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On Augmented Lagrangian Decomposition Methods For Multistage Stochastic Programs

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Abstract

A general decomposition framework for large convex optimization problems based on augmented Lagrangians is described. The approach is then applied to multistage stochastic programming problems in two different ways: by decomposing the problem into scenarios and by decomposing it into nodes corresponding to stages. Theoretical convergence properties of the two approaches are derived and a computational illustration is presented.

Keywords: Stochastic Programming, Decomposition, Augmented Lagrangian, Jacobi Method, Parallel Computation.

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1 Introduction

Multistage stochastic optimization problems are amongst the most difficult problems of mathematical programming. Their size grows very quickly with the number of stages and with the number of events (scenarios) incorporated into the model. Although problems of this type occur frequently in applications (like, e.g., investment planning problems, control of water systems or energy systems), it was a generally held opinion that they were too difficult to be solved in their full formulation. However, recent advances in the theory of stochastic programming and in computing technology make it possible to develop new methods for solving multistage stochastic programs of remarkable sizes. The purpose of this paper is to describe such an approach which has already proved successful in some applications and appears to have a potential to solve a broad class of problems.

After a brief description of the class of problems under consideration in section 2, we present the general decomposition framework in section 3. The method is applicable to general convex problems with many subproblems and many linking constraints. Our approach is based on augmented Lagrangians and has its roots in the pioneering work [25]. Following [24] we show that properties of the method heavily depend on the sparsity of the linking constraints. Next, in section 4, we apply the general framework to multistage stochastic programming problems formulated in a scenario form. The subproblems correspond to scenarios and nonanticipativity constraints are treated as linking constraints. In section 5 we apply the general framework to multistage problems decomposed into particular stages of the decision-making process. Then the equations of dynamics, which relate to the variables from different stages, are treated as linking constraints in the decomposition approach. In both cases we show that the augmented Lagrangian decomposition method has favorable properties with a broad range of parameters guaranteeing convergence. The rate of convergence estimates are better, though, for the scenario decomposition method. This is confirmed by the computational results of section 6, where a large stochastic macro-economic model is considered.

2 Multistage stochastic programming models

In a multistage optimization problem decisions are to be made in stages $t = 1, 2, \ldots, T$ and the decision vector is a collection of subvectors corresponding to successive stages,

$$x = (x(1), x(2), \ldots, x(T)).$$

Decisions in successive time stages have to satisfy two groups of relations. The first group describes the set of feasible actions for each t:

$$x(t) \in X(t), \quad t = 1, 2, \ldots, T, \tag{2.1}$$

where $X(t) \subseteq R^{m_x}$, $t = 1, 2, \ldots, T$. The second group describes the dynamics of the system and relates decisions from different time stages. In the simplest linear model they may read:

$$D(t)x(t-1) + H(t)x(t) = b(t), \quad t = 1, \ldots, T. \tag{2.2}$$

Here $D(t)$ and $H(t)$, $t = 1, \ldots, T$ are sequences of $m_b \times m_x$ matrices, $b(t)$, $t = 1, \ldots, T$, is a sequence of vectors in R^{m_b} and $x(0)$ is fixed. Obviously, the dimensions need not be the same for different t; we just use one m_x and one m_b for simplicity.

Finally, there is a cost function $c : R^{m_x T} \to R$

$$c(x) = c_1(x(1)) + c_2(x(2)) + \ldots + c_T(x(T)) \tag{2.3}$$

that needs to be minimized.

In stochastic programming, the data $X(t)$, $D(t)$, $H(t)$ and $b(t)$ are random objects defined on some underlying probability space Ω. We restrict our considerations to the case when Ω is finite:

$$\Omega = \{1, 2, \ldots, S\}$$

and we denote by p_ω the probability of an elementary event $\omega \in \Omega$. We shall call each sequence

$$s_\omega(t) = (X_\omega(t), D_\omega(t), H_\omega(t), b_\omega(t)), \quad t = 1, \ldots, T,$$

corresponding to an elementary event $\omega \in \Omega$ a *scenario*.

Realizations of the random data associated with time stage t become known at t, so it is reasonable to make the decision $x(t)$ dependent on the information that is already available. Consequently, x is a random vector itself, and (2.1) and (2.2) are relations between random variables that are assumed to hold with probability 1.

However, x cannot be an arbitrary random vector; the dependence of $x(t)$ on ω may result only from the observations carried out up to time t. This is called *non-anticipativity*: for each t decisions $x_\omega(t)$ must be equal for all scenarios ω that have common past and present.

Moreover, the cost (2.3) is a random variable itself and we need to replace it by a scalar-valued function. It is a common practice to use an expected value of the cost as the objective, although other choices are possible, too, as, e.g., mean-variance models. Since we are going to work with a general convex c, using its expectation does not seem very restrictive, because we still retain the flexibility of nonlinear utility functions.

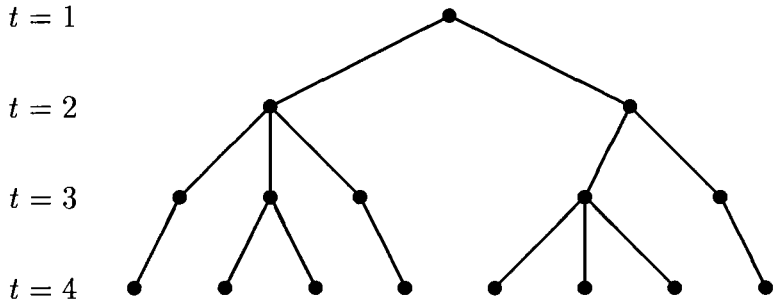

Figure 1: Scenario tree.

The problem can be now stated as follows:

$$\min \sum_{\omega\in\Omega} p_\omega \left[c_1(x_\omega(1)) + c_2(x_\omega(2)) + \ldots + c_T(x_\omega(T))\right] \tag{2.4}$$

subject to the constraints

$$D_\omega(t)x_\omega(t-1) + H_\omega(t)x_\omega(t) = b_\omega(t), \quad t = 1,\ldots,T, \ \ \omega \in \Omega, \tag{2.5}$$

$$x_\omega(t) \in X_\omega(t), \quad t = 1,\ldots,T, \ \ \omega \in \Omega, \tag{2.6}$$

with $x(0) = x_0$ fixed. The non-anticipativity constraint can be formulated as follows: for all $\omega, \zeta \in \Omega$ and any $t \in \{1,\ldots,T\}$

$$x_\omega(t) = x_\zeta(t) \ \ \textit{if} \ \ s_\omega(\tau) = s_\zeta(\tau) \ \ \textit{for} \ \ \tau = 1,\ldots,t. \tag{2.7}$$

In other words, decisions corresponding to scenarios which are indistinguishable up to time t should be equal.

