# CONDITIONAL UTILITY INDEPENDENCE AND ITS APPLICATION TO TIME STREAMS

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Abstract

The evaluation of time streams is traditionally performed by some form of discounting and even the more sophisticated approaches require some form of independence assumptions between consequences in adjacent periods. Frequently a decision maker's preferences for consequences in a given period will depend on the particular outcome in the previous and/or following period. This paper gives a simple functional form which enables such preferences to be explicitly included in a utility function for time streams.

The assessment of one dimensional, or one attribute, utility functions is fairly straightforward and there are now a number of interactive computer programs which will aid the assessment of two dimensional utility functions [e.g.6]. For higher dimensions some simplifying assumptions are required to reduce the form of the utility function so that it is only necessary to assess low dimensional functions.

A useful assumption that is often applicable is that of utility independence and Keeney [3] in particular has shown how this concept can greatly simplify the assessment of utility functions. For a problem having n attributes $Y_1, Y_2, \ldots, Y_n$ a subset $X_1 = (Y_1, Y_2, \ldots, Y_s)$ of attributes is said to be utility independent of its complement $X_2 = (Y_{s+1}, Y_{s+2}, \ldots, Y_n)$ if decisions under uncertainty, where the values of $X_2$ are known and constant, are independent of the particular constant value taken by $X_2$. That is, if

$$u(\hat{x}_1, x_2^o) = \tfrac{1}{2}u(x_1^*, x_2^o) + \tfrac{1}{2}u(x_1^o, x_2^o)$$

for some value $x_2^o$ of $X_2$ then

$$u(\hat{x}_1, x_2^*) = \tfrac{1}{2}u(x_1^*, x_2^*) + \tfrac{1}{2}u(x_1^o, x_2^*)$$

for all other values $x_2^*$ of $X_2$.

Since a utility function is unique excepting for positive linear transformations if $X_1$ is utility independent of $X_2$ then

$$u(x_1,x_2) = f(x_2) + g(x_2)\, u(x_1,x_2^o)$$

where $x_2^o$ is an arbitary value of $X_2$ and $g(\cdot) > 0$.
Keeney has shown that if, in addition, $X_2$ is utility independent of $X_1$ then

$$u(x_1,x_2) = u(x_1,x_2^o) + u(x_1^o,x_2) + k\, u(x_1,x_2^o)\, u(x_1^o,x_2) \cdots\cdot \quad (1)$$

where k is a constant and $x_1^o$ is an arbitrary value of $X_1$, where $u(\cdot,\cdot)$ is scaled so that $u(x_1^o,x_2^o) = 0$.
Note that for an assumption of utility independence to hold a subset of attributes must be independent of all the attributes in its complement. We will say (compare Section 6 in Keeney [4]) that for a situation having three disjoint complete vector attributes $X_1$, $X_2$, $X_3$ that $X_1$ is <u>conditionally utility independent</u> of $X_2$ if for <u>any</u> fixed value of $X_3$, $X_1$ is utility independent of $X_2$.
<u>Result 1</u>. If $X_1$ is conditionally utility independent of $X_2$ then

$$u(x_1,x_2,x_3) = f(x_2,x_3) + g(x_2,x_3)\, u(x_1,x_2^o,x_3)$$

where $g(\cdot,\cdot) > 0$ and $x_2^o$ is an arbitrary value of $X_2$ and if in addition $X_2$ is conditionally utility independent of $X_1$ then

$$u(x_1,x_2,x_3) = [1 - k(x_3)\, u(x_1^o,x_2^o,x_3)] [u(x_1,x_2^o,x_3) + u(x_1^o,x_2,x_3)$$

$$- u(x_1^o,x_2^o,x_3)] + k(x_3)\, u(x_1,x_2^o,x_3)\, u(x_1^o,x_2,x_3)$$

where $x_1^o$, $x_2^o$ and $x_3^o$ are arbitrary values of $X_1$, $X_2$, $X_3$.

Proof. For a fixed value of $X_3, x_3^o$ we have that

$$u(x_1,x_2,x_3^o) = f^o(x_2) + g^o(x_2)\, u(x_1,x_2^o,x_3^o) \quad .$$

for some functions $f^o$, $g^o$.

