^-(t_0)$) indicate the center and matrix of the external (or internal) ellipsoid such that

$$E^+[t_0] = \{x | (x - a^+(t_0))^T Q^+(t_0)^{-1}(x - a^+(t_0))\}.$$

In practice, because of Lemmas 2.7, 2.8, and 2.9, $E_p^+[t_0]$s (or $E_p^-[t_0]$s) are not necessarily selected as minimal (or maximal). The nonuniqueness of the trajectories of the external/internal ellipsoids was discussed in the viability case [12]. In this paper, the nonuniqueness of the trajectories is discussed due to variation of the ellipsoidal parameter and the initial ellipsoid in the nonviability case.

3 Hypercube

Definition 3.1 [15] Addresses of the n-dimensional hypercube Q_n are recursively constructed as follows:

(1) Addresses of two nodes of the one-dimensional hypercube Q_1 are 0 and 1.

(2) Let $a_{n-1}.....a_1$ be the binary address of any node of the $(n-1)$-dimensional hypercube Q_{n-1}. For same addresses $a_{n-1}.....a_1$ of two Q_{n-1}s, concatenate 0 and 1 to the leftmost bits and connect them.

Definition 3.2 [5] The graph of the n-dimensional hypercube Q_n is recursively constructed as follows:

(1) Q_0 is a trivial graph with one node.

(2) $Q_n = K_2 \times Q_{n-1}$, where K_2 is a complete graph that consists of two nodes.

Definition 3.3 [5] Subcube is a subgraph of the hypercube that satisfies definition of the hypercube. Address of the subcube is represented by symbol set 0, 1, $*$, where $*$ is a *don't care symbol* that is 0 or 1. Distance between two subcubes a and b is the Hamming distance between their addresses: $H(a,b) = \sum_{i=1}^{n} |a_i - b_i|$ where $|a_i - b_i| = 1$ if and only if $(a_i, b_i) = (0,1)$ or $(1,0)$.

Definition 3.4 [8] The n-dimensional binary-reflected Gray code (BRGC) G_n is recursively constructed as follows:

(1) $G_1 = (0,1)$,

(2) $G_n = (0G_{n-1}, 1\overline{G}_{n-1})$,

where $0G_{n-1}$ is a concatenation of 0 and G_{n-1}, and $\overline{G}_{n-1}$ is a backward-sorted code of G_{n-1}.

4 Partially Ordered Structure

4.1 Partial ordering of ellipsoids

We study partial ordering of ellipsoids, Hasse diagram of ellipsoids, and quotient set of Hasse diagram with respect to equivalence relation. The results in this section are based on the algebra of the relation by Birkoff [4]. These results lead to embedding of multiple graphs into hypercube in Section 5.

Definition 4.1 A proposition xRy is called a relation if aRb is determined true or false for each pair in the Cartesian product $X \times Y$.

Definition 4.2 A relation R is reflexive if xRx holds. R is symmetric if $xRy \rightleftharpoons yRx$. R is transitive if xRy and yRz imply xRz. R is called antisymmetric if xRy and yRx imply $x = y$. A reflexive, symmetric, and transitive relation is called an equivalence relation. A reflexive and transitive relation is called a preordering. A reflexive, transitive, and antisymmetric relation is called a partial ordering.

We now propose definitions about partial ordering between ellipsoids.

Definition 4.3 Let $E(a_1, Q_1)$, $E(a_2, Q_2)$ be n-dimensional two ellipsoids and $I_{i,j} = (a_{i,j} - \lambda_{i,j}^{\frac{1}{2}}, a_{i,j} + \lambda_{i,j}^{\frac{1}{2}})$ be interval of ellipsoid, where $a_{i,j}$ is the j-th element of vector a_i $(i = 1,2)$ and $\lambda_{i,j}$ is the j-th eigenvalue of λ_i that satisfies $Q_i x_i = \lambda_i x_i$ $(i = 1,2)$. If $I_{1,j} \subseteq I_{2,j}$ for all j, we define a partial ordering $E_1 \preceq E_2$.

