Working Paper

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Control Reconstruction for
Nonlinear Parabolic Equations

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Introduction.

The problem of reconstruction of a control for nonlinear parabolic equations is considered. Regularized solution algorithms stable with respect to informational and computational disturbances are constructed. The suggested algorithms are guided from the theory of closed-loop control [1,2]. Inaccurate measurements of current phase states are the inputs, and approximations to the real controls are the outputs for the algorithms. The algorithms are operating in real time. Estimations for convergence rate are provided. The constructions are based on the approach of [8–12]. The problem belongs to the class of inverse problems of dynamics that are being inversely studied today (see, for example, investigations [3,6,7], where the corresponding bibliography is given). These problems considered as stationary (operator) ones were deeply investigated by many authors. In the present paper solutions of control reconstruction problems are built with the help of the method of feedback control with a model from the theory of differential games [1,2] and the dynamical discrepancy method [4,5]. For other dynamical control reconstruction algorithms based on the principle of feedback for distributed-parameter systems see [13–18].

1. Control with a Model.

Let us first consider the problem of reconstruction of a control for a parabolic equation in a Hilbert space $(H, | \cdot |)$, $H = H^*$,

$$\dot{x}(t) + Ax(t) \ni Bu(t) + f(t), \tag{1.1}$$

$$t \in T, \quad x(t_0) = x_0 \in \overline{D(A)},$$

where $T = [t_0, \vartheta]$ is the time interval, and $A : D(A) \subset H \to 2^H$ is the m-accretive operator, i.e. the operator wich satisfies the following conditions:

a) for any $x_j \in D(A)$ and $y_i \in Ax_j$, $j = 1, 2$

$$(x_1 - x_2, y_1 - y_2) \geq 0,$$

b) $R(I + \lambda A) = H \ \forall \lambda > 0$.
We also assume the following notations:

$(\cdot, \cdot)$ — the scalar product in H;
$D(A)$ — the domain of A;
$\overline{D(A)}$ — the closure of $D(A)$;
$f(\cdot) \in L_2(T; H)$ — a given disturbance;

$B \in L(U; H)$ — a linear continuous operator;

$(U, | \cdot |_U)$ — an uniformly convex real Banach space.

Definition 1.1. [19,20] A function

$$t \to x(t) = x(t; t_0, x_0, u(\cdot)) : T \to \overline{D(A)}, \quad x(t_0) = x_0,$$

is called an integral solution of inclusion (1.1) on T if

a) $x(\cdot) \in C(T; H)$, $x(t) \in \overline{D(A)}$ for each $t \in T$,

b) $x(\cdot)$ satisfies the inequality:

$$|x(t) - x|^2 \le |x(s) - x|^2 +$$

$$+ 2 \int_s^t (x(\tau) - x, Bu(\tau) + f(\tau) - y)d\tau \tag{1.2}$$

$$\forall x \in D(A), \quad y \in Ax, \quad t_0 \le s \le t \le \vartheta.$$

For any $u(\cdot) \in L_2(T; U)$, $f(\cdot) \in L_2(T; H)$ there exists a unique integral solution of (1.1) (Theorem of Benilan, see Lakshmikantham and Leela, [20, Theorem 3.5.1, p. 104]).

Along with the **general** case, where A is an arbitrary multivalued m-accretive operator and the set Ax is convex for any $x \in D(A)$, consider the **special** case with the operator A of the form:

$$Ax = A_* x \quad \text{for} \quad x \in D(A) = \{x \in V : A_* x \in H\}. \tag{1.3}$$

Here $A_* : V \to V^*$ is the single-valued operator such that:

1) A_* is Lipschitz, semicontinuous and monotone; $A_* 0 = 0$;

2) $\langle A_* x - A_* y, \, x - y \rangle \ge \omega \|x - y\|^2$ for all $x, y \in V$, where $\omega > 0$ is a fixed constant;

3) $\|A_* x\|_{V^*} \le C(1 + \|x\|)$ for all $x \in V$, where C does not depend on x;

hereafter $(V, \| \cdot \|)$ is a real separable and reflexive Banach space such that $V \subset H$ and the inclusion mapping of V into H is continuous, and $\langle \cdot, \cdot \rangle$ is the duality between V and V^*.

