OBJECTIVE FUNCTIONS

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Dedicated to Karl Wirtz on the occasion of his 65th birthday

Since the early 1940's the field of operations research (OR) has played an increasingly important role. A prominent area in this field is that of optimization problems, particularly linear programming (LP). As early as 1939 Kantorovich recognized the importance of LP and made early contributions; but it was G.B. Dantzig who in 1947 made the decisive breakthrough by developing the Simplex method /1/. The significance of LP in its own right was firmly established in 1949 at the conference held by Koopmans in Chicago.

In LP problems all relations are linear. The aim is to optimize a linear objective function under a number of linear constraints. A classical example is the allocation problem in a transport task. Consider n warehouses for a certain article, and m factories producing this article. Let the specific transport costs of the article from factory i to warehouse j be C_{ij}. Then the total costs K are given by:

$$(1) \qquad K = \sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij} X_{ij} ,$$

where X_{ij} are the activities of the LP problem, the numbers of the total quantity of the article that are transported from factory i to warehouse j. There are constraints:

$$(2) \qquad \sum_{j=1}^{n} X_{ij} \underset{(<)}{=} a_i \qquad i = 1,2 \ldots\ldots m$$

and

$$(3) \qquad \sum_{i=1}^{m} X_{ij} \underset{(>)}{=} d_j , \qquad j = 1,2 \ldots\ldots n$$

i.e., the total quantity transported form factory i can (at most) correspond to its production a_i, and the total quantity arriving at warehouse j must (at least) correspond to its demand d_j. The LP problem consists in minimizing the transport costs K, where K denotes the objective function of the problem. This problem becomes non-trivial if the number of activities and constraints becomes very large. The Simplex method developed by Dantzig is designed for use with large electronic computers. The calculation effort corresponds approximately to the third power of the number of constraints. Nowadays LP problems of 30,000 activities or more are treated. In such clearly defined problems the formulation of the objective function is not difficult. More generally, in the past it was almost always the costs that were to be optimized.

Costs as the objective function need not be restricted to a given (possibly brief) time interval; longer time intervals may be considered, which then represent the time horizon of the LP problem. This is how Häfele and Manne treated the problem of transition from fossil to nuclear fuels /2/. With a time horizon of 75 years, they considered the annual composition of primary energy supply from the following sources:

fossil:	COAL, OIL + NATURAL GAS
nuclear:	LIGHT WATER REACTORS, BREEDERS, HIGH-TEMPERATURE REACTORS

Primary energy supplies from these sources are thus the activities of the LP problem. The constraints are the following:

a) The annual primary energy demand is to be met both electrically and non-electrically.

b) Cumulatively, not more than a given total amount of oil plus natural gas is to be used.

c) Cumulatively, not more than a given total amount of cheap uranium is to be used.

d) A certain annual production capacity for the reactor types considered must not be exceeded.

In d) we are dealing with a constraint limiting the activity flow: DX=X(t+1)-X(t) cannot exceed the production capacity of the reactor type designated X.

It is not the aim of this paper to repeat the reflections of Häfele and Manne, or to describe once more their relevance to the energy problem in general, as e.g. in Ref. /3/. We merely refer to the type of objective function used there. The cost objective function is:

$$(4)\quad K = \sum_{t=o}^{75} 3ß^{t} \cdot \left[\sum_{i} cur_i\, PC_i^t + (\sum_{i} cap_i\, DP_i^t)(1-TV_t)ß^{-2} \right]$$

where

ß is the present value ($ß=\frac{1}{1,1}$ with a discount rate of 10%),

$cur_i PC_i^t$ are the current costs for primary energy production in the t-th time step,

$cap_i DP_i^t$ are the capital costs,

$1-TV_t$ is the remaining value of the power plants operating at the time horizon

$ß^{-2}$ is the construction interest loss.

The objective function represents the sum of three-year steps. With $\frac{75}{3}$ = 25 steps, the LP problem considered here thus has 25*5 activities and 25*7 constraints.[1] The results obtained are strategies for the transition from fossil to nuclear fuels where, under the given constraints, the discounted value of the energy cost over the 75-year time horizon is minimized.

1) The particular problem considered in the Häfele-Manne paper is slighthly different but this has no relevance for the present discussion.

In recent years objections to the mere optimization of costs have grown. Quite rightly, non-monetary costs such as, e.g., those due to pollution are addressed more and more explicitly. It is thus natural to include costs for the retention of pollutants. With a given spatial distribution of pollutants, the emission of type p connected with each activity PC_i^t could be given by:

$$(5) \qquad EPC_i^t = e_{i,p}^t \cdot PC_i^t \quad .$$

In the case of an oil-fired power plant, p would stand for SO_2, and $e_{i,p}^t$ would have the dimension $[gSO_2/KWsec]$; hence, $EPC_{i,p}^t$ would have the dimension $[gSO_2/sec]$. A meteorological factor $s_p^{E_i \to I}$ would link emission of type p at location E_i (with which every activity PC_i^t is connected) with the immission concentration at location I. Such a meteorological factor has the dimension $[sec/m^3]$. For the immission concentration we thus have:

$$(6) \qquad IPC_i^t = s_p^{E_i \to I} \cdot e_{i,p}^t \cdot PC_i^t \left[\frac{g(p)}{m^3}\right]$$

But there are standards for the allowable immission concentration that each pollutant p must not exceed: $\frac{g(p)}{m^3}$. Thus there are additional constraints:

$$(7) \qquad \sum_i s_p^{E_i \to I} \cdot e_{i,p}^t \cdot PC_i^t \leq S_p\left[\frac{g(p)}{m^3}\right]$$

for all t,

for all I.