Definition 4.4 For two parameterized solutions, $E_{p_1}^+[t] = E(a_{p_1}^+(t), Q_{p_1}^+(t))$, $E_{p_2}^+[t] = E(a_{p_2}^+(t), Q_{p_2}^+(t))$ to the funnel equation (5), let $I_{p,j}^+(t) = (a_{p,j}^+(t) - (\lambda_{p,j}^+(t))^{\frac{1}{2}}, a_{p,j}^+(t) + (\lambda_{p,j}^+(t))^{\frac{1}{2}})$ be an interval of the external ellipsoid, where $a_{p,j}^+(t)$ is the j-th element of vector $a_p^+(t)$ and $\lambda_{p,j}^+(t)$ is the j-th eigenvalue $\lambda_p^+(t)$ that satisfies $Q_p^+(t)v_p(t) = \lambda_p^+(t)v_p(t)$, $p \in \{p_1, p_2\}$. If $I_{p_1,j}^+(t) \subseteq I_{p_2,j}^+(t)$ for all j, we define a partial ordering of the ellipsoids as follows $E_{p_1}^+[t] \preceq E_{p_2}^+[t]$:

Definition 4.5 For two parameterized solutions, $E_{p_1}^-[t] = E(a_{p_1}^-(t), Q_{p_1}^-(t))$, $E_{p_2}^-[t] = E(a_{p_2}^-(t), Q_{p_2}^-(t))$ to the funnel equation (6), let $I_{p,j}^-(t) = (a_{p,j}^-(t) - (\lambda_{p,j}^-(t))^{\frac{1}{2}}, a_{p,j}^-(t) + (\lambda_{p,j}^-(t))^{\frac{1}{2}})$ be an interval of the external ellipsoid, where $a_{p,j}^-(t)$ is the j-th element of vector $a_p^-(t)$ and $\lambda_{p,j}^-(t)$ is the j-th eigenvalue $\lambda_p^-(t)$ that satisfies $Q_p^-(t)v_p(t) = \lambda_p^-(t)v_p(t)$, $p \in \{p_1, p_2\}$. If $I_{p_1,j}^-(t) \subseteq I_{p_2,j}^-(t)$ for all j, we define a partial ordering of the ellipsoids as follows $E_{p_1}^-[t] \preceq E_{p_2}^-[t]$.

Lemma 4.6 Let $(A_1, \preceq_1)$ and $(A_2, \preceq_2)$ be two partially ordered set. Suppose R is a relation on Cartesian product $A_1 \times A_2$ such that $(a_1, a_2)R(a_1', a_2')$ holds if and only if $a_1 \preceq_1 a_1'$ and $a_2 \preceq_2 a_2'$. Then $(A_1 \times A_2, R)$ is a partially ordered set. $(A_1 \times A_2, R)$ is called the direct product of $(A_1, \preceq_1)$ and $(A_2, \preceq_2)$ and also denoted by $(A_1, \preceq_1) \times (A_2, \preceq_2)$.

Definitions 4.3, 4.4, and 4.5 are based on Lemma 4.7.

Lemma 4.7 Suppose that $(I_{p_i,1}^+(t), \cdots\cdots, I_{p_i,n}^+(t))$ is an elements of a set $C(t)$ and a set $I_j^+(t)$ satisfies $I_{p_i,j}^+(t) \in I_j^+(t)$, where p_i $(i = 1, \cdots, m)$ is a parameter. Then $(I_1^+(t), \subseteq) \times \cdots\cdots \times (I_n^+(t), \subseteq) = (I_1^+(t) \times \cdots \times I_n^+(t), R)$ is a partially ordered set and $(C(t), R)$ is a partially ordered subset.

Proof For the proof of the first part, replace A_j as I_j, $\preceq_j$ as $\subseteq$, $1 \leq j \leq n$, in the Lemma 4.6. Since $C(t) \subseteq I_1^+(t) \times \cdots \times I_n^+(t)$, the partially ordered subset is also proved.□

Remark 4.8 It is obvious that an inclusion of the attainability sets $E_{p_1}(t) \subseteq E_{p_2}(t)$ implies partial orderings $E_{p_1}^+[t] \preceq E_{p_2}^+[t]$ (and $E_{p_1}^-[t] \preceq E_{p_2}^-[t]$), but $E_{p_1}^+[t] \preceq E_{p_2}^+[t]$ (or $E_{p_1}^-[t] \preceq E_{p_2}^-[t]$) do not necessarily guarantee inclusion $E_{p_1}(t) \subseteq E_{p_2}(t)$. These partial orderings become important information for constructing a Hasse diagram that is a data structure of the ellipsoids.