In this (special) case we assume $x_0 \in H$. Then there exists a unique strong solution $x(\cdot) = x(\cdot; t_0, x_0, u(\cdot))$ of (1.1), defined as the function which satisfies (1.1) and such that $x(\cdot) \in L_2(T; V)$, $\dot{x}_t(\cdot) \in L_2(T; V^*)$ (see Lions [21, Theorem 1.2, p.173]).

Notice that the operator A of the form (1.3) is m-accretive (see Vrabie [22, Theorem 6.3]).

An example of an operator A_* is $A_* : V = H_0^1(\Omega) \to V^* = H^{-1}(\Omega)$ of the form:

$$(A_* x)(\eta) = (A_1 x)(\eta) + \beta(x(\eta)) \quad \text{for a.a.} \quad \eta \in \Omega.$$

Here $\Omega \subset \mathbf{R}^h$ is a bounded simply connected domain with the sufficiently smooth boundary; $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ are standard Sobolev spaces [21, Lions]; β is the gradient of the convex differentiable function $j : \mathbf{R} \to \mathbf{R}^+ = \{r \in \mathbf{R} : r \ge 0\}$, $A_1 : V \to V^*$,

$$\langle A_1 y, z \rangle = \int_\Omega \nabla y(\eta) \nabla z(\eta) d\eta \quad \forall y, z \in V.$$

If the function $r \to \beta(r)$ is Lipschitz and $\beta(0) = 0$, then conditions 1) – 3) hold.

Denote: X_T — the bundle of solution on (1.1), i.e.

$$X_T = \{x(\cdot; t_0, x_0, u(\cdot)) : u(\cdot) \in U_T\},$$

$$\beta(h) = \begin{cases} h, & \text{in the special case} \\ h + \delta(h)(1 + O(\Delta_h)), & \text{in the general case.} \end{cases}$$

Here

$$O(\Delta_h) = \sup\{ \sum_{i=0}^{m_h-1} O_i(\Delta_h; x(\cdot)) : x(\cdot) \in X_T \},$$

$$O_i(\Delta_h; x(\cdot)) = \sup\{ |x(\tau_{h,i}) - x(\tau)| : \tau \in [\tau_{h,i}, \tau_{h,i+1}) \}.$$

The problem in question can be explained as follows. The integral solution $x_r(\cdot) = x(\cdot, t_0, x_0, u_r(\cdot))$ of the system (1.1) (strong solution in special case) depends on a time-varying unknown control

$$u_r(\cdot) \in U_T = \{ u(\cdot) \in L_2(T; U) : u(t) \in P \text{ for a.a. } t \in T \},$$

where $P \subset U$ is a convex, closed and bounded set. The interval T is put into parts by intervals $[\tau_i, \tau_{i+1})$, $i \in [0 : m - 1]$, $\tau_{i+1} = \tau_i + \delta$, $\delta > 0$, $\tau_0 = t_0$, $\tau_m = \vartheta$. At time instants $\tau_i \in \Delta = \{\tau_i\}_{i=0}^{m}$ the $x_r(\tau_i)$ are measured approximately, i.e. the elements $\xi_{h,i} \in H$ close $x_r(\tau_i)$ are found:

$$|x_r(\tau_i) - \xi_{h,i}| \leq h.$$

Here h is the level of the informational noise. A solution $x_r(\cdot)$ is unknown. Let $U_*(x_r(\cdot))$ be the set of all controls with values in U_T, generating the $x_r(\cdot)$. The problem is to calculate an approximation to an element $u_*(\cdot) = u(\cdot; x_r(\cdot)) \in U_*(x_r(\cdot))$ syncronically with the motion process, basing on inaccurate measurements of $x_r(\tau_i)$.

To calculate approximately $u_*(\cdot)$, we use the method of closed-loop control with a model [1, 2, 8–18]. According to this approach, the problem of reconstruction of an unknown control through the measurement results $\xi_{h,i}$ is substituted by the problem of positional control for an auxiliary system M (a model). Therefore, the problem of reconstruction of $u_*(\cdot)$ is subdivided into:

  i) the problem of selecting a model (functioning synchronically with the real system),

  ii) the problem of guiding a model through a positional control algorithm.

