Working Paper

Convergence Rate of Agents' Learning in Macroeconomic Models

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WP-94-72
August 1994

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Abstract

This paper discusses agents' learning on a market. The price level evolves through a multivariable autoregressive model, which the agents learn in a least-squares sense. A theorem is stated that shows how the agents' learning might be divided into two classes with respect to the learning convergence rate. The results are exemplified by the well-known hyperinflation model. Further, for the hyperinflation model some interesting features concerning the "coupling" between the price and the learning dynamics are discussed. An explicit expression is derived for how the rate of the agents' learning depends upon this coupling.

Convergence Rate of Agents' Learning in Macroeconomic Models

*Karl Henrik Johansson**

1 Introduction

One way to introduce dynamics in macroeconomic models is to let some variables depend upon agents' expectations. The expectations can either be formed optimally out of given information in a stationary sense, so called rational expectations, Muth (1961), or they can evolve through a learning process, Bray (1982), Marcet and Sargent (1989*a*), Marcet and Sargent (1989*b*). In this paper we treat the second case. The considered models consist of two parts, see Figure 1. The first part is the price dynamics given by a linear stochastic difference equation. The second part describes the agents' learning of the price dynamics, in this case a least-squares learning algorithm. For certain model parameters, the limit of the learning process will be a rational expectation equilibrium.

Macroeconomic models embodying agents' learning have been dealt with in a great number of papers, see the survey Blume, Bray and Easley (1982). Some of these assume that the model structure is known by the agents, whereas in others the agents' model is misspecified during learning, *e. g.* Townsend (1983) and Fourgeaud, Gourieroux and Pradel (1986), respectively. In this paper we consider the latter type, and in particular we use the model set-up described in Marcet and Sargent (1989*b*). Several classical models can be rewritten to conform with this set-up, and thus, the results given below apply to all these models. The results are also applied in an extensive analysis of a hyperinflation model.

The main contribution of this paper is a theorem regarding the convergence rate of the agents' learning. It is shown that the economic models can be divided into two classes with respect to the rate of convergence. In the first class we have ordinary $1/\sqrt{t}$-convergence, while in the second the convergence is slower. If the macroeconomic model belongs to the first class, the agents' learning can be interpreted as being "sufficiently stable" in the sense that the eigenvalues determining the stability of the learning algorithm are located deep inside the stability region. In the second class the learning is less stable and the convergence is slowed down. The theorem stated is an extension to results given in Marcet and Sargent (1992).

*Department of Automatic Control, Lund Institute of Technology, Sweden. The author wants to express his gratitude to Yuri M. Kaniovski, Luigi Marengo, Henrik M. Olsson, and Helene Tordjman for their comments.

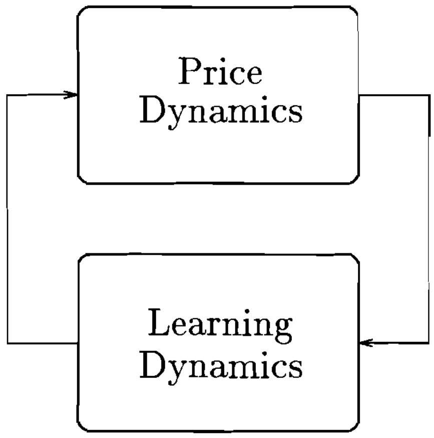

Figure 1: A block diagram that shows the interaction between the price and the learning dynamics.

Some comments are given about Cagan's classical hyperinflation model, Cagan (1956). By applying our theorem describing the agents' learning rate we are able to show new features of this model. The "coupling" between the price dynamics and the agents' learning dynamics is especially examined. An interesting nonintuitive relationship between the agents' convergence rate and this coupling is shown. It follows from the analysis that the hyperinflation model may belong to both classes described by the theorem. The coupling and the money supply model determine the class. Since the studied hyperinflation model is quite simple, it is possible to derive explicit expressions. An economic interpretation of the results is that it can be as well easier as harder for the agents to learn when the coupling between the two dynamics is tight. It is shown that the model changes from one of these behaviors to the other by a slight change of the money supply model. This nonrobustness is, of course, not acceptable for an economic model.

The agents' learning is described by a least-squares algorithm. This algorithm was chosen because of its optimality for a Gaussian set-up and because it is well-known in the macroeconomics literature. Note, however, that much of the analysis done below is not specific for least-squares learning. Similar results can be stated for other types of learning. The references Benveniste, Metivier and Priouret (1990) and Ljung (1977) contain the stochastic control theory used in this paper. Hence, the more general learning algorithms studied in these references may give ideas of how much the results in this paper can be generalized. The nonrobustness of the hyperinflation model can be related to the least-squares algorithm. If the agents' learning is modeled in a different way, the behavior of the system may change. Further, note that only the aggregated behavior of all agents is illustrated by the type of models studied here. We thus view the learning algorithm as an average of how the individual agents are learning or, in another sense, we study a representative individual, for comments on this subject see Kirman (1992).

The remainder of this paper is organized as follows. In Section 2 we introduce a hyperinflation model. This is used throughout the text to exemplify the results. A general model set-up is presented, and we also show how the hyperinflation model can be rewritten in this form. Section 3 includes a theorem on the convergence rate of the agents' learning. This result is applied to the hyperinflation model in an example. The coupling in the hyperinflation model is also treated. The conclusions are given in Section 4, followed by Appendix which includes the assumptions made and a proof of the theorem in Section 3.

2 Model Set-Up

In this section we describe a general set-up for macroeconomic models with learning dynamics. This set-up comprehend a wide range of models. The procedure of rewriting an economic model in this "standard form" is exemplified by the well-known hyperinflation model. This way of writing economic models has been developed and extensively used in Marcet and Sargent (1989*a*), Marcet and Sargent (1989*b*), and Marcet and Sargent (1992).

We start by introducing the hyperinflation model.