Definition 4.9 By "a covers b", it is meant that $a \succ b$ and $a \succ x \succ b$ is not satisfied by any x.

Lemma 4.10 Let $(A, \preceq)$ be a finite ordered set. If $a \prec b$ for $a, b \in A$, then some $x_1, \ldots.., x_n \in A$ can be selected such that $x_0 = a \prec x_1 \prec \ldots\ldots.. \prec x_n \prec x_{n+1} = b$ and x_{i+1} covers x_i for all i.

Proof Assume a set B such that $B = \{x | a \prec x \prec b\}$. Consider a case of $m = k$ based on the assumption that the proposition holds for $m \leq k-1$. Since B is not empty, there exists some c such that $a \prec c \prec b$. The number of elements of two sets $\{x | a \prec x \prec c\}$ and $\{x | c \prec x \prec b\}$ is less than k. From the assumption of the induction, there exists $y_1, ..., y_i$ and $z_1, ..., y_j$ which satisfy $a \prec y_1 \prec ..... \prec y_i \prec c$, $c \prec z_1 \prec ..... \prec z_j \prec b$ and coverage. Then $y_1, ... y_i, c, z_1, ..., z_j$ satisfies the proposition.□

Lemma 4.11 The finite partially ordered set can be represented by a Hasse diagram.

Proof Consider a finite partially ordered set (A, R). The number of elements of A is n. Since there exists maximal elements in A, we select one and call it a. Due to the assumption of the induction, a subset of the ordered set $(A - \{a\}, R)$ can be represented by a Hasse diagram. Let $b_1, ..., b_m$ be elements of $A - \{a\}$ that are covered by element a, then locate nodes of $A - \{a\}$ and connect a and b_i, $1 \leq i \leq m$, in the Hasse diagram H.□

Remark 4.12 A graph $G = (V, E)$ is connected if any two vertices of G are joined by a path in G. From the construction of the Hasse diagram shown in the proof of Lemma 4.11, the Hasse diagram is not necessarily a connected graph. Therefore, the quotient set of the Hasse diagram representing the partial ordering of ellipsoids is studied in terms of connectivity of the graph.

4.2 Quotient set of the Hasse diagram

Definition 4.13 Suppose that R is an equivalence relation. A family P of subset of a set A is called partition of A if the following holds:

(1) $A = \bigcup_{D \in P} D$

(2) $D \in P \longrightarrow D \neq \Phi$ (null set)

(3) $D, D' \in P$ and $D \neq D' \longrightarrow D \bigcap D' = \Phi$

(4) $xRy \rightleftharpoons D \in P$ exists such that $x, y \in D$.

Lemma 4.14 Let R be a relation with respect to connectivity of the graph $G = (V, E)$. Let $C(v_i) = \{v_j | v_i R v_j$ and $v_i, v_j \in V\}$ be a subset of the set V. Define a family P such that $P = \{D | D \subset V$ and $D = C(v_i)$ for some $v_i \in V\}$. Then P is a partition of V.

Proof The proof consists of two steps. First we show that a relation R with respect to connectivity of the graph is an equivalence relation. For any vertices of G, it is obvious that $v_i R v_i$ (reflexive), $v_i R v_j \rightleftharpoons v_j R v_i$ (symmetric), and $v_i R v_j, v_j R v_k$ implies $v_i R v_k$ (transitive).