The objective function now comprises the original objective function (4), which we will call K_o, and a term denoting the costs for retention measures. We thus have the following expression:

$$(8) \qquad K = K_o + \sum_{t=o}^{75} 3\beta^t \left[\sum_i k(e^t_{i,p}) \; PC^t_i \right] .$$

The costs connected with the retention factor $e^t_{i,p}$ in general do not linearly depend on e; here one must resort to piecewise linearization.

With a given geographical distribution, the method outlined permits description of an optimal cost strategy for the above problem that takes account of the concentration standards. It then becomes interesting to look at the shadow prices of such standards, i.e. to consider the effect of a change in standards on the objective function:

$$\frac{\partial K}{\partial Sp} \qquad \left[\frac{DM}{g(p)/m^3} \right] .$$

Now if, analogously to the elasticities used in econometrics, we formulate the following dimensionless expression:

$$(9) \qquad \rho_p = \frac{Sp}{K} \frac{\partial K}{\partial Sp} ,$$

we get an idea of how to approach the problem of establishing standards. K. Hoffman, in his model investigating the allocation of primary energy to secondary energy demands, considers as objective functions values other than costs /4/. He regards a minimization of pollutant concentration at a given total cost as a constraint; so also minimization of the primary energy demand with given secondary energy use.

Clearly many generalizations can be made. The International Institute for Applied Systems Analysis, among others, is working in this area. The field of model building for energy demand and production is expanding rapidly (see /5/ for an overview).

The International Institute for Applied Systems Analysis is carrying out systems analyses not only in the energy sector, but also for city systems, water systems, biomedical systems, ecology systems, and others. For our purposes it is important to consider the concept of resilience as it was developed by Holling /6/. Extensive studies on ecological equilibria, some of them based on good data covering very long time periods /7/, led Holling to consider this concept. It appears that ecological systems can absorb a finite number of perturbations, e.g., by human beings, which may radically change the system characteristics. With a subsequent intervention, however, they then collapse; i.e., they not only undergo change but break down completely. An example is the eutrophication of lakes.

It is of interest here to relate this process to the notion of safety as it is used in engineering; this is illustrated in Fig. 1. Formerly, an assessment based on engineering experience of the "realistic" expectation of accidents was in the forefront (1).

(The numbers in parentheses indicate parts of Fig. 1). This is shown by the expression MCA, the maximum credible accident, used in reactor technology. In this approach accidents against which safety measures are to be taken are considered only within limits. However, the possibility of nuclear accidents with extensive consequences cannot be excluded; so that more recently accidents without such limits are anticipated. These lead to a residual risk, which must be embedded into existing risks (3). Considerable research is required, since these questions were hardly treated in the past. Since 1974 a joint research group of the International Institute for Applied Systems Analysis and the International Atomic Energy Agency, among others, has concerned itself with this problem. Such studies reveal the difference between objective risk and subjective risk perception (4). The bulk of the residual risks cannot be treated with the traditional method of trial and error. Instead, all discussion takes place in the realm of hypotheticality on which the author elaborates in recent work /8/. Methods of decision analysis /9/ can, by means of a formalized procedure (5), help in arriving at decisions (6) on standards for the acceptability of residual risks (8), although uncertainty remains in principle. A fully developed reliability control procedure (9) must then show whether a certain technical design corresponds to these standards. This is precisely the purpose of the Rasmussen report recently published /10/. The path (2) - (10) can be described as the probabilistic approach to the treatment of accidents possibly to be expected, in contrast to the traditional approach (1) - (10). In practice the probabilistic approach is not yet quite feasible. Apart from other difficulties, this is due mainly to the inherent difficulty of establishing binding standards for residual risks. Such difficulties apply in particular to steps (3) - (6). In the author's opinion, the probabilistic treatment of accidents possibly to be expected will be emphasized in the long run.

The as yet qualitative concept of maximizing resilience developed by Holling may go one step further, so that a third level for treating accidents possibly to be expected can be envisaged.

In the following we shall try to express the resilience concept quantitatively, and formulate an appropriate objective function. A greatly simplified example will be used: we refer to the author's paper on the commemorative volume dedicated to Carl Friedrich v. Weizsäcker on the occasion of his sixtieth birthday /11/, which describes the problem of nuclear energy as consisting in an almost infinitely large benefit combined with a hypothetically almost infinitely large risk and an almost infinitely large engineering potential for insuring against this risk. Clearly the problem is one of dealing with the coupling of these almost infinitely high values. We refer to this relationship in the following.