If we define $f(x_2,x_3)$ and $g(x_2,x_3)$ to be such that $f(x_2,x_3^o) = f^o(x_2)$, $g(x_2,x_3^o) = g^o(x_2)$ for each choice of $x_3^o$ then

$$u(x_1,x_2,x_3) = f(x_2,x_3) + g(x_2,x_3)\, u(x_1 x_2^o,x_3) \quad .$$

If $X_2$ is conditionally utility independent of $X_1$ and defining $\bar{u}(x_1,x_2,x_3) = u(x_1,x_2,x_3) - u(x_1^o,x_2^o,x_3)$ then from (1)

$$\bar{u}(x_1,x_2,x_3^*) = \bar{u}(x_1,x_2^o,x_3^*) + \bar{u}(x_1^o,x_2,x_3^*) + k^*\, \bar{u}(x_1,x_2^o,x_3^*)\, \bar{u}(x_1^o,x_2,x_3^*)$$

for any fixed $x_3^*$ and hence

$$\bar{u}(x_1,x_2,x_3) = \bar{u}(x_1,x_2^o,x_3) + \bar{u}(x_1^o,x_2,x_3) + k(x_3)\,\bar{u}(x_1,x_2^o,x_3)\,\bar{u}(x_1^o,x_2,x_3) \quad .$$

Substituting for $\bar{u}$ in this expression gives the result. ||

## Application to Time Streams

For a problem involving consequences which do not all occur at the same time an outcome may be described in terms of the defining attributes X by a vector $(X_1,X_2,X_3,\ldots,X_T)$ of attributes where $x_i$ is the value of X at time period i, and where T might be infinite. Thus for a practical assessment a utility function $u(x_1,x_2,\ldots,x_T)$ must have some simplifying assumptions made concerning its form or on independence relationships between the $X_i$'s. The standard discounting assumption, that

$$u(x_1,x_2,\ldots,x_T) = u^*(\sum_{i=1}^{T} \alpha^i x_i)$$

where $u^*$ is a one dimensional utility function and $0 \leq \alpha \leq 1$ has no theoretical basis for use in situations involving uncertainty unless $u^*$ is linear. Koopmans [5] has investigated assumptions which justify the use of discounted utilities,

$$u(x_1,x_2,\ldots,x_T) = \sum_{i=1}^{T} \alpha^i u_i(x_i)$$

and Bell [1] has used a two attribute utility function $u^*(x,t)$ to approximate u and gives assumptions for the existence of a function $g(t)$ such that $u^*(x,t) = g(t)\, u^*(x,0)$.

• Meyer [7] has used the concept of utility independence to establish a form

$$u(x_1,x_2,x_3,\ldots,x_T) = \prod_{i=1}^{T} (a_i + b_i u_i^*(x_i))$$

by assuming that for each m

$$(X_1,\ldots,X_m) \quad \text{and} \quad (X_{m+1},\ldots,X_T)$$

are mutually utility independent.

All these studies assume some form of independence between preferences for $X_i$ and all other $X_j$'s. It is clear that some assumptions must be made but there are many situations where preferences for outcomes in one period are heavily dependent on the outcomes in other periods, particularly in adjacent periods. A person may be very risk averse in situations which might cause him to experience a level of consumption in one period which is lower than that in the previous period; a politician may regard it worse to raise pensions in one period and then lower them in the next than never to raise them at all.

It will be shown here that using the idea of conditional utility independence, but without assuming anything about the relationship between an outcome in one period and the outcomes in adjacent periods, can give a greatly simplified and manageable form of the utility function.

Arbitrary levels $x_1^o, x_2^o, \ldots, x_T^o$ for each period are taken and u scaled so that $u(x_1^o, x_2^o, \ldots, x_T^o) = 0$. For notational purposes an attribute which is at its arbitrary level will not be written explicitly, hence $u(x_1^o, x_2, x_3^o)$ will be written as $u(x_2)$, $u(x_1, x_2^o, x_3)$ as $u(x_1, x_3)$ and so on.

<u>Result 2</u>. Assuming that for each $i=1,\ldots,T$

(i) $X_i$ is conditionally utility independent of $X_1, X_2, \ldots, X_{i-2}, X_{i+2}, \ldots, X_T$ .