EXAMPLE—*Hyperinflation Model*
The hyperinflation model describes the relation between the money supply, the price level, and the agents' expected inflation on a market during hyperinflation. The model goes back to Cagan (1956), where it is estimated using real data. The model is further analyzed in Sargent and Wallace (1973) where also another estimation is done. The hyperinflation model has been used for theoretical studies concerning agents' learning rate and convergence to rationality in a situation of hyperinflation. These kind of works are presented in Fourgeaud *et al.* (1986), Gourieroux, Laffont and Monfort (1982), Marcet and Sargent (1989*b*), and Marcet and Sargent (1992).

Let m_t^d be the logarithm of the nominal money demand and y_t the logarithm of the price level, both at time t. The hyperinflation model says that real demand for money $m_t^d - y_t$ is mainly a linear function of expected inflation. We may write

$$m_t^d - y_t = \beta(\hat{y}_{t+1} - y_t) \tag{1}$$

where $\beta \neq 1$ is a constant and $\hat{y}_{t+1}$ denotes the agents' expectation of the price level at time $t+1$ given their information up to time t.[1] The nominal money supply m_t^s is modeled by a stochastic process $\tilde{x}_t$. Since we assume the market is clearing, the nominal money demand equals the supply, thus

$$m_t^d = m_t^s = \tilde{x}_t$$

[1]Note that in the model presented here, agents build their expectation upon old data and influence future prices. This gives a natural causality to the model, so at a certain time the price level depends on the agents' expectation and not also that the agents' expectation at the same time depends on the price level. In Marcet and Sargent (1992) this is not the case. Instead, the motion of their model will be restricted by an algebraic equation that always must be fulfilled. Some comments about causality in the original problem formulation is given in Sargent and Wallace (1973).

If we include this in (1), we get

$$y_t = \frac{\beta}{\beta - 1}\hat{y}_{t+1} - \frac{1}{\beta - 1}\tilde{x}_t$$

and after introducing a new constant λ and a new stochastic process x_t in an obvious way

$$y_t = \lambda\hat{y}_{t+1} + x_t \tag{2}$$

Based on Cagan (1956) and Sargent and Wallace (1973), we choose to examine the agents' learning for $0 < \lambda < 1$. The money supply described by x is given by a first-order autoregressive (AR) model driven by a white Gaussian process

$$x_t = \rho x_{t-1} + u_t \tag{3}$$

where $|\rho| < 1$ and $E\{u_t^2\} = \sigma_u^2$. The model (2) is unknown for the agents. Instead their expectation $\hat{y}_{t+1}$ is based upon a perceived model. Assume at time t the money supply x is known by the agents up to time t and the price level y is known up to $t-1$. The agents build their expectations on the misspecified time-invariant model

$$y_t = \theta x_{t-1} + \bar{w}_t$$

where $\bar{w}$ is the least-squares residual. The expected price level is given by

$$\hat{y}_{t+1} = \theta_{t-1}x_t \tag{4}$$

The variable θ_{t-1} is a least-squares estimate of θ derived by the agents using their known information at time t. The agents' learning is modeled by the recursive least-squares algorithm

$$\begin{array}{rcl} \theta_t & = & \theta_{t-1} + \frac{1}{t}R_t^{-1}x_{t-1}w_t \\ R_t & = & R_{t-1} + \frac{1}{t}(x_{t-1}^2 - R_{t-1}) \end{array} \tag{5}$$

where $w_t = y_t - x_{t-1}\theta_{t-1}$, and some initial values θ_0 and R_0 are given. By introducing the state $z_t = [y_t \quad w_t \quad x_t]^T$, it is possible to combine (2), (3), and (4) into the multivariable AR process

$$z_t = \begin{bmatrix} 0 & 0 & \lambda\rho\theta_{t-1} + \rho \\ 0 & 0 & \lambda\rho\theta_{t-1} - \theta_{t-1} + \rho \\ 0 & 0 & \rho \end{bmatrix} z_{t-1} + \begin{bmatrix} \lambda\theta_{t-1} + 1 \\ \lambda\theta_{t-1} + 1 \\ 1 \end{bmatrix} u_t \tag{6}$$

(*c.f.* Marcet and Sargent (1989*b*)). The algorithm (5) together with (6) summarize the evolution of the whole hyperinflation model. The model is illustrated in Figure 2, where the price dynamics essentially are given by (6) and the learning dynamics by (5). However, note that the price dynamics are given by a time-varying system, since it is influenced by the agents' estimates. This is not shown in the figure. □

By the example above we have seen how it is possible to rewrite the hyperinflation model in a standard form given by the multivariable first-order AR equation

$$z_t = T(\theta_{t-1})z_{t-1} + V(\theta_{t-1})e_t \tag{7}$$

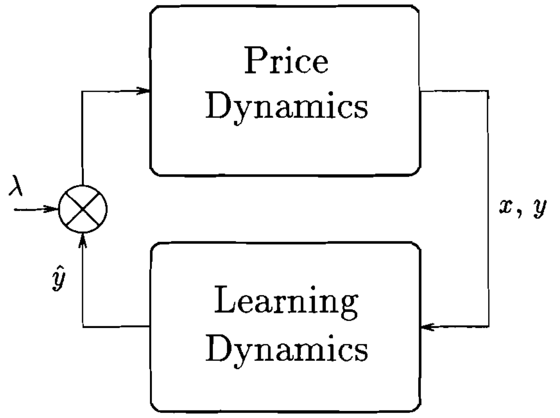

Figure 2: A block diagram that shows the interaction in the hyperinflation model. The parameter λ is multiplied to the expected price $\hat{y}_{t+1}$, and hence, can be taken as a parameter of the coupling between the two dynamics.

and the least-squares algorithm

$$\begin{aligned} \theta_t &= \theta_{t-1} + \frac{1}{t} R_t^{-1} \varphi_t w_t \\ R_t &= R_{t-1} + \frac{1}{t}(\varphi_t \varphi_t^T - R_{t-1}) \end{aligned} \tag{8}$$

where φ includes one or several of the states in (7) such that $w_t = y_t - \varphi_t^T \theta_{t-1}$ becomes the prediction error similar to w_t in the example. The vector z represents the state of the price dynamics and the vector e the applied shocks. The elements in e is assumed to be independent white Gaussian processes. In Marcet and Sargent (1989*b*) it is shown how other models, *e.g.* a model in Bray (1983) and one in Bray and Savin (1986), can be written in this form. Also a version of Townsend's model, Townsend (1983), can be transformed to fit into this notation, see Marcet and Sargent (1989*a*).