One further introductory remark: the concept of resilience can be made clear only by considering nonlinear relationships. All the ecological relations examined by Holling are highly nonlinear. We shall consider the following model of an imagined society S in this light:

1) Society S has an effective gross national product G, which can be described by a Cobb-Douglas function if only the annual consumption of energy E and of labor enter. Let labor be proportional to the total population of society S. Tintner /12/, for example, has given production functions for Austria in which the energy consumption enters explicitly as a production factor. The costs K required to reach a certain residual risk must be deducted from the production function, because it represents a part of the gross national product no longer available. We then have

$$G = A \cdot E^{\alpha} \cdot P^{\beta} - K \quad .$$

If we assume that a doubling of production factors E and P will yield only a doubling of the value of the Cobb-Douglas function, then $\alpha+\beta=1$. After Tintner, $\alpha = \frac{1}{2}$.. Thus we have:

(10) $$G = A \cdot E^{\frac{1}{2}} \cdot P^{\frac{1}{2}} - K \quad .$$

2) We assume that the standard for the residual risk applies to individuals and is inversely proportional to the expenditure k per kW year:

(11) $$r = r_o \cdot \frac{k_o}{k} = r_o \cdot \frac{K_o}{E_o} \cdot \frac{E}{K} \quad ,$$

with

(12) $$k = \frac{K}{E} \quad ,$$

where the values indexed by o represent reference values.

We obtain

(13) $$K = \frac{r_o}{r} \cdot \frac{K_o}{E_o} \cdot E \quad .$$

Note that equation (11) contains the statements about the residual risk treated in /11/: engineering safety measures can reduce the residual risk to near zero if an unlimited amount of money is spent.

3) The total energy consumption E per annum is the product of the per capita consumption per annum e and the population number P:

(14) $$E = e \cdot P \quad .$$

4) The risk acceptance of society S, e.g., as described in steps (2) - (10) of Fig. 1, is inversely proportional to the per capita consumption e raised to the power of λ . The better the individual lives, the disproportionately less ready is he to accept a residual risk:

$$\left(\frac{r}{r_o}\right) = \left(\frac{e_o}{e}\right)^{\lambda} \qquad \lambda > 1 \; .$$

For our purposes we set $\lambda=2$:

(15) $$\frac{r}{r_o} = \left(\frac{e_o}{e}\right)^2 \quad .$$

5) The availability of energy is unlimited; i.e., the total

consumption integrated over time t'

$$\int_{0}^{t} e \cdot P \ dt'$$

can rise without limit over time. Thus we implicitly include in the model an almost infinitely large benefit, the third dimension described in /11/.

6) We assume that the increase in energy consumption is proportional to the effective gross national product G:

(16) $$\frac{dE}{dt} = \mu \cdot G ,$$

where μ is the proportionality constant.

7) We assume that the population growth is positive proportional to the population number and negative proportional to the personal welfare denoted by e:

(17) $$\frac{dP}{dt} = \sigma P - \kappa e ,$$

where σ and κ are the proportionality constants.

The advantage of this very simple model of an imagined society S lies in the clearness of all the relations. It is possible to represent the model in the two-dimensional configuration space (e,P) by a common first-order differential equation. Moreover, the model is nonlinear, and as one will see allows description of the resilience concept. One can find without difficulty

$$(18)\quad \frac{de}{dP} = \frac{\mu A e^{\frac{1}{2}}\cdot P - \mu C e^{3} P - \sigma P + K e^{2}}{P(\sigma P - Ke)}$$

with

$$(19)\qquad C = \frac{Ko}{Eo}\,\frac{1}{e_o^2} \quad .$$

We now look for trajectories in the (e,P) field whose development over time can be determined by means of (16) or (17). Figure 2 shows the solutions in the (e,P) field, for which the following numerical values were assumed:

$$A = 0.25\cdot 10^{4}\ \frac{\$}{\text{year}\cdot \text{kW}^{\frac{1}{2}}\cdot \text{capita}^{\frac{1}{2}}}$$

$$\frac{Ko}{Eo} = 10\ \frac{\$}{\text{kW}}\ \text{year}$$

$$e_o = 3\ \text{kW/capita}$$

$$\mu = 24\cdot 10^{-6}\ \frac{\text{kW}}{\$}$$

$$\sigma = 2\cdot 10^{-2}\ \frac{1}{\text{year}}$$

$$k = 0.25\cdot 10^{6}\ \frac{(\text{capita})^{2}}{\text{kW year}}$$

Figure 2 was determined by the saddle point, given by $e_s = 21.9\ \frac{\text{kW}}{\text{capita}}$ and P_s = 274 million. The saddle point divides the configuration space into four completely separate regions A,B,C,D. In A both e and P rise and for $P \to \infty$, e reaches the asymptotic value of

e_∞ = 7.7 kW/capita. Personal welfare then no longer rises; the increase in active gross national product is due to the newborn only. In B, on the other hand, the trajectories leave a region of decreasing active gross national product and finally reach the stable asymptotic solution, due to the means freed by a decrease in population. The situation is different with the solutions in D. There personal welfare rises, and with it the safety requirements. Along the trajectories in D, the active gross national product soon decreases, and the necessary means can be raised only through death. The situation is similar for region C.

