HOW LIKELY ARE CATASTROPHES?

Natura Non Facit Saltus

Anonymous

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Foreword

The following paper was not written by me, but by a colleague who, for reasons that may become apparent to you, wishes to remain anonymous. Moreover, I wish to dissociate myself from his conclusions. However, his argument seemed to me sufficiently ingenious, and the local interest in catastrophe theory so keen, that the unprecedented step of publishing an internal working paper anonymously ought to be contemplated. Should any colleague wish to respond, I shall of course be glad to serve as postbox.

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[Note to readers: Because the topic of catastrophes is politically extremely sensitive, I have chosen to remain anonymous and to simply let the following results speak for themselves.]

Although catastrophes are a research topic of the greatest current importance, it can be asked: how likely is it that a given situation will contain a catastrophe? Casual observation, such as that of Aristotle cited above, indicates that catastrophes are indeed quite rare. In fact, this intuition can be quite rigorously justified if we take care to define likely.

In what follows we examine the likelihood of a catastrophe occurring in a very general dynamic model. The standard way to demonstrate the likeliness of some property is to show that it holds generically, that is on an open dense set with respect to some meaningful topology (see Feller [2]). We show that for a carefully chosen topology on the space of all dynamic models, which we call the "γ-topology", just such a statement is true for the non-existence of catastrophes.

We define a dynamic system with state space $\in R^n$ as a 2m-tuple $[f_1, \ldots, f_m, R_1, \ldots, R_m]$, where m denotes the number of agents or dynamics involved, f_i is the dynamic of the ith agent, and R_i is the constraint set of the ith dynamic or agent. In these, m is greater than or equal to one, but must be finite. (We stress that the results only apply to dynamic systems with a finite number of agents. Perhaps the powerful techniques introduced by Robinson and Brown, with the use of non-standard analysis, can extend the results to systems with an infinite number of agents.) Finally, a catastrophe is a region $C \in R^n$ such that a fold or singularity occurs (see Thom [5]).

Consider the set D of all such dynamic systems. Let D^c be the subset of D in which catastrophes occur, and let D^w be the complement of D^c. D^w is clearly non-empty, since a linear model satisfied the conditions.

<u>Definition</u>. Define the γ-topology on D be the following system of open sets:

$$[D, D^w, \emptyset]$$

where $\emptyset$ is the null set. It is easily verified that this is indeed a topology (see Bourbaki [1], p. 13).

In the following all topological notions are with respect to the γ-topology. We now state our main result.

Theorem 1. There exists an open dense set of dynamic systems which have no catastrophe surface.

Proof. D^W is clearly open and dense with respect to the γ-topology (see Bourbaki [1], p. 23).

In order to justify the γ-topology, we prove the following theorem:

Theorem 2. The γ-topology is a coarsest topology for which Theorem 1 holds.

Proof. The only topology strictly coarser than the γ-topology is the "indiscrete topology" which clearly does not have the required property (see Kelley [4], p 37-38).

It should be noted that there are other topologies with the same property and we give as examples the ∂-topology and the $\mathcal{E}$-topology. The ∂-topology in D is defined by the following system of open sets where (d) represents an arbitrary dynamic system without a catastrophe surface:

$$[D, D^W-(d), \emptyset] \quad .$$

Similarly define the $\mathcal{E}$-topology by the system of open sets:

$$[D, (d), \emptyset]$$

where d is as defined above. It seems to us that the γ-topology is in a certain sense maximal and seems also intuitively the most reasonable to use for practical purposes.

It should be clear that the method used in this remark (introduced by Grandmont et al. [3]) may be fruitfully used to obtain generic statements about a wide range of models and phenomena, however implausible the results might seem at first blush. Once the essential definition of "likelihood" is grasped, and when the applicability of the theorems to the particular problem at hand is understood, then the reader will appreciate the full significance of the results and others that can be obtained in the same way.

Further, it seems that given the <u>unlikelihood</u> of the catastrophe models, more thought should be given to the immense scientific effort currently engaged in this line of research. It must be asked, in light of their mathematical unlikelihood, whether some darker motives impel research into this area.

List of References

[1] Bourbaki, N., *Eléments de Mathématique Topologie Générale*, Paris, Herman, 1965.

[2] Feller, W., *Foundations of Probability Theory*, Vol. II, Wiley, 1964.

[3] Grandmont, J. M., A. P. Korman, and W. Neufeind, "A New Approach to Uniqueness of Equilibrium" *Review of E. Stud.*, April 1974.

[4] Kelley, J. L. *General Topology*, Nostrand, Princeton, 1955.

[5] Thom, R., *Stabilité Structurelle et Morphogenèse*, Benjamin, New York, 1972.