An important problem is if the agents' learning process will converge to an rational equilibrium or not. This problem has been extensively studied in many papers, and is one of the main problems in the literature of rational expectations. In Marcet and Sargent (1989*b*) it is shown that if the assumptions in Appendix A hold, then θ_t will converge to a unique equilibrium θ_f almost surely as $t \to \infty$ for a large class of models described by (7) and (8). In the remainder of this paper we will assume that $\theta_t \to \theta_f$ almost surely and that the assumptions in Appendix A hold if nothing else is mentioned.

3 Convergence Rate of Agents' Learning

In this section we concentrate on the the agents' learning process. The convergence rate of the learning algorithm is discussed, the differential equation associated with the algorithm is introduced, and a theorem concerning the asymptotic behavior of the agents' estimates is given. We also continue the example in the previous section.

Consider the agents' learning algorithm (8) again. To analyze the behavior of this algorithm, it is convenient to introduce the associated differential equation. This approach was suggested in Ljung (1977), see also Ljung and Söderström (1983). We start by heuristically explaining the ideas behind it. For sufficiently large t, $1/t$ in the algorithm (8) will be small. Thus, if we view the last terms on the right hand sides in (8) as corrections to θ_{t-1} and R_{t-1}, these corrections will be small if t is sufficiently large. Hence, for large t it is reasonable to assume that θ_t and R_t vary slowly. We can then approximate them over a small time interval by their averaged values θ_a and R_a, respectively. The approximate updating rules are

$$\begin{aligned} \theta_t &= \theta_{t-1} + \frac{1}{t} R_a^{-1} f(\theta_a) \\ R_t &= R_{t-1} + \frac{1}{t}[g(\theta_a) - R_a] \end{aligned} \tag{9}$$

where

$$\begin{aligned} f(\theta_a) &= \lim_{t\to\infty} E\{\varphi_t(y_t - \varphi_t^T \theta_a)\} \\ g(\theta_a) &= \lim_{t\to\infty} E\{\varphi_t \varphi_t^T\} \end{aligned}$$

E denotes the expected value with respect to the distribution of the states z for a fixed value θ_a. If the assumptions in Appendix A hold, the algorithm (9) will act almost like (8) in a neighborhood to θ_a and R_a for sufficiently large t. With a change of time scale this new algorithm can be interpreted as a difference approximation to the differential equations

$$\begin{aligned} \frac{d\theta_d}{dt} &= R_d^{-1} f(\theta_d) \\ \frac{dR_d}{dt} &= g(\theta_d) - R_d \end{aligned} \tag{10}$$

The discrete time variables θ and R will asymptotically follow the trajectories θ_d and R_d of these associated differential equations. Simulations describing the learning in an economic system using the differential equations instead of the original algorithm are shown in Marcet and Sargent (1992).

To study the behavior of these nonlinear differential equations at equilibrium, we linearize them around their stationary point. For the differential equations (10) the linearized system is

$$\frac{d}{dt}\begin{bmatrix} \theta_d - \theta_f \\ R_d - R_f \end{bmatrix} = \begin{bmatrix} R_d^{-1}\frac{df}{d\theta_d} & 0 \\ * & -I_{\eta\times\eta} \end{bmatrix}_{\substack{\theta_d=\theta_f \\ R_d=R_f}} \begin{bmatrix} \theta_d - \theta_f \\ R_d - R_f \end{bmatrix} \tag{11}$$

where (θ_f, R_f) is the stationary point, $*$ denotes elements we are not interested in, and $I_{\eta\times\eta}$ is the $\eta \times \eta$ identity matrix. The system matrix above has $\eta = \dim\theta$ eigenvalues at -1 and η eigenvalues determined by

$$R_f^{-1} \frac{df}{d\theta_d}\bigg|_{\theta_d=\theta_f} = g^{-1}(\theta_f) f_{\theta_d}(\theta_f) \tag{12}$$

We introduce some notations. The function H denotes the last term excluding $1/t$ in the θ_t-equation of (8)

$$H(\theta_{t-1}, z_t) = R_t^{-1}\varphi_t(y_t - \varphi_t^T\theta_{t-1})$$

Define

$$h(\theta) = \lim_{t\to\infty} E\{H(\theta_{t-1}, z_t)\} = R_f^{-1}f(\theta)$$

This means that $h_\theta(\theta_f)$ is equal to (12). Thus, the eigenvalues of $h_\theta(\theta_f)$ determines the stability of the linear system (11). Further, denote the covariance matrix of H at the equilibrium point by D, *i. e.*

$$D = \lim_{t\to\infty} E\{H(\theta_f, z_t)H^T(\theta_f, z_t)\}$$

The notation $\mu_i(A)$ is used for the ith eigenvalue of the matrix A, and finally

$$\alpha = \max_i \mathrm{Re}\{\mu_i(h_\theta(\theta_f))\}$$

The following theorem is an extension to a theorem stated in Marcet and Sargent (1992).