Figures 3 and 4 show the development over time of P(t), e(t), K(t), r(t), and G(t) for initial conditions P_o= 220 million, e_o= 10 kw/capita, and initial conditions P_o= 75 million, and e_o= 2.7 kW/capita, respectively.

The model presented here is greatly simplified. The results may quantitatively serve as food for thought. The desire for absolute safety may lead to collapse. It is clear that for a quantitatively relevant model, many more relations must be considered. For this reason we will not go into details of the model; instead we will try to define the concept of resilience. Such a definition will of course be much more general than the model discussed.

The relevant point in a more general consideration is the following: the different areas for solution are sharply divided by the two separatrices which traverse the saddle point. Initial conditions in the neighborhood of separatrix S_1, which may be very close to each other, can lead to qualitatively different final conditions far apart from each other. In case a state in region A of the human-ecological system considered lies close to S_1, there may be some danger that it will be changed across S_1 to a state in D, owing to an event not specified and not described by equations (10) - (17), or owing to inexact knowledge about the position of separatrix S_1. In the framework of this model this would mean collapse, since, for $t \to \infty$, we then have $P \to o$.

It should be mentioned that value judgments enter: A is considered more desirable than D. Other value judgements would lead to a different preference structure.

On the basis of such a value judgement it becomes natural to make the distance from the separatrix dividing the desirable from the feared as large as possible, and the time spent in its neighborhood as short as possible. Consider Figure 5: for a given line segment (1 or 2), let a(s) be the distance from the separatrix. a(s) is a function of time. We now define a value R:

$$\frac{1}{R} = \int_{s_o}^{s_1} \frac{ds}{\frac{ds}{dt} \cdot a(s)} \qquad (20)$$

For a given section of a line segment between s_o and s_1, R increases with an increase in the distance from the separatrix and in the speed with which a line segment is travelled. We are speaking of line segments here for the sake of generality. In a completely deterministic model such as the one presented, the line segments are parts of trajectories. If additional influences not covered in the original equations are permitted, the line segments may also cross trajectories. The two line segments 1 and 2 can be quantitatively compared via the value R. R should then be a measure of resilience, for the following reasons:

a) In contrast to the engineering and the probabilistic approaches to the treatment of accidents possibly to be expected, here the result which leads to a change across separatrix S is not explicitly anticipated. An implicit anticipation is given only by the assumption that the closer one is to the separatrix and the longer one remains near it, the greater the danger.

b) A maximization of R, say in the framework of a suitably formulated LP model, also covers the case that one is somewhat uncertain about the validity of the relations in reality. In maximizing R one implies merely that in reality a kind of separatrix lies somewhere in the vicinity of the separatrix predicted by the model.

The supposition is thus permissible that the maximization of R might in fact be a process on the third, lowest, level of Fig.1; i.e. that it might be possible to take safety precautions beyond the explicit anticipation of accidents.

One further point should be made here. The definition of a distance in configuration space requires a metric. In the example presented here we would have:

$$(21) \qquad A^2 = \underset{e_s, P_s}{\mathrm{Min}} \left(\frac{K}{\sigma}(e-e_s)\right)^2 + m^2 \cdot (P-P_s)^2 \quad ,$$

where e_s and P_s are the coordinates of the separatrix.

The constants K and σ, as in differential equation (17), make the dimensions compatible. The determination of m implies the metric which cannot be deduced from the formalism and must be defined separately. The explicit treatment of the problem of accidents considered possible then reduces to this more precisely structured problem. m relates changes in population to those in the per-capita energy consumption to one another.

The quantitative resilience concept, as we have said, is independent of the model presented here, whose details are unimportant. This is due to the differential-topological relationships which are typical of nonlinear problems. In general, many more than two variables will describe the relevant configuration space. The examples investigated by Holling suggest many thousands of state variables.

In conclusion, let us consider once more the model of an imagined society S. As it is presented here, it is totally deterministic. No variable is free for optimization. The following approach is now possible. Let energy production take place in two ways: nuclear (Index 2) and fossil (Index 1). Then

$$(22) \qquad e = e_1 + e_2 \quad .$$

In our model let us assume that fossil energy production is riskless (which is not the case in reality). Then instead of equation (12), the following equation applies:

(12a) $$r = r_o \cdot \frac{k_o}{k} \cdot \frac{e_2}{e} .$$

Now of course

(23) $$V_1 - \int_0^t e_o \cdot P \cdot dt \geq 0 ,$$

since the fossil reserves V_1 are finite. It is now natural to use the freedom given by factor $\frac{e_2}{e}$ for optimization in an LP program. In this case, with (22) as a constraint, resilience R as given in equation (20) would be the objective function. The resulting strategies for the transition from fossil to nuclear fuels should then be compared to the strategies resulting from using the discounted present value of the total costs, as e.g. given by equation (4) or (8).