We shall assume throughout this paper that the sets $X_\omega(t)$, $t = 1,\ldots,T$, $\omega \in \Omega$ are convex and closed and the functions c_t, $t = 1,\ldots,T$, are convex, which makes (2.4)-(2.6) a convex optimization problem.

Since Ω is finite, with the set of scenarios $s_\omega(t)$, $t = 1,\ldots,T$, $\omega \in \Omega$, we can associate a tree $\mathcal{T} = \{\mathcal{N}, \mathcal{A}\}$, where $\mathcal{N}$ is a set of nodes and $\mathcal{A}$ is a set of arcs of $\mathcal{T}$. The set of nodes $\mathcal{N}$ is divided into subsets (levels) $\mathcal{N}_t$, $t = 1,\ldots,T$; the nodes $n \in \mathcal{N}_t$ at level t correspond to different subscenarios $\{s^n(1),\ldots,s^n(t)\}$. At level 1 there is only one node $n = 1$ (the data for stage 1 are known). At level 2 there are as many nodes as different realizations of $s(2)$ can occur; at level 3 the nodes correspond to different pairs $\{s(2), s(3)\}$, etc. The number of nodes at level T is equal to the number of scenarios S. The arcs join nodes from neighboring levels in such a way that a node n at level t corresponding to subscenario $s^n = \{s^n(1),\ldots,s^n(t)\}$ is connected with all nodes m at level $t+1$ whose subscenarios $s^m = \{s^m(1),\ldots,s^m(t+1)\}$ equal s^n up to time t. An example of such a tree for an 8-scenario problem is shown in Fig. 1.

Problems with finitely many scenarios are more amenable for computer solutions, but many difficulties still remain.

First of all, one has to note the remarkable size of the problem. If the scenarios introduced to the model are to reflect uncertainties that occur at successive time stages,

then the number S of scenarios grows exponentially with the increase of the time horizon T. Even for relatively small T the dimension of (2.4)-(2.7) may be so large that the whole problem will become intractable by direct solvers.

However, (2.4)-(2.7) has a very special structure which creates a number of possibilities for developing special solution methods.

Existing computational methods for multistage stochastic programming problems can be divided into two main groups. First, there are versions of general-purpose algorithms in which special features of stochastic problems are used to improve data structures and solution strategies [10, 9]. Secondly, we have a number of special *decomposition methods* which exploit the structure of the problem to split it into manageable pieces and coordinate their solution [27]. One can distinguish two classes: *primal* decomposition methods that work with subproblems which are assigned to time stages [4, 8, 21, 22, 26] and *dual* methods, in which subproblems correspond to scenarios [13, 23, 19].

In this paper we shall use the general theory of augmented Lagrangian decomposition of [24] to develop and analyze two new decomposition methods for multistage stochastic programs. The first one is a dual method proposed for linear multistage stochastic programs in [13] and further developed in [14]. We shall show how to deal with convex objectives and we shall present some results on its convergence and rate of convergence. The second approach is a primal method based on the concept of nodal decomposition. Again, we shall use the theory developed in [24] to obtain convergence and rate of convergence results for the method. Alternative decomposition approaches based on augmented Lagrangians are discussed in [3, 7, 19, 23].

3 General decomposition framework

The purpose of this section is to briefly describe the general augmented Lagrangian decomposition method for partially separable convex problems. The approach will then be used in later sections to develop specific methods for multistage stochastic problems.

Let $X_1, X_2, \ldots, X_L$ be non-empty closed convex subsets of $R^{n_1}, R^{n_2}, \ldots, R^{n_L}$, respectively, and let $f_i : R^{n_i} \to R$, $i = 1, 2, \ldots, L$ be convex functions. Next, let A_i be matrices of dimension $m \times n_i$, $i = 1, 2, \ldots, L$ and let $b \in R^m$. We consider the convex programming problem:

$$\min \sum_{i=1}^{L} f_i(x_i) \tag{3.1}$$

$$\sum_{i=1}^{L} A_i x_i = b, \tag{3.2}$$

$$x_i \in X_i, \ i = 1, 2, \ldots, L. \tag{3.3}$$

The augmented Lagrangian for this problem is defined as:

$$\Lambda(x, \pi) = \sum_{i=1}^{L} f_i(x_i) + \langle \pi, b - \sum_{i=1}^{L} A_i x_i \rangle + \frac{\rho}{2} \left\| b - \sum_{i=1}^{L} A_i x_i \right\|^2, \tag{3.4}$$

with some penalty parameter $\rho > 0$. As usual, we define the dual functional

$$g(\pi) = \inf_{x \in X} \Lambda(x, \pi)$$

with $X = X_1 \times X_2 \times \cdots \times X_L$, and the dual problem:

$$\max_{\pi \in R^m} g(\pi). \tag{3.5}$$

There are many theoretical and computational advantages of the augmented Lagrangian approach over the ordinary Lagrangian (with $\rho = 0$). For the duality to hold, it is sufficient that the following condition is satisfied.

Constraint Qualification Condition. At least one of the following conditions holds:

(i) at some feasible point x^0

$$\text{ri}\,\{d: \ \exists \alpha > 0 \ \textit{such that} \ x^0 + \alpha d \in X\} \cap \{d: \ Ad = 0\} \neq \emptyset;$$

(ii) X is a polyhedral set.

The fundamental duality result can be formulated as follows.

Proposition 3.1 *Assume that (3.1)-(3.3) has an optimal solution and the Constraint Qualification Condition is satisfied. Then (3.5) has an optimal solution and*

(a) for every optimal solution $\hat{x}$ of (3.1)-(3.3) and every optimal solution $\hat{\pi}$ of (3.5)

$$f(\hat{x}) = g(\hat{\pi});$$

(b) for every optimal solution $\hat{\pi}$ of (3.5) a point $\hat{x}$ is a solution of (3.1)-(3.3) if and only if

$$\Lambda(\hat{x}, \hat{\pi}) = \min_{x \in X} \Lambda(x, \hat{\pi}). \tag{3.6}$$

The above proposition is a re-formulation of generally known facts and can be easily proved as follows. First, from [16], thm. 2.10 (for example), we deduce the existence of Lagrange multipliers $\hat{\pi}$. Then, by [17], thm. 28.3, we obtain assertion (a), because (3.4) is the ordinary Lagrangian for the problem having its objective augmented by the quadratic penalty term. The latter observation yields (b) as well.