Then from Definition 4.2, R is an equivalence relation. Second, we show that P is a partition of V. Conditions (1) and (2) of Definition 4.13 are obviously satisfied. Next suppose that $D, D' \in P$ and $D \neq D'$. There exist v, v' such that $D = C(v)$ and $D' = C(v')$. Let v be an element such that $d \in D \cap D'$. Relations vRd and $v'Rd$ hold. Relations dRv' (symmetric) and vRv' (transitive) also hold. Furthermore, a relation $v'Rv$ (symmetric) holds. Now suppose $x \in C(v)$, then $v'Rx$ holds using vRx and $v'Rv$. Then $x \in C(v')$ which implies $C(v) \subseteq C(v')$. Similarly $C(v') \subseteq C(v)$ holds. Since $C(v) \neq C(v')$ is a contradiction, condition (3) is satisfied. Condition (4) is proved by supposing $x, y \in D = C(z)$. Since zRx and zRy, xRy holds using symmetric and transitive laws. *Box*

Definition 4.15 Partition P of set V is called quotient set with respect to equivalence relation R. It is denoted by V/R. The element of the partition P is called an equivalence class. $C(v_i)$ is also called the equivalence class of v_i.

Lemma 4.16 A mapping f from V into V/R such that $v \in V \longrightarrow C(v) \in V/R$ is surjection.

Proof Suppose for $c \in V/R$ there is no $v \in V$ such that $c = f(v)$. By definition of V/R, $c = C(v')$ for some $v' \in V$. This is a contradiction.□

Lemma 4.17 Suppose a partition P is given. Define a relation R on V of a graph $G = (V, E)$ such that $v_i R v_j$ if and only if there exists some $D \in P$ such that $v_i, v_j \in D$. Then R is an equivalence relation.

Proof From condition (1) of Definition 4.13, for any $v \in V$, there exists some $D \in P$ such that $v \in D$, then vRv holds. From the definition of the Lemma 4.17, relation R satisfies the symmetric law. Suppose that xRy and yRz hold. There exist D and D' such that $x, y \in D$ and $y, z \in D'$. Since $D \cap D'$ is not empty due to existence of y, then $D = D'$ holds from condition (3) of Definition 4.13. Then xRz holds. Thus the transitivity law is also satisfied. □

Lemma 4.18 Let f be a mapping from V into R^1, where $G = (V, E)$. Define a relation R such that vRv', if and only if, $f(v) = f(v')$, where $v, v' \in V$. Then R is an equivalence relation. A mapping from V/R into $f(V)$ is bijection.

Proof Let $C(v)$ be an equivalence class that includes $v \in V$. Suppose g represents a mapping from V/R to $f(V) \subseteq R^1$ such that $g(C(v)) = f(v)$. By assumption, $C(v) = C(v')$, that is, vRv', is satisfied if and only if $f(v) = f(v')$. This implies $vRv' \rightleftharpoons v'Rv$ (symmetric). By replacing $v = v'$, vRv (reflexive) is obtained. By adding $C(v'') = C(v')$, that vRv' and $v'Rv''$ imply vRv'' (transitive) is obtained. Then R is an equivalence relation. Again, if $f(v) = f(v')$, then vRv', that is, $C(v) = C(v')$. This implies that mapping g is injection. If $c \in f(V)$, there exists some $v \in V$ such that $f(v) = c$. Since $g(C(v)) = f(v) = c$, mapping g is surjection. The mapping g that is injection and surjection implies bijection. □

Lemma 4.19 From a preordering R, an equivalence relation $\sim$ can be defined by $v \sim v' \rightleftharpoons (vRv'$ and $v'Rv)$ in V, where $G = (V, E)$. Then a partial ordering R^* can be defined in quotient set $V/\sim$ by $C(v)R^*C(v') \rightleftharpoons vRv'$, where $C(v)$ is an equivalence class of v with respect to $\sim$.