(ii) For each value $x_i$ of $X_i$ there exist values $x_{i-1}^*$ of $X_{i-1}$ and $x_{i+1}^*$ of $X_{i+1}$ such that

$$u(x_{i-1}^*, x_i) \neq u(x_{i-1}^o, x_i)$$

$$u(x_i, x_{i+1}^*) \neq u(x_i, x_{i+1}^o)$$

then for $T \geq 4$ either

a) $$u(x_1, x_2, \ldots, x_T) = \sum_{i=1}^{T-1} u(x_i, x_{i+1}) - \sum_{i=2}^{T-1} u(x_i)$$

or

b) $$u(x_1, x_2, \ldots, x_T) = \left[\prod_{i=2}^{T-1} (w + u(x_i))\right]^{-1} \left[\prod_{i=1}^{T-1} (w + u(x_i, x_{i+1}))\right] - w$$

where w is a constant which may be taken as $\pm 1$.

<u>Proof</u>. The result is actually true, trivially, for T = 2 but for T = 3 we have attributes $X_1$, $X_2$, $X_3$ with the assumptions that $X_1$ and $X_3$ are mutually conditionally utility independent

which from Result 1 gives that

$$u(x_1,x_2,x_3) = [1 - k(x_2)\, u(x_2)]\,[u(x_1,x_2) + u(x_2,x_3) - u(x_2)]$$

$$+ k(x_2)\, u(x_1,x_2)\, u(x_2,x_3) \quad . \qquad (2)$$

For T = 4 we have that $\{X_1, X_2\}$ are mutually conditionally utility independent with $X_4$ and $X_1$ is mutually conditionally independent with $\{X_3, X_4\}$. Regarding $X_1$, $X_2$ as one vector attribute we may use Result 1 to give that

$$u(x_1,x_2,x_3,x_4) = [1 - s(x_3)\, u(x_3)]\,[u(x_1,x_2,x_3) + u(x_3,x_4) - u(x_3)]$$

$$+ s(x_3)\, u(x_1,x_2,x_3)\, u(x_3,x_4) \qquad (3)$$

and regarding $X_3$, $X_4$ as a single attribute Result 1 gives

$$u(x_1,x_2,x_3,x_4) = [1 - k(x_2)\, u(x_2)]\,[u(x_1,x_2) + u(x_2,x_3,x_4) - u(x_2)]$$

$$+ k(x_2)\, u(x_1,x_2)\, u(x_2,x_3,x_4) \quad . \qquad (4)$$

for some functions $s(x_3)$ and $k(x_2)$.
Substitution of $X_1 = x_1^o$ in (3) gives

$$u(x_2,x_3,x_4) = [1 - s(x_3)\, u(x_3)]\,[u(x_2,x_3) + u(x_3,x_4) - u(x_3)]$$

$$+ s(x_3)\, u(x_2,x_3)\, u(x_3,x_4) \quad . \qquad (5)$$

and $X_4 = x_4^o$ in (4) gives

$$u(x_1,x_2,x_3) = [1 - k(x_2)\ u(x_2)]\,[u(x_1,x_2) + u(x_2,x_3) - u(x_2)]$$

$$+ k(x_2)\ u(x_1,x_2)\ u(x_2,x_3) \quad . \qquad (6)$$

Now substitute (5) into (4) and (6) into (3), then subtraction of (4) from (3) gives that

$$A(x_2,x_3)\,[-u(x_1)\ u(x_3) + u(x_3)\ u(x_1,x_2) + u(x_2)\ u(x_3,x_4) - u(x_1,x_2)u(x_3,x_4)] \equiv 0 \qquad (7)$$

where

$$A(x_2,x_3) = s(x_3) - k(x_2) - s(x_3)\ k(x_2)\,[u(x_2) - u(x_3)] \quad .$$

Suppose that there exist values of $X_2$, $X_3$, say $x_2^*$, $x_3^*$, such that

$$A(x_2^*,x_3^*) \neq 0$$

then it must be that

$$- u(x_2^*)\ u(x_3^*) + u(x_3^*)\ u(x_1,x_2^*) + u(x_2^*)\ u(x_3^*,x_4)$$

$$- u(x_1,x_2^*)\ u(x_3^*,x_4) \equiv 0 \qquad \text{for all } x_1,\ x_4.$$

By assumption we may choose a value $X_4$, $x_4^*$ such that $u(x_3^*,x_4^*) \neq u(x_3^*,x_4^o)$ hence from (8)

$$[u(x_1,x_2^*) - u(x_2^*)]\ [u(x_3^*,x_4^*) - u(x_3^*,x_4^o)] = 0$$

which implies that

$$u(x_1,x_2^*) = u(x_1^o,x_2^*)$$

for all $x_1$ a contradiction to assumption (ii).