An important advantage over the usual Lagrangian duality is that (3.6) is sufficient for primal recovery when the dual solution is known. The major computational advantage is the possibility of solving the dual problem by the following *method of multipliers*:

$$x^k = \arg\min_{x \in X} \Lambda(x, \pi^k), \tag{3.7}$$

$$\pi^{k+1} = \pi^k + \rho(b - Ax^k), \ k = 0, 1, 2, \ldots. \tag{3.8}$$

The following two propositions summarize the fundamental properties of the method of multipliers (see [18], thms. 4 and 6).

Proposition 3.2 *Let the Constraint Qualification Condition be satisfied. Then the sequence* $\{\pi^k\}$ *generated by the method of multipliers is convergent to a solution* $\hat{\pi}$ *of (3.5).*

Proposition 3.3 *Assume that* f_i, $i = 1, 2, \ldots, L$ *are convex polyhedral functions,* X_i, $i = 1, 2, \ldots, L$ *are convex polyhedral sets and (3.1)-(3.3) has a solution. Then the method of multipliers is convergent in finitely many iterations.*

The simplicity of iteration (3.8) makes the method of multipliers especially attractive for problems with many linking constraints (3.2), where column generation techniques stemming from [6] fail. However, a serious disadvantage is that (3.4) is not separable, so problem (3.7) cannot be split into independent subproblems for x_i, $i = 1, 2, \ldots, L$.

To overcome this difficulty we introduce for $i = 1, 2, \ldots, L$ the functions

$$\Lambda_i(x_i, \tilde{x}, \pi) = f_i(x_i) - \langle A_i^T \pi, x_i \rangle + \frac{\rho}{2} \left\| b - A_i x_i - \sum_{j \neq i} A_j \tilde{x}_j \right\|^2, \tag{3.9}$$

where $\tilde{x} \in R^n$ is an additional parameter, $n = \sum_{i=1}^{L} n_i$. The main idea of our approach is to replace problem (3.7) with L problems

$$\min_{x_i \in X_i} \Lambda_i(x_i, \tilde{x}, \pi^k), \quad i = 1, 2, \ldots, L, \tag{3.10}$$

and to iteratively update the parameter $\tilde{x}$ by making steps towards the solutions of (3.10). It is not difficult to see that (3.10) is equivalent to minimizing (3.4) with respect to x_i while keeping x_j, $j \neq i$, frozen at $\tilde{x}_j$. However, we are not going to use (3.10) in a sequential fashion, but we shall rather solve it for each i in parallel and then update $\tilde{x}$. This approach is called a *nonlinear Jacobi algorithm.*

We are now ready to describe the method in detail. It should be noted that it is a sub-algorithm for carrying out the minimization step (3.7) of the method of multipliers in a decomposed fashion. In what follows $\tau \in (0, 1)$ is a parameter of the method.

Jacobi Method

Step 0. Set $\tilde{x}^{k,0} = x^{k-1}$ and $r = 0$.

Step 1. For $i = 1, 2, \ldots, L$ solve (3.10) getting points $x_i^{k,r}$.

Step 2. If $A_i x_i^{k,r} = A_i \tilde{x}_i^{k,r}$, $i = 1, 2, \ldots, L$, then stop; otherwise set for $i = 1, 2, \ldots, L$

$$\tilde{x}_i^{k,r+1} = \tilde{x}_i^{k,r} + \tau(x_i^{k,r} - \tilde{x}_i^{k,r}), \tag{3.11}$$

increase r by 1 and go to Step 1.

Let us now pass to conditions under which the Jacobi method generates sequences $\{x^{k,r}\}_{r=0}^{\infty}$ and $\{\tilde{x}^{k,r}\}_{r=0}^{\infty}$ whose accumulation points are solutions of (3.7). They involve the measure of sparsity of the linking constraints (3.2) defined as follows. For every matrix A_i, let A_{ji} denote its jth row and let

$$V(i, j) = \{k \neq i : \ A_{ji}^T A_{jk} \neq 0\};$$

i.e., $V(i,j)$ is the set of other blocks linked with block i via row j. We can now define the *maximum number of neighbors* as

$$N = \max_{i,j} |V(i,j)|. \tag{3.12}$$

In other words, N is the maximum number of blocks linked by any single constraint, decremented by one. The theorems to follow show that convergence properties of the Jacobi method depend heavily on the number of neighbors N.

Theorem 3.1 *Assume that the assumptions of Proposition 3.1 are satisfied and the sets $X_i, i = 1,2,\ldots,L$ are bounded. If in the Jacobi method the under-relaxation coefficient satisfies the inequalities*

$$0 < \tau < \frac{1}{N}, \tag{3.13}$$

where N is given by (3.12), then:

(a) for all $i = 1,2,\ldots,L$ $\lim_{r\to\infty} A_i(x_i^{k,r} - \tilde{x}_i^{k,r}) = 0$;

(b) each accumulation point of the sequence $\{x^{k,r}\}_{r=0}^{\infty}$ is a solution of (3.7).

To estimate the speed of convergence, we need the following assumption on the growth rate of the augmented Lagrangian function ($\hat{X}(\pi)$ denotes the set of solutions of (3.7)).

Quadratic Growth Condition. There exist $\gamma > 0$ and $\delta > 0$ such that for every $x \in X$ with $\operatorname{dist}(x, \hat{X}(\pi)) < \delta$ we have $\Lambda(x,\pi) - \hat{\Lambda}(\pi) \geq \gamma[\operatorname{dist}(x, \hat{X}(\pi))]^2$.

It is clear that this condition is satisfied by linear and quadratic problems (3.1)-(3.3).

We can now formulate our main result on the speed of convergence.

Theorem 3.2 *Let the assumptions of Theorem 3.1 and the Quadratic Growth Condition be satisfied. Then, for all $r = 0,1,2,\ldots$ the following inequality holds*

$$\Lambda(\tilde{x}^{k,r+1}, \pi^k) - \hat{\Lambda}(\pi^k) \leq q\left[\Lambda(\tilde{x}^{k,r}, \pi^k) - \hat{\Lambda}(\pi^k)\right] \tag{3.14}$$

with

$$q = 1 - \frac{\tau(1-\tau N)}{2\rho\alpha^2 N^2 \gamma^{-1} + 1}, \tag{3.15}$$

and

$$\alpha = \max_{1\leq i\leq L} \|A_i\|. \tag{3.16}$$

Theorems 3.1 and 3.2 have been proved in [24]. We can also find there further refinements of these results for the case when the subproblems (3.10) are not solved till optimality, but with dynamically determined stopping criteria.