Proof It is proved that $\sim$ implies equivalence relation. By assumption $v \sim v' \rightleftharpoons (vRv'$ and $v'Rv)$, $v \sim v$ (reflexive), $v \sim v'$ and $v' \sim v"$ implies $v \sim v"$ (transitive), and $v \sim v' \rightleftharpoons v' \sim v$ (symmetric) hold. Next it is proved that R^* is a partial ordering. Since R is preordering, then vRv (reflexive) and vRv' and $v'Rv''$ implies vRv'' (transitive). By assumption $C(v)R^*C(v') \rightleftharpoons vRv'$, $C(v)R^*C(v)$ (reflexive) and that $C(v)R^*C(v')$ and $C(v')R^*C(v'')$ implies $C(v)R^*C(v")$ (transitive) hold. Suppose that $C(v)R^*C(v')$ and $C(v')R^*C(v)$ are satisfied. By assumption, vRv' and $v'Rv$, that is, $v \sim v'$ hold. Then $C(v) = C(v')$, which implies R^* is antisymmetric.□

Remark 4.20 Lemmas 4.14-4.19 give fundamental theories so that the Hasse diagram representing partial ordering may be decomposed into multiple-connected graphs. In Section 5, we study embedding these multiple connected graphs into the hypercube with preserving adjacency.

5 Embedding of Hasse Diagram into Hypercube

Definition 5.1 [5] Relaxed-squashed (RS) embedding is a node-to-subcube distance preserving mapping from source graph to the hypercube.

Definition 5.2 [5] The dimension of the minimal cube required for the RS embedding of a source graph $G = (V, E)$ is called the weak cubical dimension $wd(G)$ of the graph.

Definition 5.3 For graph $G = (V, E)$, an induced subgraph $ind_G(V_S)$ of G with a node set $V_S \subseteq V$ is the maximal subgraph with the node set V_S.

Lemma 5.4 [5] Let $G = (V, E)$ be a connected graph and let $G_S = (V_S, E_S)$ be a subgraph of G. Suppose that the induced subgraph $ind_G(V_S)$ can be RS embedded into m-dimensional hypercube Q_m, and the removal of all edges in E_S from G results in $|V_S|$ disjoint graphs, $G_i = (V_i, E_i)$, $1 \leq i \leq |V_S|$. Then $wd(G) \leq \max_{1 \leq i \leq |V_S|} wd(G_i) + m$.

Proof Let v_i, $1 \le i \le |V_S| = k$, be the nodes in $V_S \cap V_i$ of $G_S \cap G_i$. The notation $address_{G_S}(v_i)$ represents the encoding of v_i in V_S in order that G_S can be RS embedded into Q_m. The notation $address_{G_i}(v)$ represents the encoding of $v \in V_i$ for the RS embedding of G_i into $Q_{wd(G_i)}$, where $address^j_{G_i}(v)$ is the j-th bit of $address_{G_i}(v)$. The RS embedding generates $address_G(w)$ for each w in G by the following procedures:

Algorithm 5.5: RS Embedding of single graph

Step 1 For each $w \in V_i$, $1 \le i \le k = |V_S|$,
$address^j_G(w) \longleftarrow address^j_{G_S}(v_i)$, where $1 \le j \le m$.

Step 2 For each $w \in V_i$, $1 \le i \le k = |V_S|$,
if $address^j_{G_S \cap G_i}(v_i) = 1$
then $address^j_G(w) \longleftarrow \overline{address^{j-m}_{G_i}(v)}$
(overline indicates complement of binary code)
else $address^j_G(w) \longleftarrow address^{j-m}_{G_i}(v)$
where $m + 1 \le j \le wd(G_i) + m$.

Step 3 For each $w \in V_i$, $1 \le i \le k = |V_S|$,
$address^j_G(w) \longleftarrow *$ *(don't care symbol)*
where $wd(G_i) + m + 1 \le j \le \max_{1 \le i \le k} wd(G_i) + m$.

As shown in the Algorithm 5.5, the weak cubical dimension $wd(G)$, that is, the dimension of the minimum cube required for the RS embedding of the graph $G = (V, E)$, is smaller than $\max_{1 \le i \le |V_S|} wd(G_i) + m$.□

We propose a theorem and its algorithm about RS embedding of multiple graphs.