This is the direction of research at the International Institute for Applied Systems Analysis in Laxenburg near Vienna.

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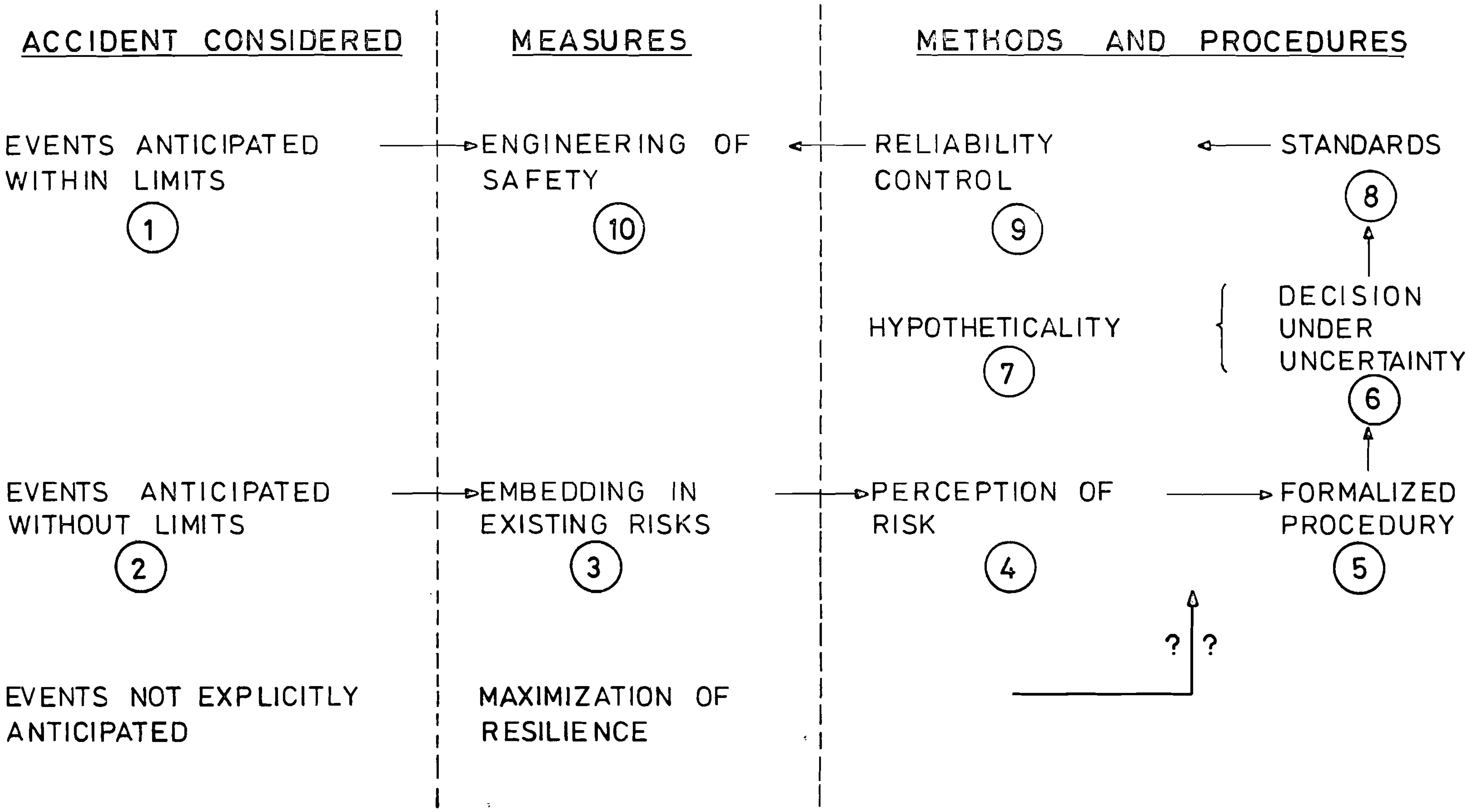