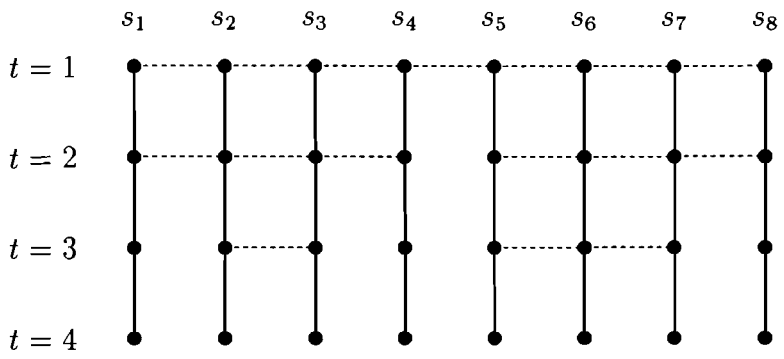

Figure 2: Sequences of decisions and nonanticipativity.

4 Scenario decomposition

We shall now apply the general framework of the previous section to problem (2.4)-(2.7) with the following assignments:

- subproblems correspond to scenarios $i = 1, \ldots, S$ with decision vectors

$$x_i = (x_i(1), x_i(2), \ldots, x_i(T));$$

- relations (2.5) and (2.6) are used to describe the sets X_i in (3.3):

$$x_i \in X_i = \{x_i : \ D_i(t)x_i(t-1) + H_i(t)x_i(t) = b_i(t), \\ x_i(t) \in X_i(t), \ t = 1, \ldots, T\}; \tag{4.1}$$

- non-anticipativity constraints are treated as linking constraints (3.2).

Let us develop a formulation of non-anticipativity constraints which is convenient for our decomposition approach. We define the *last common stage* of scenarios ω and ξ by

$$t^{\max}(\xi, \omega) = \max\{t : \ s_\xi(\theta) = s_\omega(\theta), \ \theta = 1, \ldots, t\}.$$

We shall now order scenarios in Ω by assigning to them numbers $i = 1, \ldots, S$ in such a way that for every i scenario $i+1$ has the largest last common stage with i among all scenarios $j > i$:

$$t^{\max}(i, i+1) = \max\{t^{\max}(i, j) : \ j > i\}.$$

Scenarios in Fig. 2 for the tree of Fig. 1 are ordered in this way.

It is easy to observe that with such an ordering, the bundles of scenarios which are indistinguishable up to some time t form connected subsets of {1,...,S}. In Fig. 2, they are joined by horizontal dotted lines.

Next, for every scenario i and every time period t, we define the *sibling* of i at t as

$$\nu(i,t) = \begin{cases} i+1 & \text{if } t^{\max}(i, i+1) \geq t, \\ \min\{j : \ t^{\max}(i,j) \geq t\} & \text{otherwise.} \end{cases}$$

Time stage	Scenario							
	1	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8	1
2	2	3	4	1	6	7	8	5
3	1	3	2	4	6	7	5	8
4	1	2	3	4	5	6	7	8

Table 1: Siblings of scenarios.

Let us note that a scenario may have different siblings at different time stages. For the example of Fig. 1 and Fig. 2, siblings of scenarios are shown in Table 1.

For every t, the mapping $\nu(i,t)$ defines a permutation of Ω, which maps bundles of indistinguishable scenarios onto themselves. It is easy to observe that $\nu(i,t) \neq i$, if the bundle of scenario i at stage t contains more than one member. The inverse permutation will be denoted by $\nu^{-1}(i,t)$.

Using the mapping $\nu(i,t)$ we can describe the non-anticipativity condition by the constraints:

$$x_i(t) = x_{\nu(i,t)}(t) \quad \text{for all} \quad (i,t) \quad \text{such that} \quad i \neq \nu(i,t). \tag{4.2}$$

There is still some redundancy in this set (we can remove one equation for each bundle), but we shall keep all equations (4.2) for convenience.

Thus, the whole problem has the following structure:

$$\min \sum_{i=1}^{S} p_i \sum_{t=1}^{T} c_t(x_i(t)) \tag{4.3}$$

subject to (4.1) and (4.2). This corresponds exactly to the general model (3.1)-(3.3).

The augmented Lagrangian function for (4.1)-(4.3) has the form

$$\begin{aligned} \Lambda(x,\pi) \;=\; & \sum_{i=1}^{S} p_i \sum_{t=1}^{T} c_t(x_i(t)) + \sum_{i=1}^{S} \sum_{t=1}^{T-1} \langle \pi_i(t), x_i(t) - x_{\nu(i,t)}(t) \rangle \\ & + \frac{1}{2}\rho \sum_{i=1}^{S} \sum_{t=1}^{T-1} \| \, x_i(t) - x_{\nu(i,t)}(t) \, \|^2 \, . \end{aligned} \tag{4.4}$$

Subproblems (3.10) take on the form:

$$\begin{aligned} \min_{x_i \in X_i} [\Lambda_i(x_i, \tilde{x}, \pi) \;=\; & p_i \sum_{t=1}^{T} c_t(x_i(t)) + \sum_{t=1}^{T-1} \langle \pi_i(t) - \pi_{\nu^{-1}(i,t)}(t), x_i(t) \rangle + \\ & \frac{1}{2}\rho \sum_{t=1}^{T-1} \{ \| \, x_i(t) - \tilde{x}_{\nu(i,t)}(t) \, \|^2 + \| \, x_i(t) - \tilde{x}_{\nu^{-1}(i,t)}(t) \, \|^2 \}]. \end{aligned} \tag{4.5}$$

In other words, the augmented Lagrangian is minimized with respect to the variables associated with scenario i assuming that other variables are temporarily fixed at their values $\tilde{x}_j$ for all $j \neq i$. This is done in parallel for each scenario.

Jacobi Method

Step 0. Set $\pi = \pi^k$, $\tilde{x}^{k,r} = x^{k-1}$ *and* $r = 1$.

Step 1. For $i = 1, \ldots, S$ solve (4.5) with $\tilde{x} = \tilde{x}^{k,r}$ obtaining new points $x_i^{k,r}$.

Step 2. If $x_i^{k,r}(t) = \tilde{x}_i^{k,r}(t)$ for all (i,t) such that $i \neq \nu(i,t)$, then stop; otherwise set

$$\tilde{x}_i^{k,r+1}(t) = \tilde{x}_i^{k,r}(t) + \tau(x_i^{k,r}(t) - \tilde{x}_i^{k,r}(t)), \tag{4.6}$$

increase r by 1 and go to Step 1.

Let us now pass to convergence conditions and to the speed of convergence. We immediately see that each constraint (4.2) links variables from only two scenarios. Therefore, the number of neighbors in (3.12) equals $N = 1$. By Theorem 3.1, apart from the Constraint Qualification Condition, it is sufficient for convergence that the under-relaxation coefficient in (4.6) satisfies the inequalities

$$0 < \tau < 1.$$

This is a very mild requirement.