Theorem 5.6 Let $G_j = (V_j, E_j)$ be a connected graph and let $G_{j,S} = (V_{j,S}, E_{j,S})$ be a subgraph of G_j, $1 \le j \le m$. For each G_j, suppose that the induced subgraph $ind_{G_j}(V_{j,S})$ can be RS embedded into $wd(G_{j,S})$-dimensional hypercube $Q_{wd(G_{j,S})}$, and the removal of all edges in $E_{j,S}$ from G_j results in $|V_{j,S}|$ disjoint graphs, $G_{j,i} = (V_{j,i}, E_{j,i})$, $1 \le i \le |V_{j,S}|$. The multiple graphs, G_j, $1 \le j \le m$, are assumed to be RS embedded into n-dimensional hypercube Q_n that satisfies $2^{n-1} < \sum_{j=1}^{m} 2^{d_j} \le 2^n$, where $d_j = \max_{1 \le i \le |V_{j,S}|} wd(G_{j,i}) + wd(G_{j,S})$. Then, there exists an addressing scheme so that d_j-dimensional original addressing for each graph G_j, $1 < j \le m$, can be mapped into n-dimensional address of $G_1 \cup \cdots \cup G_m$ without any change of the original address.

Proof The RS embedding generates $address_{G_1 \cup \cdots \cup G_m}(w)$ for each $w \in G_1 \cup \cdots \cup G_m$ by the following procedures:

Algorithm 5.7: RS Embedding of multiple graphs

Step 1 Sort d_j, $1 \leq j \leq m$, in descendant order such that $d_1' \geq ..... \geq d_r'$, where $G_k' = (V_k', E_k')$ corresponds to the k'-th graph due to sorting. **Step 2** For each node $w \in V_k'$ in G_k', $1 \leq k \leq m$, encode $address_{G_1 \cup \cdots \cup G_m}(w)$ as follows:
$address_{G_1 \cup \cdots \cup G_m}(w) \longleftarrow address_{G_k'}(w) + binary(\sum_{j=1}^{k-1} 2^{d_j'})$,
where $address_{G_k'}(w)$ is the embedded encoding into Q_k'.

Binary representation of original encoding, $address_{G_k'}(w)$ is $a_1 \cdots a_{d_k}'$, where $a_i = 0, 1$ or $*$ *(don't care symbol)* $1 \leq i \leq d_k'$. On the other hand, binary representation of $\sum_{j=1}^{k-1} 2^{d_j'}$ in Q_n, i.e., $binary(\sum_{j=1}^{k-1} 2^{d_j'})$ is $b_1 \cdots b_{n-d_{k-1}'} 0 \cdots 0$ where $b_j = 0$ or 1, $1 \leq j \leq n - d_{k-1}'$, and the number of the rightmost 0s is d_{k-1}'. Since $d_{k-1}' \geq d_k'$, consider two cases, $d_{k-1}' > d_k'$ and $d_{k-1}' = d_k'$. In the case of $d_{k-1}' > d_k'$, binary representation of $address_{G_1 \cup \cdots \cup G_m}(w)$ becomes $b_1 \cdot \cdot b_{n-d_{k-1}'} 0 \cdot \cdot 0 a_1 \cdot \cdot a_{d_k'}$. In the case of $d_{k-1}' = d_k'$, binary representation of $address_{G_1 \cup \cdots G_m}(w)$ becomes $b_1 \cdot \cdot b_{n-d_{k-1}'} a_1 \cdot \cdot a_{d_k'}$. Thus, d_j-dimensional original addressing for each graphs G_j, $1 \leq j \leq m$, is mapped into n-dimensional addresss of $G_1 \cup \cdots G_m$ without any change of the original address. □

6 Parallel Detection of Inclusion between Ellipsoids

We propose two theorems and their relevant algorithms about parallel detection of inclusion between ellipsoids that are external or internal estimates of the attainability set in the differential inclusion. The first theorem and its relevant algorithm is concerned with a single-connected graph that is a Hasse diagram representing partial ordering of the ellipsoids, the second theorem ones is concerned with multiple-connected graphs.

Theorem 6.1 Assume the condition of Lemma 5.4 is satisfied. The number of vertices of the connected graph $G = (V, E)$ satisfies an inequality $2^{n-1} <$ number of $V \leq 2^n$. Then, there exists an addressing scheme that enables allocation of every elements $(v_i, v_j) \in V \times V$ in d-dimensional hypercube Q_d, where $\max_{1 \leq i \leq |V_S|} G_i + wd(G_S) + n$.