Hence $A(x_2,x_3) \equiv 0$ .

Thus

$$s(x_3) = k(x_2)/[1 - k(x_2)\ u(x_2) + k(x_2)\ u(x_3)] \quad ,$$

so that if $k(x_2^*) = 0$ for some $x_2^*$ then $s(x_3) \equiv 0$ (similarly $s(x_3^*) = 0$ implies $k(x_2) \equiv 0$) otherwise

$$s(x_3) = 1/[k(x_2)^{-1} - u(x_2) + u(x_3)]$$

implying that

$$k(x_2)^{-1} - u(x_2) = \text{constant} = w \quad \text{say,}$$

or $\quad k(x_2) = (w + u(x_2))^{-1},$ (8)

and $\quad s(x_3) = (w + u(x_3))^{-1}$ . (9)

Substituting (6), (8) and (9) into (3) gives

$$u(x_1,x_2,x_3,x_4) =$$

$$[w + u(x_1,x_2)]\,[w + u(x_2,x_3)]\,[w + u(x_3,x_4)]/[w + u(x_2)]\,[w + u(x_3)] - w \quad .$$

If $k(x_2) \equiv s(x_3) \equiv 0$ then

$$u(x_1,x_2,x_3,x_4) = u(x_1,x_2) + u(x_2,x_3) + u(x_3,x_4) - u(x_2) - u(x_3) \tag{11}$$

Now the proof for $T \geq 5$ may proceed by induction on T. For $X_1,\ldots,X_{T+1}$, by Result 1

$$u(x_1,\ldots,x_{T+1}) = (1 - s(x_T)\ u(x_T))\,[u(x_1,\ldots,x_T) + u(x_T,x_{T+1}) - u(x_T)] + s(x_T)\ u(x_1,\ldots,x_T)\ u(x_T,x_{T+1}) \tag{12}$$

and

$$u(x_1,\ldots,x_{T+1}) = [1 - k(x_2)\ u(x_2)]\,[u(x_1,x_2) + u(x_2,\ldots,x_{T+1}) - u(x_2)] + k(x_2)\ u(x_1,x_2)\ u(x_2,\ldots,x_{T+1}) \tag{13}$$

By induction we may assume that each of $u(x_1,\ldots,x_T)$ and

$u(x_2,\ldots,x_{T+1})$ has either the additive or multiplicative form and by substituting $X_i = x_i^o$ for all but i = 2,4 it may be seen that either both are additive or both are multiplicative with the same parameter w.

Suppose both are additive.

Then $u(x_1,x_2,x_3) = u(x_1,x_2) + u(x_2,x_3) - u(x_2)$ but from (13)

$$u(x_1,x_2,x_3) = [1 - k(x_2)\, u(x_2)]\,[u(x_1,x_2) + u(x_2,x_3) - u(x_2)]$$

$$+ k(x_2)\, u(x_1,x_2)\, u(x_2,x_3) \qquad (14)$$

Hence

$$-k(x_2)\, u(x_2)\,[u(x_1,x_2) + u(x_2,x_3) - u(x_2)] + k(x_2)\, u(x_1,x_2)\, u(x_2,x_3) \equiv 0$$

implying that $k(x_2) \equiv 0$ and

$$u(x_1,x_2,\ldots,x_{T+1}) = \sum_{i=1}^{T} u(x_i,x_{i+1}) - \sum_{i=2}^{T} u(x_i) \quad .$$

If both are multiplicative

$$u(x_1,x_2,x_3) = (w + u(x_1,x_2))(w + u(x_2,x_3))/(w + u(x_2))-w$$

and comparing with (14) we have that

$$k(x_2) = (w + u(x_2))^{-1}$$

Similarly $s(x_T) = (w + u(x_T))^{-1}$

and .

$$u(x_1,x_2,\ldots,x_{T+1}) = \left[\sum_{i=2}^{T} (w + u(x_i))\right]^{-1} \left[\sum_{i=1}^{T} (w + u(x_i,x_{i+1}))\right] - w$$

Notice that we may assume that $w = \pm 1$ for if $w > 0$ then make the substitution $u = w\bar{u}$ and if $w < 0$ make the substitution $u = -w\bar{u}$, then the results will be of the required form. ||

Note that putting $X_4 = x_4^o$ into (10) gives a special case of (2). It is not possible to infer that this special case is always valid for $u(x_1,x_2,x_3)$. To be more precise, let us call the utility function for $X_1,\ldots,X_n$, $u_n$. Then $u_n$ has the additive or multiplicative form for $n \geq 4$ but not necessarily for $n = 3$. Thus, it may be the case that

$$u_4(x_1,x_2,x_3,x_4^o) \neq u_3(x_1,x_2,x_3) \quad .$$

It is important to realize that in the proof of Result 2 $u_3$ only appeared in equation (2), $u_4$ for equations (3) to (11) and $u_T$ for equations (12) to the end of the proof. The difference occurs because of the assumption that $X_4$ was not a degenerate attribute (see assumption (ii)).