Assuming additionally the Quadratic Growth Condition, from Theorem 3.2, we obtain the guaranteed ratio of convergence:

$$q = \frac{\tau(1-\tau)}{4\rho\gamma^{-1} + 1}. \tag{4.7}$$

The number 4 in the denominator follows from the observation that the constraint matrix of (4.2) has submatrices A_i which, after removing empty rows and columns, can be permuted to the form

$$A_i = \begin{bmatrix} 1 & & & \\ -1 & & & \\ & 1 & & \\ & -1 & & \\ & & \ddots & \\ & & & 1 \\ & & & -1 \end{bmatrix}.$$

Thus $\|A_i\| \leq \sqrt{2}$ so $\alpha^2 \leq 2$ in (3.15). The best estimate of the ratio (4.7) can be obtained for $\tau = \frac{1}{2}$:

$$q = \frac{1}{16\rho\gamma^{-1} + 4}. \tag{4.8}$$

For polyhedral cost functions c_t and polyhedral sets $X_i(t)$, $t = 1, \ldots, T$, $i = 1, \ldots, S$, we can additionally observe that (locally) $\gamma = \beta^{-1}\rho$ with some $\beta > 0$ independent of ρ. Then the ratio becomes independent of the penalty parameter ρ:

$$q = \frac{1}{16\beta + 4}. \tag{4.9}$$

The above results constitute a promising theoretical fundament for an efficient practical method for convex multistage stochastic problems. The computational results of [14] and [1] provide practical evidence for that.

5 Nodal decomposition

We shall now apply the general framework of section 3 to problem (2.4)-(2.7) with the following assumptions:

- explicit non-anticipativity constraints are removed from the problem by decreasing the number of decision variables;
- equations of dynamics (2.5) are treated as linking constraints.

Let us start by removing explicit non-anticipativity constraints. To achieve that we shall use the scenario tree $\mathcal{T} = \{\mathcal{N}, \mathcal{A}\}$, as described in section 2 and illustrated in Fig. 1. We denote by $a(n)$ the *ancestor* of node n, i.e. the node at the previous level with which n is connected and by $\mathcal{S}(n)$ the set of *successors* of n, $\mathcal{S}(n) = \{m : n = a(m)\}$.

A node n at level t of the tree corresponds to the bundle Ω_n of scenarios which are indistinguishable up to time t. By the non-anticipativity condition (2.7), all decisions $x_\omega(t)$, $\omega \in \Omega_n$, must be equal. We denote their value by x_n.

Next, for each node $n \in \mathcal{N}$, we define probability $\bar{p}_n$ as follows: for each *terminal node* $n \in \mathcal{N}_T$ we set $\bar{p}_n = p_\omega$, where $\omega \in \Omega$ is the event that corresponds to leaf n. For other nodes we define $\bar{p}_n = \sum_{m \in \mathcal{S}(n)} \bar{p}_m$.

Finally, with a slight abuse of notation, for a node n corresponding to event ω at stage t we define:

$$\begin{aligned} D_n &= D_\omega(t), \\ H_n &= H_\omega(t), \\ b_n &= b_\omega(t), \\ X_n &= X_\omega(t), \\ c_n(\cdot) &= c_t(\cdot). \end{aligned}$$

Using this notation we can rewrite (2.4)-(2.7) as follows:

$$\min \sum_{n \in \mathcal{N}} \bar{p}_n c_n(x_n) \tag{5.1}$$

$$D_n x_{a(n)} + H_n x_n = b_n, \; n \in \mathcal{N}, \tag{5.2}$$

$$x_n \in X_n, \; n \in \mathcal{N}, \tag{5.3}$$

where $x_{a(1)} = x(0)$. This corresponds again to the general format (3.1)-(3.3).

The augmented Lagrangian for (5.1)-(5.3) has the form:

$$\begin{aligned}\Lambda(x,\pi) &= \sum_{n\in\mathcal{N}} \bar{p}_n c_n(x_n) + \sum_{n\in\mathcal{N}} \bar{p}_n \langle \pi_n, b_n - D_n x_{a(n)} - H_n x_n\rangle \\ &\quad + \frac{\rho}{2}\sum_{n\in\mathcal{N}} \bar{p}_n \|b_n - D_n x_{a(n)} - H_n x_n\|^2. \end{aligned} \tag{5.4}$$

The introduction of scaling factors $\bar{p}_n$ simplifies subproblems (3.10)

$$\begin{aligned}\min_{x_n\in X_n}[\Lambda_n(x_n,\tilde{x},\pi) &= c_n(x_n) - \langle H_n^T\pi_n + \sum_{m\in\mathcal{S}(n)} p_{m|n} D_m^T \pi_m, x_n\rangle \\ &\quad + \frac{\rho}{2}\|b_n - D_n\tilde{x}_{a(n)} - H_n x_n\|^2 \\ &\quad + \frac{\rho}{2}\sum_{m\in\mathcal{S}(n)} p_{m|n}\|b_m - D_m x_n - H_m\tilde{x}_m\|^2], \end{aligned} \tag{5.5}$$

where $p_{m|n} = \bar{p}_m/\bar{p}_n$ is the probability of getting to node m from node n.

Jacobi Method

Step 0. Set $\pi = \pi^k$, $\tilde{x}^{k,r} = x^{k-1}$ *and* $r = 1$.

Step 1. For $n \in \mathcal{N}$ solve (5.5) with $\tilde{x} = \tilde{x}^{k,r}$ obtaining new points $x_n^{k,r}$.

Step 2. If $D_n x_{a(n)} + H_n x_n = b_n$ for all $n \in \mathcal{N}$ then stop; otherwise set for $n \in \mathcal{N}$

$$\tilde{x}_n^{k,r+1} = \tilde{x}_n^{k,r} + \tau(x_n^{k,r} - \tilde{x}_n^{k,r}), \tag{5.6}$$

increase r by 1 and go to Step 1.