In [15] it was noticed that for some cases of parabolic systems a copy of the real system can serve as a model. Examples of such concrete systems were provided. We take M as a copy of the real system (1.1) for the system equations with the m-accretive operator. This copy is:

$$\dot{w}(t) + Aw(t) \ni Bv^h(t) + f(t), \qquad t \in T, \tag{1.4}$$

$$x(t_0) = x_0.$$

Describe the above mentioned algorithm.

First, a family $\Delta_h = \{\tau_{h,i}\}_{i=0}^{m_h}$, $\tau_{h,0} = t_0$, $\tau_{h,m_h} = \vartheta$, $\tau_{h,i} = \tau_{h,i-1} - \delta(h)$ of partitions of the interval T with diameters $\delta(h)$, and a function $\alpha(h)$ are fixed. The functions $\delta(h)$ and $\alpha(h)$ are chosen so as:

$$\delta(h) \to 0, \qquad \alpha(h) \to 0, \qquad \beta(h)/\alpha(h) \to 0 \text{ as } h \to 0.$$

Before the initial time of the process, values h, $\alpha(h)$ and a partition $\Delta = \{\Delta_h\}_{i=0}^{m}$, $m = m_h$ are fixed. The work of the algorithm staring at time t_0 is decomposed into m-1

identical steps. At the i-th step run during the time interval $\delta_i = [\tau_i, \tau_{i+1})$, $\tau_i = \tau_{h,i}$, the following operations are carried out. At time τ_i a control

$$v^h(t) = v_i^h \qquad \text{for a.a. } t \in \delta_i,$$

$$v_i^h = \arg\min\{l_\alpha(s_i, v) : v \in P\}, \tag{1.5}$$

$$l_\alpha(s_i, v) = 2(s_i, v) + \alpha(h)|v|_U^2, \qquad s_i = w(\tau_i) - \xi_{h,i}$$

is calculated. A control $v^h(t)$ is a piece-wise constant function formed by the feedback:

$$v_i^h = v_i^h(w(\tau_i), \xi_{h,i}).$$

Finally, the state $w(\tau_i)$ of the model (1.4) is transformed into $w(\tau_{i+1})$. The procedure stops at time ϑ.

Let $u_*(\cdot) = u_*(\cdot; x_r(\cdot))$, $x_r(\cdot) \in X_T$, be the element of the set $U_*(x_r(\cdot))$ whose $L_2(T; U)$-norm is minimal.

Theorem 1.1. Let $-A$ generate the nonexpansive compact $(\forall t > 0)$ semigroup $S(t)$. In the general case

$$v^h(\cdot) \to u_*(\cdot, x_r(\cdot)) \text{ in } L_2(T; U) \text{ as } h \to 0.$$

Proof. From the inequality (1.2), Benilan's [20, Theorems 3.6.1 and 3.5.1] and Vrabie's [23, Theorem 2] result follow the propositions:

a) For every $t_*, t^* \in T$, $t_* < t^*$, $x_* \in H$, $u(\cdot) \in U_{t_*,t^*} = \{u(\cdot) \in L_2([t_*, t^*]; U) : u(t) \in P$ for a.a. $t \in [t_*, t^*]\}$ the solution $x(\cdot; t_*, x_*, u(\cdot))$ is continuous.

b) (semigroup property) For every $t_* \in T$, $t^* \in (t_*, \vartheta]$, $u_{t_0,t_*}(\cdot) \in U_{t_0,t_*}$, $u_{t_*,t^*}(\cdot) \in U_{t_*,t^*}$ and $t \in [t_*, t^*]$ the equality

$$x(t; t_0, x_0, u_{t_0,t^*}(\cdot)) = x(t; t_*, x_*, u_{t_*,t^*}(\cdot))$$

is true, where $x_* = x(t_*; t_0, x_0, u_{t_0,t^*}(\cdot))$,

$$u_{t_0,t^*}(t) = \begin{cases} u_{t_0,t_*}(\cdot) \text{ for a.a. } \tau \in [t_0, t_*] \\ u_{t_*,t^*}(\cdot) \text{ for a.a. } \tau \in [t_*, t^*]. \end{cases}$$

c) For every $t_*, t^* \in T$, $t_* < t^*$, $x_1, x_2 \in H$, $u_1(\cdot), u_2(\cdot) \in U_{t_*,t^*}$ functions $\tau \to L(x_1(\tau), u_j(\tau)) : [t_*, t^*] \to \mathbf{R}$, $L(x, u) = 2(x, u)$, where $j = 1, 2$, $x_j(\cdot) = x(\cdot, t_*, x_j, u_j(\cdot))$ are summable and