THEOREM 1
If the assumptions in Appendix A hold, then if

$\underline{\alpha < -1/2}$

$$\sqrt{t}(\theta_t - \theta_f) \xrightarrow{w} \mathcal{N}(0, P), \quad t \to \infty$$

where $\mathcal{N}$ denotes the normal distribution and the covariance matrix P satisfies the Lyapunov equation

$$\left(\frac{1}{2}I + h_\theta(\theta_f)\right)P + P\left(\frac{1}{2}I + h_\theta(\theta_f)\right)^T = -D$$

If

$\underline{-1/2 < \alpha < 0}$

$$t^\gamma(\theta_t - \theta_f) \xrightarrow{P} 0, \quad t \to \infty$$

where γ is arbitrary in the range $(0, -\alpha)$. □

Proof: The first part is proven in Benveniste *et al.* (1990) (see Theorem 3, p. 110) and the second in Appendix B. □

The convergence notations above are "weakly" and "in probability," respectively, see Appendix B for definitions.

Note that in Marcet and Sargent (1992) a simulation method is suggested to determine the convergence rate in the second part of Theorem 1. We give an analytical proof showing that γ might be chosen arbitrary in $(0, -\alpha)$. The largest γ in this interval may be interpreted as the convergence rate, but since the interval is open such γ does not exist.

The theorem above divides the agents' learning into two classes. In the first one the learning algorithm can be interpreted as being "sufficiently stable"; all real parts of the eigenvalues of $h_\theta(\theta_f)$ are less than $-1/2$. Then the usual $1/\sqrt{t}$-convergence holds and the estimation errors tend to be normally distributed. In the second class the learning process still converges, but for this case the convergence is slower than $1/\sqrt{t}$-convergence. There is a whole range of sequences t^γ that satisfies $t^\gamma(\theta_t - \theta_f) \to 0$.

From the discussion above it follows that the eigenvalues of $h_\theta(\theta_f)$ are crucial for the behavior of the learning algorithm. Of course, it would be interesting to connect each economic model embodying learning dynamics to a certain class. However, in general this is not a simple task, since given a model the eigenvalues of $h_\theta(\theta_f)$ are determined in an implicit way. Nevertheless, sometimes, as in the example below, we are able to derive explicit expressions. Also, as it is shown below, an economic model might belong to more than one class.

Now we continue the example in the previous section. The associated differential equations for the hyperinflation model are derived, and it is shown how this model can be classified using Theorem 1.

EXAMPLE—*Hyperinflation Model (cont'd)*
In the previous part of this example we ended up with a price equation of the form

$$z_t = T(\theta_{t-1})z_{t-1} + V(\theta_{t-1})u_t$$

Let us for fixed θ introduce the covariance matrix

$$M = \begin{pmatrix} M_{yy} & M_{yw} & M_{yx} \\ M_{wy} & M_{ww} & M_{wx} \\ M_{xy} & M_{xw} & M_{xx} \end{pmatrix} = \lim_{t\to\infty} E\{z_t z_t^T\} \tag{13}$$

then M satisfies the Lyapunov equation

$$M = T(\theta)MT^T(\theta) + V(\theta)\sigma_u^2 V^T(\theta)$$

The equilibrium θ_f is easily derived for the hyperinflation model. Since

$$y_t = \lambda\hat{y}_{t+1} + x_t = (\lambda\theta_{t-1} + 1)\rho x_{t-1} + (\lambda\theta_{t-1} + 1)u_t$$

and the agents' misspecified model is

$$y_t = \theta x_{t-1} + \bar{w}_t$$

we get the unique equilibrium

$$\theta_f = \frac{\rho}{1 - \lambda\rho}$$

In Marcet and Sargent (1989*b*) it is shown that θ converges to θ_f almost surely. The associated differential equations are given by (10) and

$$\begin{aligned} f(\theta) &= \lim_{t\to\infty} E\{\varphi_t(y_t - \varphi_t^T\theta)\} = [(\lambda\rho - 1)\theta + \rho]M_{xx} \\ g(\theta) &= \lim_{t\to\infty} E\{\varphi_t\varphi_t^T\} = M_{xx} \end{aligned} \tag{14}$$

To apply Theorem 1 to the hyperinflation model h_θ and D must be derived.

$$h(\theta) = R_f^{-1}f(\theta) = R_f^{-1}[(\lambda\rho - 1)\theta + \rho]M_{xx} = (\lambda\rho - 1)\theta + \rho \tag{15}$$

where the last equality follows from (10) and (14). Then,

$$h_\theta = \lambda\rho - 1 \tag{16}$$

Since $y_t - \theta_f x_{t-1} = w_t$, we get

$$D = \lim_{t\to\infty} E\{R_f^{-1}\varphi_t(y_t - \varphi_t^T\theta_f)^2\varphi_t^T R_f^{-T}\} = \lim_{t\to\infty} E\{x_{t-1}^2 w_t^2/R_f^2\}$$

By taking conditional expectations and using the equality $E\{Y\} = E\{E\{Y|X\}\}$ where X denotes information (σ-field) up to time $t-1$, we get

$$D = \lim_{t\to\infty} E\{x_{t-1}^2\}E\{w_t^2\}/R_f^2 = M_{ww}/R_f \tag{17}$$

In Theorem 1 the agents' learning is divided into two classes. Since h_θ is given by the simple expression (16), we directly see that the hyperinflation model belongs to the first class

$$\alpha < -1/2, \qquad \text{if } \lambda\rho < 1/2$$

and to the second class

$$-1/2 < \alpha < 0, \qquad \text{if } 1/2 < \lambda\rho < 1$$

The feature of the agents' convergence within the first class, *i. e.* when the learning algorithm is "sufficiently stable", is given by the asymptotic covariance P (in this example a scalar). The smaller α the faster is the agents' convergence in the sense that P is smaller, see Marcet and Sargent (1992). Solving the Lyapunov equation in Theorem 1 gives

$$P = \frac{M_{ww}}{M_{xx}(1 - 2\lambda\rho)} \tag{18}$$

Hence, equation (18) gives a measure of the convergence rate if $\lambda\rho < 1/2$. Recall the price evolvement in the hyperinflation model given by

$$y_t = \lambda\hat{y}_{t+1} + x_t$$

The positive parameter λ can be interpreted as the coupling between the price dynamics and the learning dynamics, see Figure 2. This means that if λ is small, the agents' expectations of the future price have minor influence on the price level.