Let us now pass to convergence conditions and to the speed of convergence. We immediately see that each constraint (5.2) links variables from only two nodes, so the number of neighbors in (3.12) equals $N = 1$. By Theorem 3.1, similar to scenario decomposition, it is sufficient for convergence that the under-relaxation coefficient in (5.6) satisfies the inequalities

$$0 < \tau < 1$$

and the Constraint Qualification Condition holds. Assuming additionally the Quadratic Growth Condition, from Theorem 3.2 we obtain the guaranteed ratio of convergence:

$$q = \frac{\tau(1-\tau)}{2\alpha^2\rho\gamma^{-1} + 1} \tag{5.7}$$

with α defined as in (3.16). Let us estimate α. Assume that $\mathcal{S}(n) = \{m_1, m_2, \ldots, m_l\}$. The submatrix A_n of the constraint matrix of (5.2), after removing empty rows, has the form

$$A_n = \begin{bmatrix} H_n \\ D_{m_1} \\ D_{m_2} \\ \vdots \\ D_{m_l} \end{bmatrix}.$$

Thus

$$\|A_n\|^2 \leq \|H_n\|^2 + \sum_{m \in \mathcal{S}(n)} \|D_m\|^2.$$

Therefore it is sufficient to use in (5.7)

$$\alpha^2 = \max_{n \in \mathcal{N}} \left[\|H_n\|^2 + \sum_{m \in \mathcal{S}(n)} \|D_m\|^2 \right]. \tag{5.8}$$

The best estimate of the ratio (4.7) can be obtained for $\tau = \frac{1}{2}$:

$$q = \frac{1}{8\alpha^2 \rho \gamma^{-1} + 4}.$$

Again, for polyhedral cost functions c_t and polyhedral sets $X_i(t)$, $t = 1, \ldots, T$, $i = 1, \ldots, S$, we can additionally observe that (locally) $\gamma = \beta^{-1}\rho$ with some $\beta > 0$ independent of ρ. Then the ratio becomes independent of the penalty parameter ρ:

$$q = \frac{1}{8\alpha^2 \beta + 4}.$$

These estimates differ from (4.8) and (4.9) mainly by the factor α. We should, in general, expect slower convergence of the nodal decomposition method than the scenario decomposition method, unless α^2 given by (5.8) is smaller than 2.

6 Computational Results

To test the behavior of both the scenario and nodal versions of the augmented Lagrangian decomposition algorithm on an actual numerical example, we consider an eight scenario nonlinear (convex) stochastic macroeconomic energy model based on the Global 2100 model developed by Alan Manne and Richard Richels [11], [12]. The stochastic model was developed at IIASA [20] in collaboration with the Environmentally Compatible Energy Strategies group and the models eight scenarios represent alternative futures of energy resource commodity prices and properties of the production function that drives nonenergy economic output in the model.

When solved using scenario decomposition the model has eight subproblems and when solved via nodal decomposition, seventy one. We solve the problems within GAMS [5], [15] using the looping and solve facilities provided therein. As such, we do not report any timings, as this method of implementation takes no advantage of the obvious parallelism. Instead, we report only the numbers of outer and inner iterations that were required to meet certain levels of accuracy. Though this makes the results incomparable to the solution times of alternative solution methods, it does provide basic algorithmic information about the behavior of both forms of the algorithm when applied to a fairly large model of practical relevance.

Constraints	398
Nonlinear Constraints	25
Variables	610
Nonlinear Variables	85
Multipliers	408

Table 2: Scenario Model Characteristics

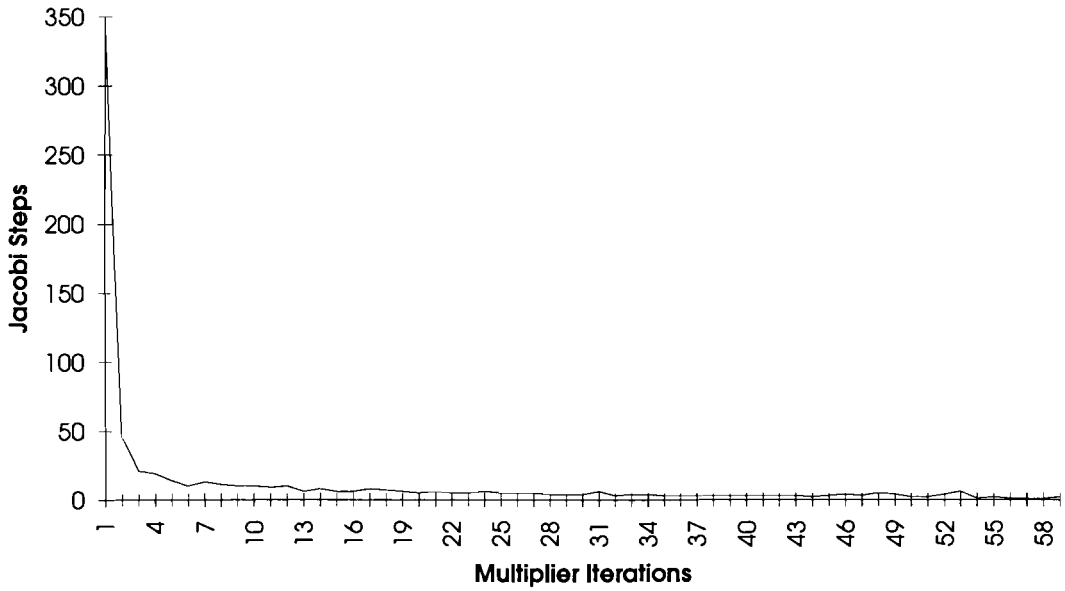

Figure 3: Scenario decomposition. Number of Jacobi steps in each outer loop.

6.1 Scenario Decomposition

The characteristics of each of the individual scenario models is summarized in table 2. We use a stopping criteria for the outer loop of $\epsilon = 10^{-4}$ and a stopping criteria for each variable in the inner loop of one half of the error in the corresponding nonanticipativity constraint in the outer loop. We start from a cold start (multipliers and Jacobi updates all set to zero) and obtain, in a representative run, 59 outer loops and 705 inner loops, each inner loop requiring the solution of the eight separate GAMS scenario models. As is clear from figure 3, the inner loops occur with greatest frequency at the beginning of the algorithm because of the cold start. The maximum relative error in nonanticipativity over the course of the algorithm is displayed in figure 4. The algorithm initially converges linearly as the changing multipliers on the relaxed nonanticipativity constraints move the solution along linear constraints. The progress is later impeded by the presence of the nonlinear constraints and variables whose properties slow the search for the appropriate penalties. Finally, figure 5 plots the number of nonanticipativity constraints violated at