$$|x_1(t^*) - x_2(t^*)|^2 \le |x_1(t_*) - x_2(t_*))|^2 +$$

$$+ \int\limits_{t_*}^{t^*} L(x_1(\tau) - x_2(\tau), u_1(\tau)) - u_2(\tau))d\tau,$$

d) The set $U_*(x_r(\cdot))$ is convex, bounded and closed in $L_2(T; U)$.

e) The bundle of motion X_T is uniformly bounded and semicontinuous in $C(T; H)$.

The proof of the theorem follows from the result of [15]. □

Consider the special case. Let the following condition hold.

Condition 1.1. Let $U = V$, V be a real Hilbert space, B be the canonical embedding of V into H, $X_b \subset X_T$ be the set of all solutions of the system (1.1) such that for any $x_r(\cdot) \in X_b$ the full variation of the control $u_*(\cdot) = u_*(\cdot; x_r(\cdot)) \in U_T$ generating $x_r(\cdot)$ is bounded by a $b \in (0, +\infty)$.

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Then the following theorem is true.

Theorem 1.2. In the special case, under Condition 1.1, the following estimation for the convergence rate is true:

$$\|v^h(\cdot) - u_*(\cdot; x_r(\cdot))\|^2_{L_2(T;U)} \leq$$

$$\leq k\{(h + \delta(h) + \alpha(h))^{1/2} + (h + \delta(h))/\alpha(h)\},$$

where $k = k(X_b) = \text{const} \in (0, +\infty)$.

The constant k is found explicitly. The proof of Theorem 1.2 follows from Lemma 1.1.

Lemma 1.1. Let Y be a Banach space, $v_1 \in L_\infty(T; Y^*)$, $v_2(t) \in Y$ $\forall t \in T$ be a function with the bounded variation,

$$|\int_0^t v_1(\tau)d\tau|_{Y^*} \leq \gamma, \ |v_2(t)|_Y \leq d \ \forall t \in T.$$

Then the bound

$$|\int_0^\vartheta \langle v_1(\tau), v_2(\tau) \rangle_Y d\tau| \leq \gamma(d + \text{var}_Y v_2(\cdot))$$

holds.

Here $\langle \cdot, \cdot \rangle_Y$ is the duality between Y and Y^*, $\text{var}_Y v_2(\cdot)$ stands for the full variation of a function $v_2(\cdot)$ on the time interval T.

Proof. The proof of the lemma is analogous to that of Lemma 1 [24]. Let $\varepsilon_k \to 0$ as $k \to \infty$,

$$r(t) = \int_0^t v_1(\tau)d\tau.$$

It holds

$$|\int_0^\vartheta \langle v_1(\tau), v_2(\tau) \rangle_Y d\tau| = \rho_k + \beta_k.$$

Here

$$\rho_k = \sum_{i=0}^{m_k-1} \int_{\tau_i^k}^{\tau_{i+1}^k} \langle v_1(\tau), v_2(\tau_i^k) \rangle_Y d\tau,$$

$$\beta_k = \sum_{i=0}^{m_k-1} \int_{\tau_i^k}^{\tau_{i+1}^k} \langle v_1(\tau), v_2(\tau) - v_2(\tau_i^k) \rangle_Y d\tau,$$

$\tau_i^k = \tau_{h_k,i}$, $m_k = m_{h_k}$. Due to the inequality $|v_1(\cdot)|_{L_\infty(T;Y^*)} < p_0 < \infty$, we have

$$|\beta_k| \leq \text{var}_y v_2(\cdot) p_0 \delta(h_k).$$

Note that

$$\rho_k = \sum_{i=0}^{m_k-1} \langle r(\tau_{i+1}^k) - r(\tau_i^k), v_2(\tau_i^k) \rangle_Y =$$

$$= \sum_{i=1}^{m_k-1} \langle r(\tau_i^k), v_2(\tau_{i-1}^k) - v_2(\tau_i^k) \rangle_Y + \langle r(\vartheta), v_2(\tau_{m_k-1}^k) \rangle_Y.$$