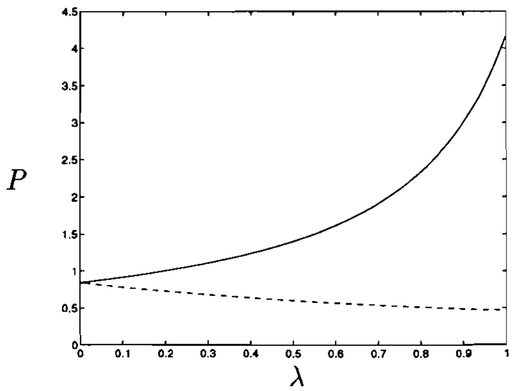

Figure 3: The asymptotic variance P shown as a function of the coupling λ for two different money supply models. The dashed line is for $\rho = -0.4$ and the solid for $\rho = 0.4$.

The price is almost completely determined by the money supply x. On the other hand, if λ is large agents' expectations have large impact on the price level. In general Theorem 1 gives an implicit formula for how λ influences the asymptotic convergence rate in the case $\alpha < -1/2$, and in our example the explicit formula (18) describes this. Thus, by applying Theorem 1 to the hyperinflation model we show how the coupling in the model influences the convergence rate of the agents' learning when the learning is sufficiently stable.

We might expect from (18) that the shape of P as a function of λ highly depends on ρ. (Notice that M_{ww} depends on ρ.) This is also the case. In Figure 3 $P = P(\lambda)$ is shown when $\sigma_u^2 = 1$ and $\rho = \pm 0.4$. For the case when $\rho = -0.4$, P is a decreasing function (dashed). This means that the asymptotic variance of the parameter estimate derived by the agents is decreasing with the amount of connection in the hyperinflation model. An interpretation of this is that it is easier for the agents to learn if the coupling is higher. However, $\rho = 0.4$ gives an increasing function P (solid line). Hence, in contradiction to the first case, it is now harder for the agents to learn when they have high influence on the price. Our conclusion from these two parameter choices is that the money supply model is crucial in determining if the agents' learning is gaining or not from a tight coupling between the price and the learning dynamics. □

Similar analysis as in the example above can be done for other models.[2] By the same method expressions like (16) and (17) can be derived, and then Theorem 1 applies for a classification. Also, the dependency of the convergence rate on other parameters can be examined. Note that in general P is a matrix. Then the convergence might be studied by examining the trace of P, $\sum \mu_i(P)$, which captures the essential feature of the corresponding Gaussian distribution.

The particular model studied in this example showed that the money supply

[2] In Johansson (1993) the slightly more complex hyperinflation model in Marcet and Sargent (1992) is considered. Numerical examples similar to the example in this paper are shown.

model influences the agents' learning in a dramatic way. It is *not* only the connection between the learning and the price dynamics which determines the learning rate. Especially, varying the money supply model influences the dependencies of the coupling in a qualitative way. This result might be considered in two directions. Firstly, if the agents are assumed to learn, *e. g.* , faster when the price and learning dynamics are tightly connected, then we could by an obvious procedure determine which money supply models are reasonable by studying the covariance functions $P = P(\lambda)$ which they give. Without any economic interpretation of why these money supply models were chosen, this is probably an unsatisfactory approach. Secondly, the result can be added to the list of criticism to the literature discussing agents' convergence to rationality. The following statement is cited from Frydman and Phelps (1983): "The critical fault of the [rational expectation] hypothesis is not its unrealism but rather its lack of robustness." For further critical reviews of the rational expectation hypothesis see Frydman and Phelps (1983) and Shiller (1978).

Above the agents were assumed to be learning in a least-squares sense. If the agents' learning is modeled by another algorithm, the behavior of the system is different. Hence, the results shown are dependent on the combination of the price dynamics and the agents' learning.

If a more complex model than in the example above is studied, explicit expressions, as these between the coupling and the money supply model, should not be expected. Instead only numerical examples for certain parameter settings can be derived. Of course, these do not give as much understanding of a model as the expressions above gave about the relationship between the money supply, the coupling, and the agents' learning.

4 Conclusions

In this paper we have discussed agents' learning on a market. The market is such that the price level depends on the expectation of the agents. A quite general set-up based upon a first-order multivariable AR process was used for the price dynamics, and the agents learned according to a least-squares learning process. In an example we showed how the classical hyperinflation model could be rewritten into this form.

For the set-up described above a theorem concerning the convergence rate of the agents' learning was stated. It declared that considering the learning, the models can be divided into two classes. In the first class the convergence speed was the ordinary $1/\sqrt{t}$, while in the second class the convergence was slower. The theorem was applied to the hyperinflation model to show a relation between the coupling price–learning dynamics and the agents' learning. The relation was heavily depending on the model of the money supply, and for some models the relation was nonintuitive.

A Assumptions

We recall the modified least-squares algorithm and the assumptions given in Marcet and Sargent (1989*b*). The modification of the least-squares algorithm is done for technical reasons to assure convergence.