Result 2 can be specialized to the case where preferences for $X_i$ are conditionally utility independent of everything but $X_{i-1}$. In this case we have in addition that

$$u(x_i, x_{i+1}) = u(x_i) + k_{i+1}(x_{i+1})\ u(x_i) \quad .$$

Stationarity

Using Result 2 the derivation of $u(x_1, x_2, \ldots, x_T)$ requires the assessment of T-1 two attribute utility functions

$$u_1(x,y) = u(x,y,x_3^o,\ldots,x_T^o)$$

$$u_2(x,y) = (x_1^o,x,y,x_4^o,\ldots,x_T^o)$$

and so on, with the additional constraint that

$$u_i(x_i^o,y) = u_{i+1}(y,x_{i+2}^o) \qquad \text{for all } i = 1,\ldots,T-2 \quad .$$

For small values of the time horizon T this might be reasonable to do directly but for large T (and in particular for infinite T), some other assumption is required. The concept of stationarity of preferences is often appropriate, or at least reasonable, and greatly reduces the amount of assessment required. The idea is that if a decision maker is willing to accept some uncertain gamble then if the resolution of the uncertainty and all payments, receipts connected with the gamble are delayed by some fixed amount of time, the decision maker should still be willing to accept the gamble. It does not say anything about his absolute preferences for the gamble, only that his relative preferences are unaffected.

We will assume that the decision maker's preferences regarding decisions under uncertainty affecting two adjacent periods, with all other periods fixed at their arbitrary level, are independent of the particular two periods chosen, that is, tradeoffs between two periods are utility independent of time. This assumption is likely to be reasonable if $x_i^o = x^o$ for all i and the decision maker has no deadlines or important dates which make certain periods special in some way. It ensures that

$$u(x_1^o, x_2^o, \ldots, x_{i-1}^o, x, y, x_{i+2}^o, \ldots, x_T^o) = \alpha_i \, u(x, y, x_3^o, \ldots, x_T^o)$$

for some constant $\alpha_i$, for all i.

Result 3. Combining the assumptions of Result 2 and of stationarity, and assuming $x_i^o = x^o$ for all i then either

$$u(x_1, x_2, \ldots, x_T) = \sum_{i=1}^{T-1} \alpha^{i-1} u^*(x_i, x_{i+1}) - \sum_{i=2}^{T-1} \alpha^{i-1} u^*(x_i, x^o)$$

or

$$u(x_1, x_2, \ldots, x_T) = \left[\prod_{i=2}^{T-1} (w + \alpha^{i-1} u^*(x_i, x^o))\right]^{-1} \left[\prod_{i=1}^{T-1} (w + \alpha^{i-1} u^*(x_i, x_{i+1}))\right] - w$$

where $\alpha$ is constant and $u^*(x^o, y) = \alpha u^*(y, x^o)$.

Proof. Let $u(x, y, x_3^o, \ldots, x_T^o) = u^*(x, y)$ then

$$u(x_1^o, x_2^o, \ldots, x_{i-1}^o, x, y, x_{i+2}^o, \ldots, x_T^o) = \alpha_i u^*(x, y) \tag{15}$$

Now for all i

$$\alpha_i u^*(x^o, y) = \alpha_{i+1} u^*(y, x^o)$$

since both equal

$$u(x_1^o, \ldots, x_{i-1}^o, y, x_{i+1}^o, \ldots, x_T^o) \quad . \tag{16}$$

Thus $\alpha_{i+1} = \alpha\alpha_i$ for all i for some $\alpha$.
Substituting (15) and (16) in Result 2 gives Result 3. ||

## Summary

We have shown that it is possible to take explicit account of time preferences where there is considerable dependence between preferences of adjacent periods. If stationarity is assumed also, the problem of assessing the time utility function reduces to that of assessing one two dimensional utility function and one "discount" constant $\alpha$.

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