FIGURE 1: TREATMENT OF ACCIDENTS POSSIBLY TO BE EXPECTED

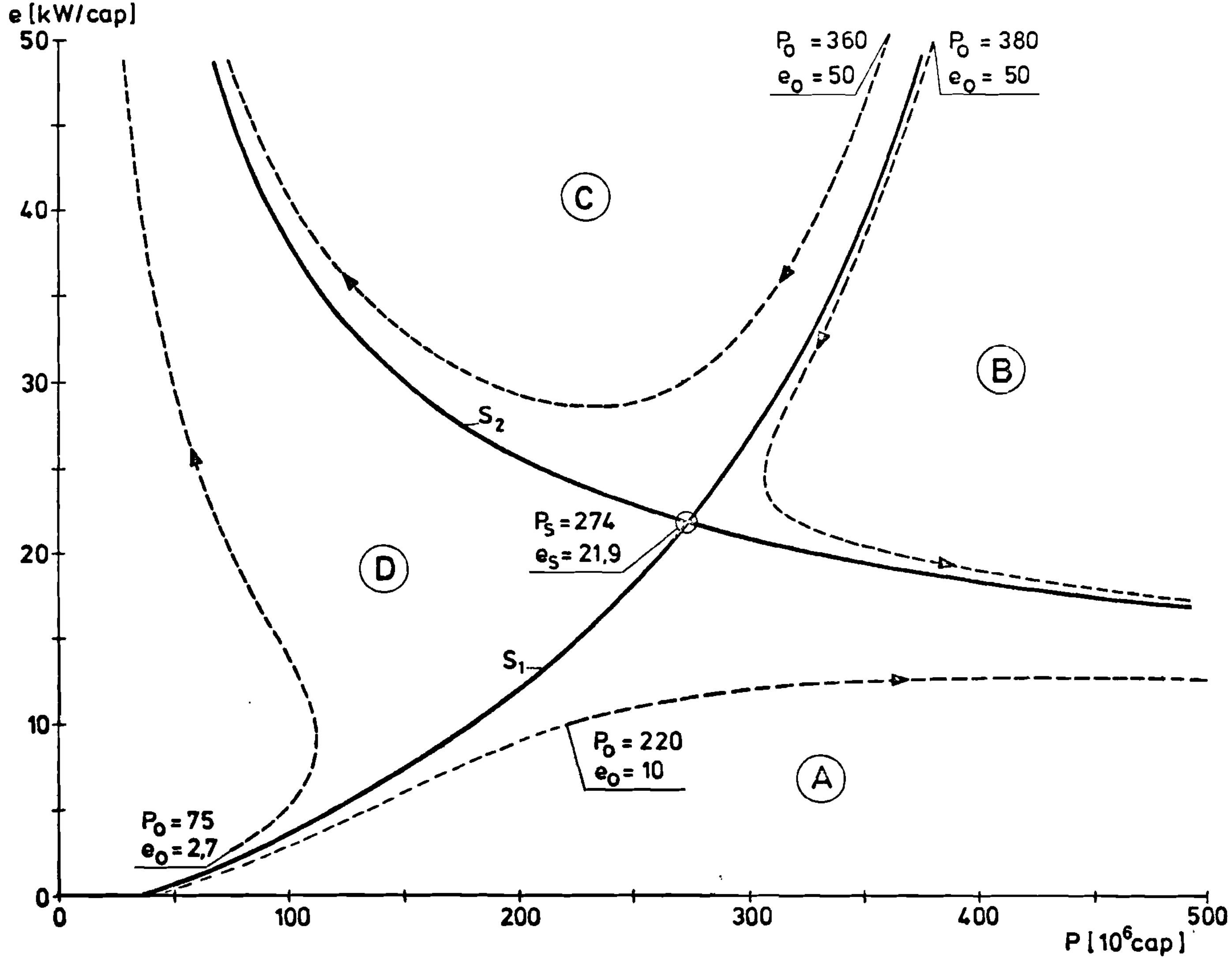
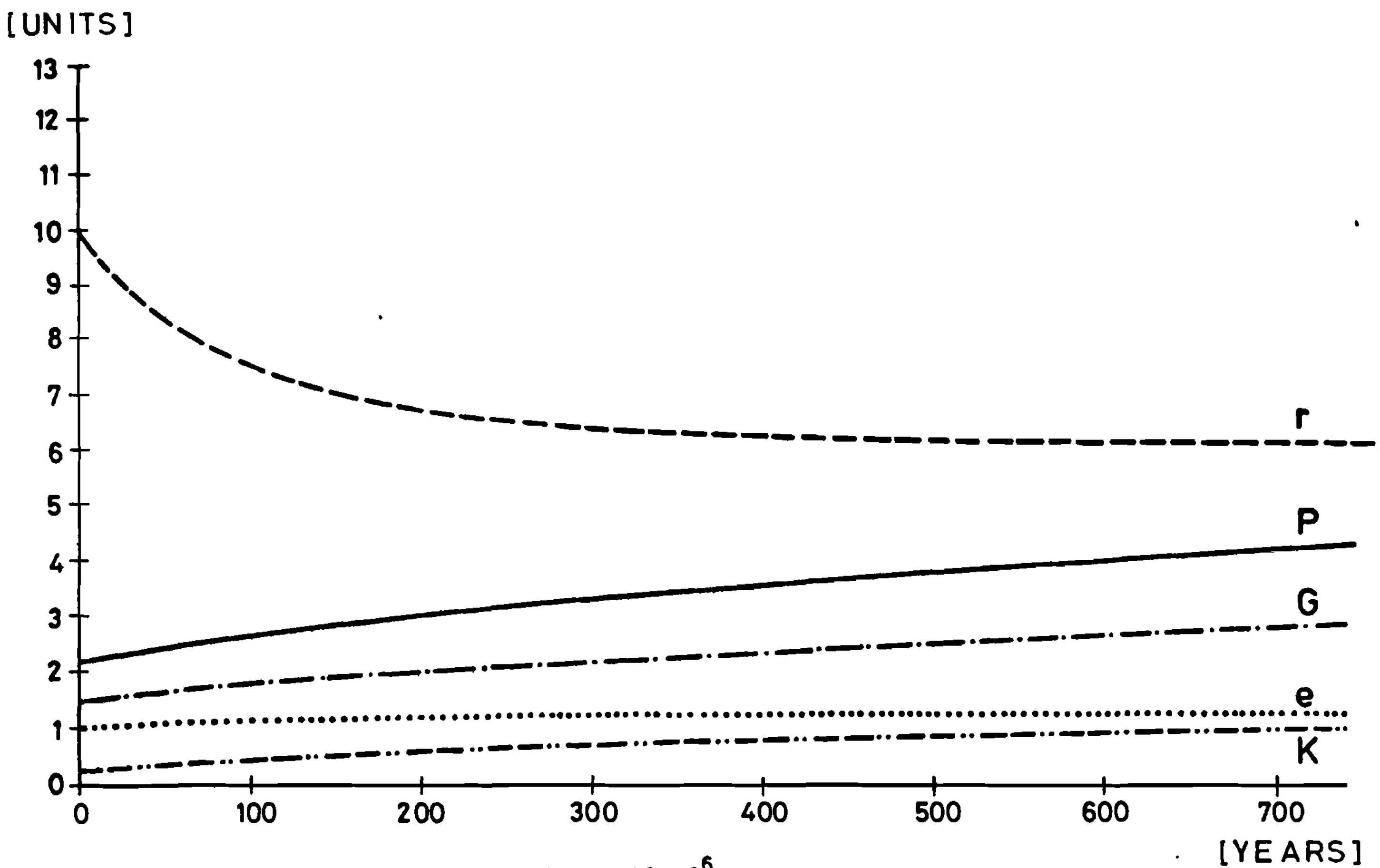