Define a set D_s that consists of the parameters θ that make the AR process (7) stable, *i. e.*

$$D_s = \{\theta \mid |\mu_i(T(\theta))| < 1 \ \forall i\}$$

Also, define the open and bounded set D_1 and the closed set D_2 by the two relations $D_2 \subset D_1$ and

$$(\theta, R) \in D_1 \quad \Rightarrow \quad \theta \in D_S$$

The modified learning algorithm is given by

$$\begin{aligned} \hat{\theta}_t &= \theta_{t-1} + \frac{1}{t} R_t^{-1} \varphi_t (y_t - \varphi_t^T \theta_{t-1}) \\ \hat{R}_t &= R_{t-1} + \frac{1}{t}(\varphi_t \varphi_t^T - R_{t-1}) \\ (\theta_t, R_t) &= \begin{cases} (\hat{\theta}_t, \hat{R}_t), & (\hat{\theta}_t, \hat{R}_t) \in D_2 \\ (\theta_{t-1}, R_{t-1}), & (\hat{\theta}_t, \hat{R}_t) \notin D_2 \end{cases} \end{aligned} \tag{19}$$

This modification assures us that the estimates always stay in the set D_1.

The following assumptions are assumed to hold.

1. The equilibrium point θ_f is unique and belongs to D_S.

2. Each element in $T(\theta)$ is two times differentiable and each element in $V(\theta)$ is one time differentiable for all $\theta \in D_S$.

3. M which is defined similar to (13) in Section 3 has full rank.

4. For $\{e_t\}$ in (7) it is true that $E\{|e_t|^p\} < \infty$ for all $p > 1$.

5. There exists a subset Ω_0 of the sample space such that $\Pr\{\Omega_0\} = 1$. There also exists two random variables $F_1(\omega)$ and $F_2(\omega)$ and a subsequence $\{t_k\}$ such that

$$\begin{aligned} |z(t_k)| &< F_1(\omega) \\ |R(t_k)| &< F_2(\omega) \end{aligned}$$

for all $\omega \in \Omega_0$ and $k = 1, 2, \ldots$.

6. The trajectories of the associated differential equations (10) with initial conditions $(\theta_0, R_0) \in D_2$ do not leave D_1.

B Proof of Theorem 1

In this appendix we prove the second part of Theorem 1 in Section 3.

We need the definitions for almost sure convergence, convergence in probability, convergence in quadratic mean, and weak convergence. Recall

DEFINITION 1
$\{\theta_t\}$ converges *almost surely* to θ_f if $\forall\varepsilon > 0$ and $\forall\delta > 0$, $\exists N(\varepsilon,\delta)$:

$$\Pr\{||\theta_t - \theta_f|| > \varepsilon, t \geq m\} < \delta, \quad \forall m \geq N(\varepsilon,\delta) \tag{20}$$

where $||\cdot||$ is the Euclidean norm.

$\{\theta_t\}$ converges *in probability* to θ_f if $\forall\varepsilon > 0$ and $\forall\delta > 0$, $\exists N(\varepsilon,\delta)$ such that if $t \geq N(\varepsilon,\delta)$

$$\Pr\{||\theta_t - \theta_f|| > \varepsilon\} < \delta$$

$\{\theta_t\}$ converges *in quadratic mean* to θ_f if

$$E\{||\theta_t - \theta_f||^2\} \to 0, \qquad t \to \infty$$

$\{\theta_t\}$ converges *weakly* to θ_f if the associated sequence of probability functions f_{θ_t} converges weakly to f_{θ_f}, *i. e.* in all continuity points of f_{θ_f}.

We use the notations

$$\theta_t \xrightarrow{a.s.} \theta_f \qquad \theta_t \xrightarrow{P} \theta_f \qquad \theta_t \xrightarrow{q.m.} \theta_f \qquad \theta_t \xrightarrow{w} \theta_f$$

□

Recall the learning algorithm

$$\theta_t = \theta_{t-1} + \frac{1}{t}H(\theta_{t-1}, z_t) = \theta_{t-1} + \frac{1}{t}h(\theta_{t-1}) + \epsilon_t \tag{21}$$

where θ_0 is given, and H and h are defined as in Section 3. Bounds for ϵ_t are extensively discussed in Benveniste *et al.* (1990) (Part II, Section 1.3). Denote as earlier

$$\alpha = \max_i \text{Re}\ \{\mu_i(h_\theta(\theta_f))\}$$

We are going to show that if $\alpha \in (-1/2, 0)$ then

$$t^\gamma(\theta_t - \theta_f) \xrightarrow{P} 0, \qquad t \to \infty$$

where γ is arbitrary in the range $(0, -\alpha)$.

We call a matrix A stable if all real parts of its eigenvalues are less than zero. Recall the following well-known result due to Lyapunov.

LEMMA 1
Assume A is a stable matrix. Then for every positive definite matrix Q there exists a symmetric positive definite matrix P, such that

$$A^T P + PA = -Q$$

□

Given two vectors x and y introduce the inner product

$$\langle x, y \rangle := x^T y$$

Define a second inner product from the first and the matrix P in Lemma 1.

$$[x, y] := \langle Px, y \rangle \tag{22}$$

We then have the following lemma.

LEMMA 2
Assume the matrices in Lemma 1 exist, then for all x

$$[Ax, x] < 0$$

□

Proof: From Lemma 1 we have

$$\langle A^T Px, x \rangle + \langle PAx, x \rangle = -\langle Qx, x \rangle$$

Since

$$\langle A^T Px, x \rangle = \langle Px, Ax \rangle = [x, Ax] = [Ax, x]$$

we conclude that

$$2[Ax, x] = -\langle Qx, x \rangle < 0$$

□

Note that $h_\theta(\theta_f)$ is a stable matrix, and that $h_\theta(\theta_f) + \gamma I$ is also stable for all $\gamma \in (0, -\alpha)$. Let $A = h_\theta(\theta_f) + \gamma I$ in Lemma 1 and use the corresponding P to define the inner product $[\cdot, \cdot]$ we will work with. Lemma 2 gives that there exists ε such that

$$[h_\theta(\theta_f)(\theta - \theta_f), (\theta - \theta_f)] < -\gamma[(\theta - \theta_f), (\theta - \theta_f)], \qquad \|\theta - \theta_f\| \leq \varepsilon \tag{23}$$