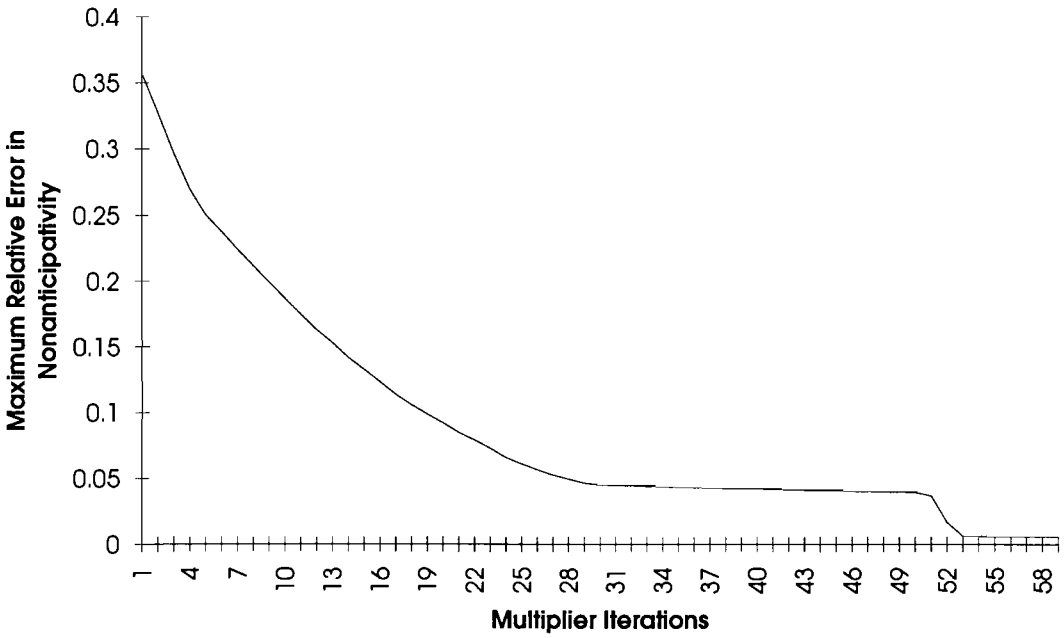

Figure 4: Scenario decomposition. Maximum relative error

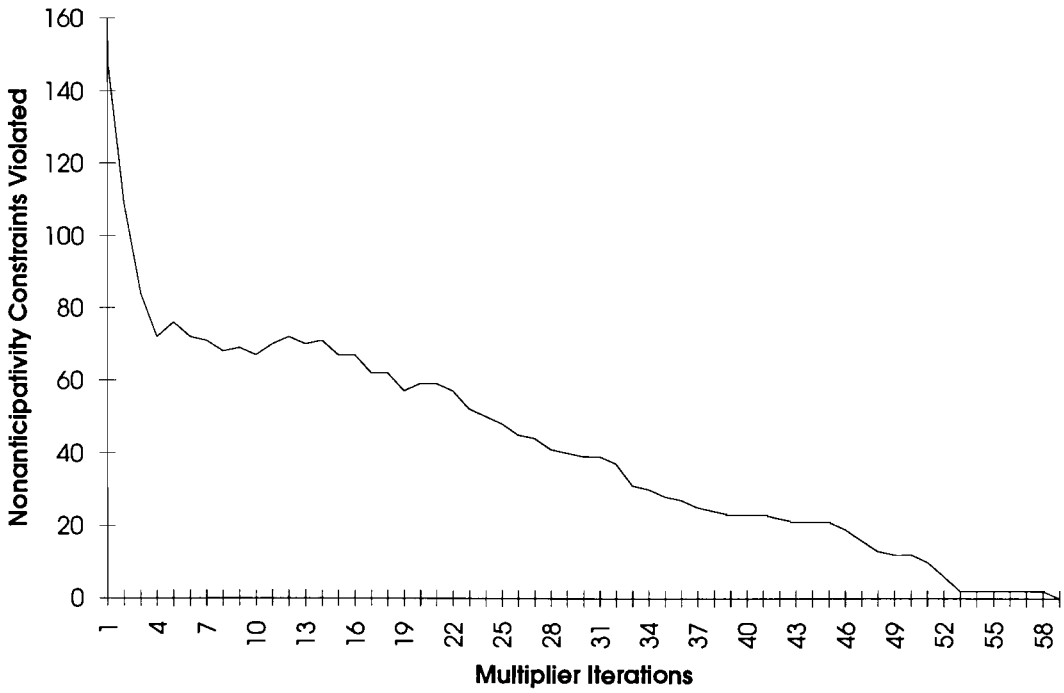

Figure 5: Scenario decomposition. Number of violated nonanticipativity constraints.

Constraints	14
Nonlinear Constraints	2
Variables	60
Nonlinear Variables	9
Multipliers	122

Table 3: Nodal Model Characteristics

each multiplier step. Again, very fast progress at the start is followed by a slower linear rate of convergence until the stopping criteria is fulfilled.

6.2 Nodal Decomposition

We summarize the characteristics of the typical individual nodal model in table 3 (initial period and terminal period nodes have slightly different characteristics). In this formulation, each subproblem represents one of the nodal problems from the stochastic tree. No nonanticipativity constraints exist to be relaxed as they are implicit in the structure of the tree. Instead, all dynamic linking constraints are relaxed and incorporated into the objective function as previously discussed.

We, again, use a stopping criteria for the outer loop of $\epsilon = 10^{-4}$. The stopping criteria for the inner loop is a little more difficult to determine as each variable may be a part of multiple relaxed constraints. Also, the relationship between the stopping criteria used for a particular constraint and the stopping criteria that should be used for each component variable of the constraint may be very complicated. Both of these properties of the problem formulation make practical implementation more difficult to achieve. For our tests, we used a stopping criteria for each inner loop variable that required successive values of the variable to differ relatively by no more than the minimum of the relative error of all constraints of which that particular variable was a part. This proved to be too rigorous of a stopping criteria, though, as the algorithm repeatedly became trapped in one of the inner loop cycles. We overcame this problem by making the stopping criteria be the maximum of this minimum relative error and an arbitrary small number (0.005 in our test). Using this, we achieve very poor convergence when compared to what we saw with the simpler scenario disaggregation. Results appear in figures 6, 7, and 8.

In contrast to the previous case where the number of Jacobi steps required declined rapidly, approaching a low level close to zero, the number of inner loops exhibits erratic behavior, jumping from high to low at irregular intervals. Also, average relative error in the slowly declining population of constraints that don't meet the stopping criteria actually seems to increase during the course of the algorithm.

It would appear from this rather simple numerical experiment that the augmented Lagrangian decomposition algorithm performs best when nonanticipativity constraints of the type used in the scenario decomposition formulation of the problem are the constraints