Proof The following parallel detection algorithm determines an addressing scheme.

Algorithm 6.2: Parallel detection of inclusion between ellipsoids (single graphs)

Step 1 For each $v_i \in V$, allocate $address_G(v_i)0 \cdots 0$, where $address_G(v_i)$ is the RS-embedded encoding of $(\max_{1 \leq i \leq |V_S|} G_i + wd(G_s))$ dimension and the number of 0s is n.

Step 2 Broadcast ellipsoidal information $E_i^+[t]$ (or $E_i^-[t]$) from $address_G(v_i)0 \cdots 0$ to all nodes in the subcube $address_G(v_i) * \cdots *$, where the number of $*$ *(don't care symbol)* is n. This broadcast is parallel for every $v_i \in V$.

Step 3 Broadcast ellipsoidal information $E_i^+[t]$ (or $E_i^-[t]$) from $address_G(v_i)0\cdots 0$ to all $address_G(v_j)0\cdots 0$, where $v_j(\neq v_i) \in V$. This broadcast is also parallel for each $v_i \in V$.
Step 4 For each subcube $address_G(v_i) *\cdots*$, allocate received ellipsoidal information $E_j^+[t]$ (or $E_j^-[t]$) from $address_G(v_i)0\cdots 0$ to an $address_G(v_i)G_n(j)$, where $G_n(j)$ is the j-th encoding of n-dimensional binary-reflected Gray code (BRGC). This allocation is carried out for every $v_j \in V$ in the subcube $address_G(v_i) *\cdots*$.
Step 5 Detect inclusion between couple of ellipsoids, $E_i^+[t]$, $E_j^+[t]$ (or $E_i^-[t]$, $E_j^-[t]$), at every $address_G(v_i)G_n(j)$ in the hypercube.

As shown in Algorithm 5.5, in the subcube $address_G(v_i) *\cdots*$, there exists an addressing scheme $address_G(v_i)G_n(v_j)$ with $(\max_{1\leq i\leq |V_S|} G_i + wd(G_S) + n)$ dimension that is allocated to every elements $(v_i, v_j) \in V \times V$. □

Theorem 6.3 Assume that the condition of Theorem 5.6 is satisfied. The number of vertices of the k-th connected graph $G_k = (V_k, E_k)$ satisfies inequality $2^{n_k-1} <$ number of $V_k \leq 2^{n_k}$, $1 \leq k \leq m$. Then, there exists an addressing scheme that enables allocation of every elements $(v_{k,i}, v_{k,j}) \in V_k \times V_k$ into $(n + \max_{1\leq k\leq m} n_k)$-dimensional hypercube that satisfies $2^{n-1} < \sum_{k=1}^{m} 2^{d_k} \leq 2^n$, where $d_k = \max_{1\leq i\leq |V_{k,S}|} wd(G_{k,i}) + wd(G_{k,S})$.

Proof The following parallel detection algorithm determines an addressing scheme.

Algorithm 6.4: Parallel detection of inclusion between ellipsoids (multiple graphs)