Given this ε, introduce the function

$$\tilde{h}(\theta) = \begin{cases} h(\theta), & \|\theta - \theta_f\| \leq \varepsilon \\ h_\theta(\theta_f)(\theta - \theta_f), & \|\theta - \theta_f\| > \varepsilon \end{cases} \tag{24}$$

Consider the stochastic process $\{\tilde{\theta}_t\}$

$$\tilde{\theta}_t = \tilde{\theta}_{t-1} + \frac{1}{t}\tilde{H}(\tilde{\theta}_{t-1}, z_t) = \tilde{\theta}_{t-1} + \frac{1}{t}\tilde{h}(\tilde{\theta}_{t-1}) + \epsilon_t \tag{25}$$

where $\tilde{\theta}_0 = \theta_0$. We are not interested in $\tilde{H}$. If $\tilde{\theta}$ is close to θ_f, this algorithm is the same as the original one, and otherwise it is an approximation. From the assumptions we know that $\theta_t \xrightarrow{a.s.} \theta_f$. We will show that $\tilde{\theta} \xrightarrow{P} \theta_f$. Then, for sufficiently large t given by the definitions of almost sure convergence and in probability, (25) will act like (21) with probability at least $1 - \delta$. The constant δ can be chosen arbitrarily small. Note that the algorithm (25) is dependent on both ε and δ.

The rest of the proof is outlined as follows. We show that $\{\tilde{\theta}_t\}$ converges in quadratic mean. This implies that $\{\tilde{\theta}_t\}$ converges in probability. This fact is then used for showing convergence in probability for $\{\theta_t\}$.

To show convergence in quadratic mean, we introduce the conditional variance

$$\Delta_t = E\{\|\tilde{\theta}_t - \theta_f\|^2 \mid F_{t-1}\}$$

where F_{t-1} is the σ-field generated by the sequence $R_0, \theta_0, z_0, z_1, \ldots, z_{t-1}$. Let us refer to the introductory part of the proof of Theorem 24 (p. 246) in Benveniste *et al.* (1990) for our learning algorithm. If the assumptions in Appendix A hold, we can state an inequality similar to (1.10.16) in Benveniste *et al.* (1990) by using (23):

$$\Delta_t \leq (1 - \frac{2\gamma}{t})\Delta_{t-1} + \frac{C_1}{t^2} + \frac{2}{t}E\{f_{t-1} - f_t\} \tag{26}$$

where the norm used in the definition of Δ_t is the one defined above. Throughout the appendix C_i denotes positive constants. The stochastic variable $\{f_t\}$ is defined in Benveniste *et al.* (1990), where it also is shown that

$$E\{|f_t|\} \leq C_2 \tag{27}$$

By iterating (26), we get

$$\begin{aligned}
\Delta_t &\leq \Delta_0 \prod_{i=1}^{t}(1 - \frac{2\gamma}{i}) + C_1 \sum_{i=1}^{t} \frac{1}{i^2} \prod_{j=i+1}^{t} (1 - \frac{2\gamma}{j}) \\
&\quad + 2\sum_{i=1}^{t} \frac{1}{i} E\{f_{i-1} - f_i\} \prod_{j=i+1}^{t} (1 - \frac{2\gamma}{j}) \\
&=: T_1 + T_2 + T_3 \leq |T_1| + |T_2| + |T_3|
\end{aligned}$$

Above as well as below, we follow the convention that $\prod_{j=i}^{t}(\cdot) = 1$ if $i > t$. We treat the three terms T_1, T_2, and T_3 separately. Before we estimate them we recall some inequalities: For large t we have

$$\prod_{i=1}^{t}(1 - \frac{2\gamma}{i}) \leq \exp\Big(- 2\gamma \sum_{i=1}^{t} \frac{1}{i}\Big) \leq \exp(-2\gamma \ln t + C_3) = C_4 t^{-2\gamma} \tag{28}$$

which gives

$$\sum_{i=1}^{t} \frac{1}{i^2} \prod_{j=i+1}^{t} (1 - \frac{2\gamma}{j}) \leq \sum_{i=1}^{t} \frac{C_5}{i^2} \cdot \frac{t^{-2\gamma}}{i^{-2\gamma}} = C_5 t^{-2\gamma} \sum_{i=1}^{t} i^{2\gamma-2} \tag{29}$$

and for $\beta < 0$

$$\sum_{i=1}^{t} i^{\beta} \leq C_6 \int_1^t i^{\beta} di \leq C_6 \frac{t^{\beta+1}}{\beta+1} \tag{30}$$

Using (28), an upper bound for T_1 is obtained as

$$T_1 \leq \Delta_0 C_4 t^{-2\gamma}$$

Since $\gamma \in (0, 1/2)$, we have $\beta := 2\gamma - 2 < 0$. Thus, (29) and (30) give

$$T_2 \leq C_7 t^{-1}$$

The term T_3 needs more detailed investigation. Firstly, note that

$$\begin{aligned}
&\sum_{i=1}^{t} \frac{1}{i}[f_{i-1} - f_i] \prod_{j=i+1}^{t} (1 - \frac{2\gamma}{j}) \\
&= \sum_{i=1}^{t} \frac{1}{i}[f_{i-1} - f_i](1 - \frac{2\gamma}{i+1}) \prod_{j=i+2}^{t} (1 - \frac{2\gamma}{j}) \\
&= (f_0 - f_1)(1-\gamma) \prod_{j=3}^{t} (1 - \frac{2\gamma}{j}) - \sum_{i=2}^{t} \frac{f_{i-1}}{i(i-1)} \prod_{j=i+2}^{t} (1 - \frac{2\gamma}{j}) \\
&+ \sum_{i=2}^{t} \left(\frac{f_{i-1}}{i-1} - \frac{f_i}{i} \right) \prod_{j=i+2}^{t} (1 - \frac{2\gamma}{j}) \\
&- 2\gamma \sum_{i=1}^{t} \left(\frac{f_{i-1}}{i(i+1)} - \frac{f_i}{i(i+1)} \right) \prod_{j=i+2}^{t} (1 - \frac{2\gamma}{j}) \\
&=: S_1 + S_2 + S_3 + S_4
\end{aligned}$$