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Then

$$\left| \int_0^\vartheta \langle v_1(\tau), v_2(\tau) \rangle_Y d\tau \right| \le$$

$$\le \varlimsup_{k \to \infty} \{ \left| \sum_{i=1}^{m_k - 1} \langle r(\tau_i^k), v_2(\tau_{i-1}^k) - v_2(\tau_i^k) \rangle_Y + \right.$$

$$\left. + \langle r(\vartheta), v_2(\tau_{m_k-1}^k) \rangle_Y \right| + |\beta_k| \} \le$$

$$\le \varlimsup_{k \to \infty} \gamma \sum_{i=1}^{m_k - 1} |v_2(\tau_{i-1}^k) - v_2(\tau_i^k)|_Y + \gamma d \le$$

$$\gamma(d + \operatorname{var}_Y v_2(\cdot)).$$

the lemma is proved. $\square$

Proof of Theorem 1.2. It is easy to establish the relations:

$$\sup\{ |x(\cdot)|_{C(T;H)} + |x(\cdot)|_{L_2(T;V)} : x(\cdot) \in X_T \} \le k_1,$$

$$\sup\{ |w(\cdot)|_{C(T;H)} + |w(\cdot)|_{L_2(T;V)} : u(\cdot) \in U_T \} \le k_2,$$

$$\|x_r(t_1) - x_r(t_2)\|_{V^*} \le \int_{t_1}^{t_2} \{ k_3 \|x_r(t)\| + k_4 \} dt,$$

$$\|w(t_1) - w(t_2)\|_{V^*} \le \int_{t_1}^{t_2} \{ k_5 \|w(t)\| + k_6 \} dt,$$

$$w(\cdot) = w(\cdot; t_0, x_0, u(\cdot)), \quad u(\cdot) \in U_T,$$

$$\varepsilon_h(t) + \omega \int_{\tau_i}^t \|x_r(\tau) - w(\tau)\|^2 d\tau \le$$

$$\le k_7(t - \tau_i)\{ h + \Phi_i(\Delta_h, x_r(\cdot)) + \Phi_i(\Delta_h, w(\cdot)) \},$$

$$k_j \in (0, +\infty), \quad j \in [1:7].$$

Here

$$\Phi_i(\Delta_h; x_r(\cdot)) = \sup\{ \|x_r(\tau_{h,i}) - x_r(\tau)\|_{V^*} : \tau \in [\tau_{h,i}, \tau_{h,i+1}] \},$$

$$\Phi_i(\Delta_h; w(\cdot)) = \sup\{ \|w(\tau_{h,i}) - w(\tau)\|_{V^*} : \tau \in [\tau_{h,i}, \tau_{h,i+1}] \},$$

$$\varepsilon_h(t) = \sup_{\tau \in [t_0, t]} |x_r(\tau) - w(\tau)|^2 +$$

$$+ \alpha(h) \int_{t_0}^t \{ |v^h(\tau)|_U^2 - |u_*(\cdot; x_r(\cdot))|_U^2 \} d\tau.$$

Therefore

$$\varepsilon_h(t) + \omega \int_{t_0}^t \|x_r(\tau) - w(\tau)\|^2 d\tau \le k_8(h + \delta(h)), \quad t \in T.$$

Consequently, $\forall t_1, t_2 \in T, \ t_1 < t_2$

$$\left\| \int_{t_1}^{t_2} (u_*(t) - v^h(t)) dt \right\|_{V^*} \le k_9(h + \delta(h) + \alpha(h))^{1/2}, \tag{1.6}$$

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$$u_*(\cdot) = u_*(\cdot; x_\tau(\cdot)).$$

Note that Lemma 1.1 and inequality (1.6) imply the inequallity

$$|u_*(\cdot) - v^h(\cdot)|^2_{L_2(T;H)} \le$$

$$\le 2|u_*(\cdot)|^2_{L_2(T;H)} - 2\int_{t_0}^{\vartheta} (u_*(t), v^h(t))dt + k_{10}(h + \delta(h))/\alpha(h) \le$$

$$\le k\{h + \delta(h) + \alpha(h)^{1/2} + (h + \delta(h))/\alpha(h)\},$$

where $k = k(x_b)$. The theorem is proved. $\square$

If there exists a test function $t \to u_0(t) = u_0 \in U$, $t \in T$, and one has to compute an element from $U_*(x_\tau(\cdot))$ nearest in $L_2(T; U)$ to u_0, then in (1.5)