FIGURE 2: SOLUTIONS IN THE (e, P) – FIELD



INITIAL CONDITIONS : $P_o = 220 . 10^6$ cap
e_o = 10 kW/cap

SCALE : e[10 kW/cap / UNIT]
P[100.10^6 cap / UNIT]
K,G [10^{12} $ / UNIT]
r [0.1 / UNIT]

FIGURE 3 : DEVELOPMENT OVER TIME OF SOCIETY S

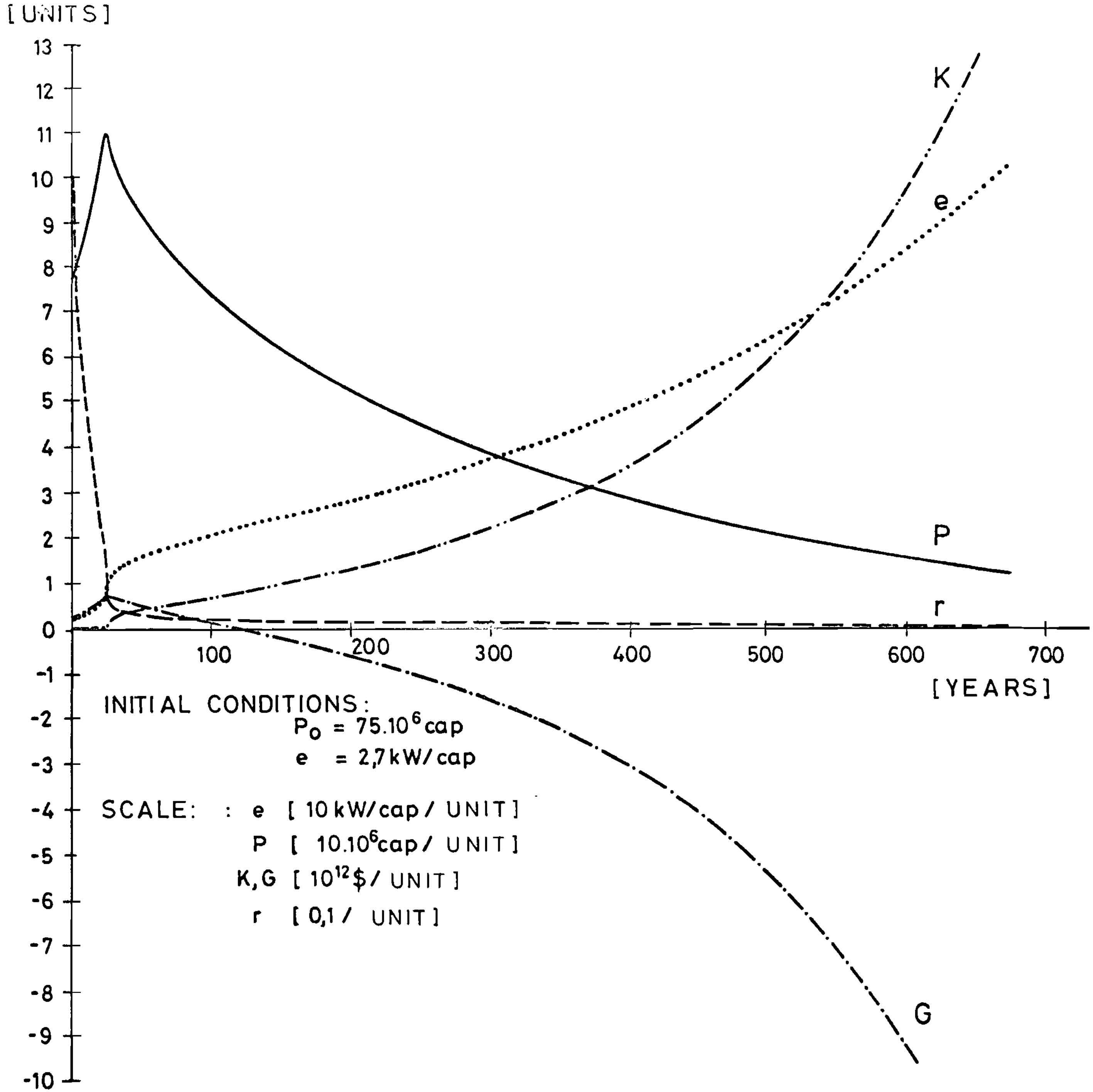


FIGURE 4.DEVELOPMENT OVER TIME OF SOCIETY S

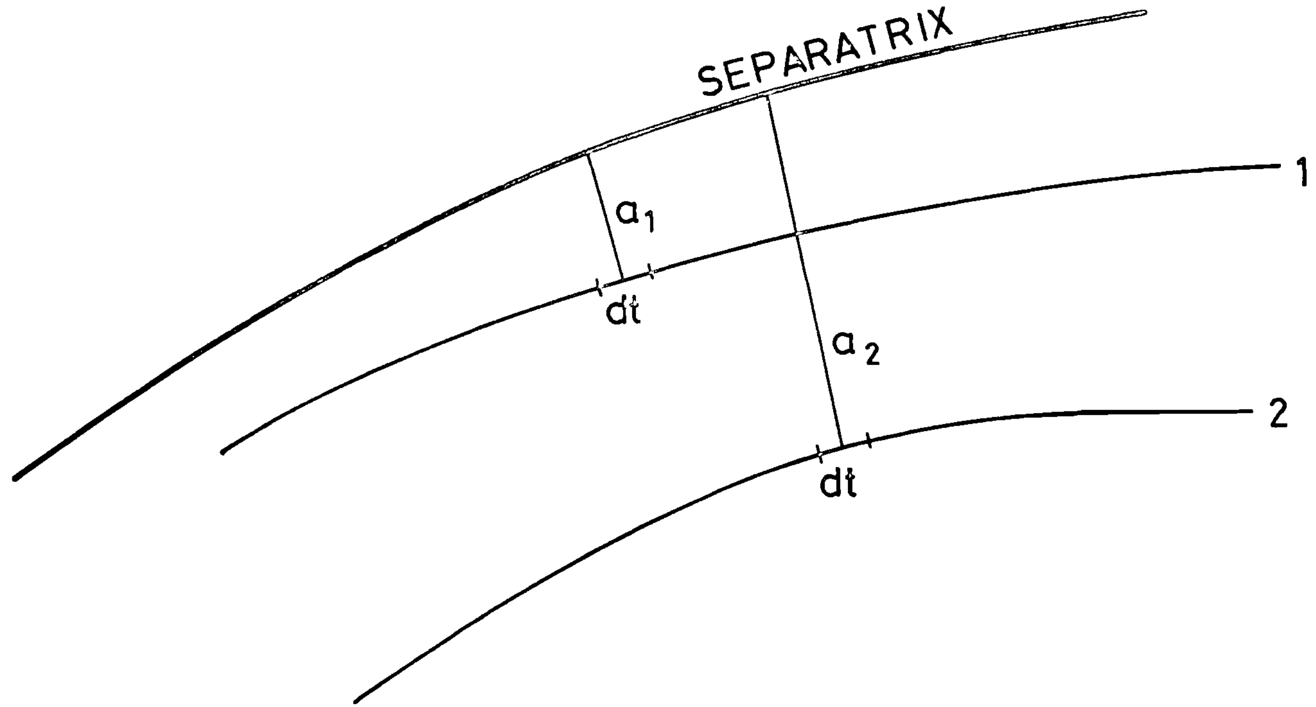


FIGURE 5: DISTANCE OF LINE SEGMENTS FROM THE SEPARATRIX