Step 1 For each $v_{k,i} \in V_k$, $1 \leq k \leq m$, allocate $address_{G_1\cup\cdots\cup G_m}(v_{k,i})0\cdots 0$ where $address_{G_1\cup\cdots\cup G_m}(v_{k,i})$ is the encoding obtained using Algorithm 5.7 that is the RS embedding of multiple graphs, and the number of 0s is $\max_{1\leq k\leq m} n_k$.
Step 2 Broadcast ellipsoidal information $E_{k,i}^+[t]$ (or $E_{k,i}^-[t]$) from $address_{G_1\cup\cdots\cup G_m}(v_{k,i})0\cdots 0$ to all nodes in the subcube $address_{G_1\cup\cdots\cup G_m}(v_{k,i})0\cdots 0 *\cdots*$, where number of 0s is $(\max_{1\leq k\leq m} n_k) - n_k$ and number of $*$ (*don't care symbol*) is n_k. This broadcast is parallel for every $v_{k,i} \in V_k$, $1 \leq k \leq m$.
Step 3 Broadcast ellipsoidal information $E_{k,i}^+[t]$ (or $E_{k,i}^-[t]$) from $address_{G_1\cup\cdots\cup G_m}(v_{k,i})0\cdots 0$ to all $address_{G_1\cup\cdots\cup G_m}(v_{k,j})0\cdots 0$, where $v_{k,j}(\neq v_{k,i}) \in V_k$. This broadcast is also parallel for each $v_{k,i} \in V_k$, $1 \leq k \leq m$.
Step 4 For each subcube $address_{G_1\cup\cdots\cup G_m}(v_{k,i})0\cdots 0 *\cdots*$ used in Step 2, allocate ellipsoidal information $E_{k,j}^+[t]$ (or $E_{k,j}^-[t]$) from $address_{G_1\cup\cdots\cup G_m}(v_{k,i})0\cdots 0$ ($\max_{1\leq k\leq m} n_k$ 0s) to $address_{G_1\cup\cdots\cup G_m}(v_{k,i})0\cdots 0G_{n_k}(j)$ (($\max_{1\leq k\leq m} n_k) - n_k$ 0s), where $G_{n_k}(j)$ is the j-th encoding of n_k-dimensional binary reflected Gray code (BRGC). This allocation is carried

out in the subcube $address_{G_1\cup\cdots\cup G_m}(v_{k,i})0\cdots0*\cdots*$ for every $v_{k,j}\in V_k$, $1\le k\le m$.

Step 5 Detect inclusion between couple of ellipsoids, $(E^+_{k,i}[t], E^+_{k,j}[t])$ or $(E^-_{k,i}[t], E^-_{k,j}[t])$, at every $address_{G_1\cup\cdots\cup G_m}(v_{k,i})0\cdots0G_{n_k}(j)$ in the hypercube.

As shown in Algorithm 5.7, there exists an addressing scheme $b_1\cdots b_{n-d'_k}a_1\cdots a_{d'_k}0\cdots 0G_{n_k}(j)$ for $(v_{k,i}, v_{k,j})\in V_k\times V_k$, where $b_1\cdots b_{n-d'_k}a_1\cdots a_{d'_k}$ implies binary representation of $address_{G_1\cup\cdots\cup G_m}(v_{k,i})$ (see proof of Theorem 5.6), and the number of 0s is $(\max_{1\le k\le m} n_k)-n_k$. □

7 Concluding Remarks

This paper presents a hypercube parallel processing for detection of inclusion between ellipsoids in the differential inclusion. The approach is characterized by (1) constructing data structure about partially ordering of the ellipsoids, (2) embedding this data structure into hypercube with preserving adjacency and (3) detecting inclusion of the ellipsoids in parallel manner. An alternative approach may be considered that is characterized by the embedding of multidimensional array, $(E^+_i[t], E^+_j[t])$, into hypercube. This approach can be developed from Lemma 7.10.

Lemma 7.1 [14] The $2^n\times 2^n$ mesh can be embedded into $2n$-dimensional hypercube Q_{2n} by the following recursive manner:

$$A^{2^{i+1}} = \begin{pmatrix} G_2(0)A^{2^i} & G_2(1)(A^{2^i}_{\pi/2})^T \\ G_2(3)(A^{2^i}_{3\pi/2})^T & G_2(2)A^{2^i}_{\pi} \end{pmatrix}$$

$$A^{2^1} = \begin{pmatrix} G_2(0) & G_2(1) \\ G_2(3) & G_2(2) \end{pmatrix},$$

where $A^{2^i}_\theta$ stands for rotation of A^{2^i} by θ radian and $G_2(i)$, $0\le i\le 3$ is the i-th encoding of two-dimensional BRGC.

When we compare these two approaches, our approach surpasses the alternative approach due to embedding with preserving adjacency in a sense of partial ordering of ellipsoids. Because, although adjacent ellipsoids do not necessarily imply inclusion between the ellipsoids, the possibilities of inclusion between adjacent ellipsoids are good, thus low-cost communication is expected between adjacent nodes or subcubes leading to efficient parallel computation.

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