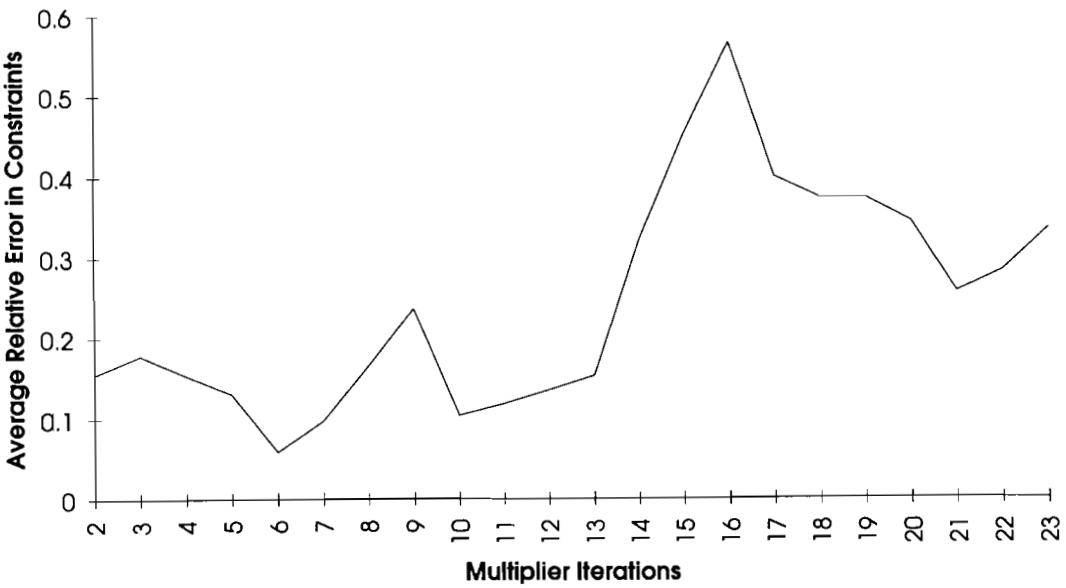

Figure 6: Nodal decoposition. Average relative error.

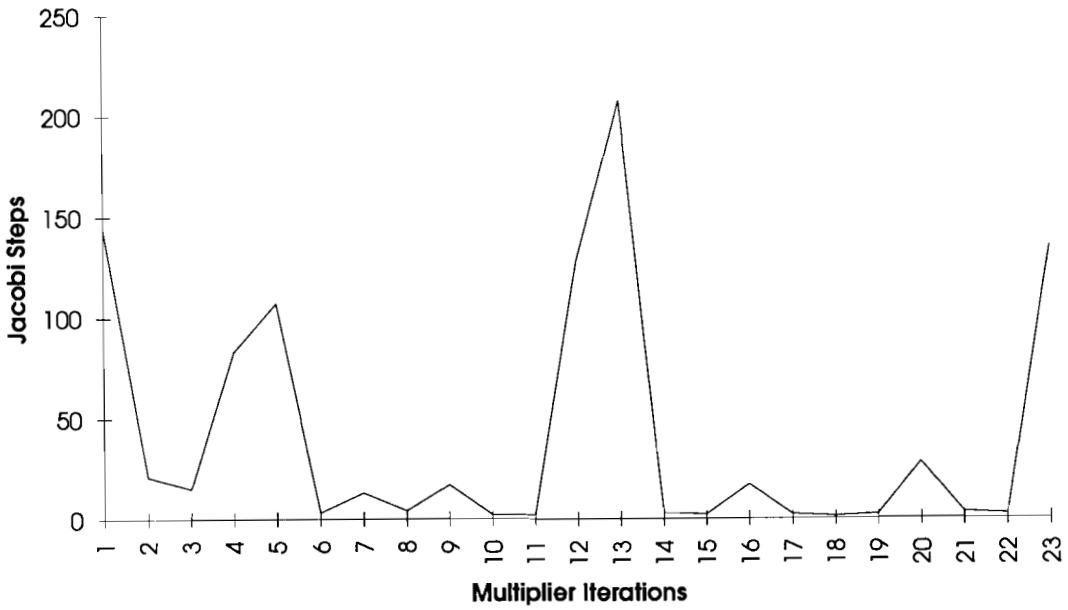

Figure 7: Nodal decomposition. Number of Jacobi steps in each outer loop.

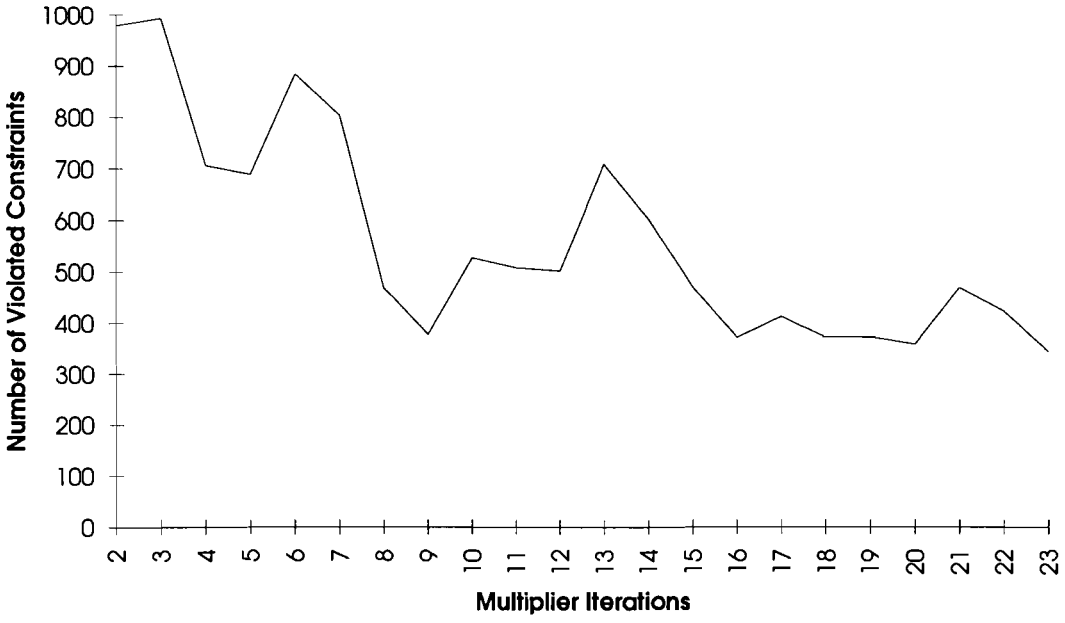

Figure 8: Nodal decomposition. Number of violated linking constraints.

that are relaxed. Theory suggests that this would be the case as one of the principal factors that governs the rate of convergence of the general augmented Lagrangian approach is the norm of the submatrices that make up the relaxed constraints. The greater this norm, the slower the convergence. In the scenario decomposition approach, this norm is small as each submatrix of a subproblem consists of an individual nonanticipativity constraint which is a simple matrix with of most two nonzero elements per column. In contrast, the matrices that describe general dynamic linking constraints may have much more complicated structures with correspondingly larger norms. These greater norms may slow convergence of the method, as described earlier in the paper.

In addition to the nodal decomposition method being less attractive as a solution technique by virtue of its poor convergence, it is also less attractive from a modeling point of view. This is so because the decomposition of many specialized dynamic linking constraints that have problem specific structure is a complicated procedure, much less amenable to any automated procedures that might be applied in the scenario case where the structure of the nonanticipativity constraint makes decomposition of the problem essentially indistinguishable from application to application.

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