S_1 can be treated like T_1:

$$|S_1| \leq C_8 \prod_{j+3}^{t} (1 - \frac{2\gamma}{j}) \leq C_9 t^{-2\gamma}$$

By using (27) we get

$$\begin{aligned}
E\{|S_2|\} &= E\left\{ \left| \sum_{i=2}^{t} \frac{f_{i-1}}{i(i-1)} \prod_{j=i+2}^{t} (1 - \frac{2\gamma}{j}) \right| \right\} \\
&\leq \sum_{i=1}^{t} \frac{C_{10}}{i^2} \prod_{j=i+1}^{t} (1 - \frac{2\gamma}{j}) \leq C_{11} t^{-1}
\end{aligned}$$

Further,

$$E\{|S_3|\} = E\left\{ \left| \sum_{i=2}^{t} \frac{f_{i-1}}{i-1} \prod_{j=i+2}^{t} (1 - \frac{2\gamma}{j}) - \sum_{i=2}^{t} \frac{f_i}{i} \prod_{j=i+2}^{t} (1 - \frac{2\gamma}{j}) \right| \right\}$$

$$\leq f_1 \prod_{j=4}^{t}(1-\frac{2\gamma}{j}) + E\Big\{\Big|\sum_{i=2}^{t}\frac{f_i}{i}\prod_{j=i+3}^{t}(1-\frac{2\gamma}{j}) - \sum_{i=2}^{t}\frac{f_i}{i}(1-\frac{2\gamma}{i+2})\prod_{j=i+3}^{t}(1-\frac{2\gamma}{j})\Big|\Big\}$$

thus,

$$\begin{aligned} E\{|S_3|\} &\leq |f(1)|\prod_{j=4}^{t}(1-\frac{2\gamma}{j}) + C_{12}\sum_{i=2}^{t}\frac{1}{i^2}\prod_{j=i+3}^{t}(1-\frac{2\gamma}{j}) \\ &\leq C_{13}t^{-2\gamma} + C_{14}t^{-1} \end{aligned}$$

The term S_4 is treated in the same way as S_2

$$\begin{aligned} E\{|S_4|\} &\leq E\Big\{\Big|\sum_{i=1}^{t}\frac{f_{i-1}}{i(i+1)}\prod_{j=i+2}^{t}(1-\frac{2\gamma}{j})\Big| + \Big|\sum_{i=1}^{t}\frac{f_i}{i(i+1)}\prod_{j=i+2}^{t}(1-\frac{2\gamma}{j})\Big|\Big\} \\ &\leq C_{15}\sum_{i=1}^{t}\frac{1}{i^2}\prod_{j=i+2}^{t}(1-\frac{2\gamma}{j}) \leq C_{16}t^{-1} \end{aligned}$$

Hence,

$$|T_3| \leq |S_1| + |S_2| + |S_3| + |S_4| \leq C_{17}t^{-1} + C_{18}t^{-2\gamma}$$

To conclude, we have shown that

$$\Delta_t \leq T_1 + T_2 + T_3 \leq C_{19}t^{-2\gamma} + C_{20}t^{-1} \tag{31}$$

For large t, the first term of the right hand side in (31) dominates. Thus,

$$\lim_{t\to\infty}\sup t^{2\gamma}\Delta_t \leq C_{19}$$

which is equal to say that $\{\tilde{\theta}_t\}$ converges in quadratic mean.

We now finally show that the convergence in quadratic mean of $\{\tilde{\theta}_t\}$ implies convergence in probability of $\{\theta_t\}$. Thus for sufficiently large t the algorithm (25) will act like (21) with probability at least $1-\delta$. Firstly,

$$t^{\gamma}(\tilde{\theta}_t - \theta_f) \xrightarrow{q.m.} 0, \quad t\to\infty$$

implies

$$t^{\gamma}(\tilde{\theta}_t - \theta_f) \xrightarrow{P} 0, \quad t\to\infty$$

or equivalently that $\forall\sigma > 0$ and $\forall\mu > 0$, $\exists M(\sigma,\mu)$ such that

$$\Pr\{\|t^{\gamma}(\tilde{\theta}_t - \theta_f)\| > \sigma\} < \mu/2, \quad t \geq M(\sigma,\mu)$$

For $t \geq \max\{N(\varepsilon, \delta), M(\sigma, \mu)\}$ we have

$$\begin{aligned}\Pr\{\|t^\gamma(\theta_t - \theta_f)\| > \sigma\} &= \Pr\{\|t^\gamma(\theta_t - \theta_f)\| > \sigma,\ \theta_t = \tilde{\theta}_t\} \\ &\quad + \Pr\{\|t^\gamma(\theta_t - \theta_f)\| > \sigma,\ \theta_t \neq \tilde{\theta}_t\} \\ &\leq \Pr\{\|t^\gamma(\theta_t - \theta_f)\| > \sigma,\ \theta_t = \tilde{\theta}_t\} + \Pr\{\theta_t \neq \tilde{\theta}_t\} \\ &\leq \Pr\{\|t^\gamma(\tilde{\theta}_t - \theta_f)\| > \sigma\} + \delta = \mu/2 + \delta \leq \mu\end{aligned}$$

where the last inequality follows if we in the definition of almost sure convergence choose $\delta \leq \mu/2$. Hence, we have shown that

$$t^\gamma(\theta_t - \theta_f) \xrightarrow{P} 0, \quad t \to \infty$$

where $\gamma \in (0, -\alpha)$ is arbitrary. Notice that since $(0, -\alpha)$ is an open interval there exist no largest γ.

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