$$l_\alpha(s_i, v) = 2(s_i, v) + \alpha(h)|v - u_0|^2_U.$$

2. Dynamical Discrepancy Method.

In this section we consider a dynamical modification of the discrepancy method from the theory of ill-posed problems [4,5] and provide upper and lower bounds for its convergence rate. Denote: V and H — real Hilbert spaces, V^* and H^* — spaces dual to V and H respectively, $\langle \cdot, \cdot \rangle$ — the duality between V and V^*, $(\cdot, \cdot)$ — the scalar product in H, $\| \cdot \|$ and $|\cdot|$ — the norms in V and H respectively, $P \subset H$ — a convex, bounded and closed bundle. We identify spaces H and H^* and suppose that V is densely and continuously imbedded in H. Let Δ_h be a partition of the interval T with the diameter $\delta(h)$.

Consider a parabolic system whose evolution is described by

$$\dot{x}(t) + A_1 x(t) = u(t) + f(t), \qquad \text{for a.a. } t \in T, \qquad (2.1)$$

$$x(t_0) = x_0.$$

Here $A_1 = A + A_2$, $A : V \to V^*$ is a linear, continuous and symmetric operator satisfying the coercivity condition

$$\langle Ay, y \rangle + a|y|^2 \ge \omega \|y\|^2 \qquad \forall y \in V$$

for certain $\omega > 0$ and $a \in \mathbf{R}$; A_2 is the gradient of a convex differentiable function $\varphi : H \to \overline{\mathbf{R}} = [0, +\infty]$, $f(\cdot) \in W^{1,\infty}(T; H) = \{v(\cdot) \in L_2(T; H) : v_t(\cdot) \in L_\infty(T; H)\}$.

The problem considered is analogous to that described in Section 1. An unknown control $u_* = u_*(\cdot; x_\tau(\cdot)) \in U_T$ acts upon the system (2.1) and generates a motion $x_\tau(\cdot)$. At time instants $\tau_i = \tau_{h,i} \in \Delta_h$, the history $x_{\tau_{i-1}, \tau_i}(\cdot)$ of the motion is measured approximately, i.e. a piece-constant function $p_i^h = \xi_{\tau_{i-1}, \tau_i}^h(\cdot)$ being an approximation of $x_{\tau_{i-1}, \tau_i}(\cdot)$ is memorized:

$$|\xi^h(\tau_i) - x_\tau(\tau_i)| \le h, \qquad |\int_{\tau_{i-1}}^{\tau_i} A(x_\tau(t) - \xi^h(t))dt| \le h. \qquad (2.2)$$

An algorithm for computing an approximation to $u_*(\cdot)$ is to be found.

Assume the following

Condition 2.1. $x_0 \in D(A_H) = \{x \in V : Ax \in H\}$, $u_*(\cdot) \in U_T = \{v(\cdot) \in L_2(T;H) : v(t) \in P, \ |\dot{v}(t)| \leq K \ \text{for a.a.} \ t \in T\}$, $K < +\infty$, and the operator A_2 is Lipschitz.

By Theorem 4.3 [25] (see also [26]), for every $u(\cdot) \in U_T$ there exists a unique solution $x(\cdot) = x(\cdot; t_0, x_0, u(\cdot))$ of (2.1) such that $x(\cdot) \in W^{1,\infty}(T;H) \cap C(T;V)$, $t \mapsto Ax(t) \in L_2(T;H)$. Introduce the sets

$$V_{k,i}^{h,\delta}(p_i^h) = \{u \in P : |u - \psi(\tau_{i-1}, \tau_i; \xi_{\tau_{i-1},\tau_i}^h(\cdot))| \leq k(h\delta^{-1} + \delta)\},$$

$$i \in [1 : m_h],$$

$$\psi(\tau_{i-1}, \tau_i; \xi_{\tau_{i-1},\tau_i}^h(\cdot)) = (\xi_i^h - \xi_{i-1}^h)\delta^{-1} + \psi_i^h + A_2\xi_{i-1}^h - f(\tau_{i-1}),$$

where

$$\psi_i^h = \delta^{-1} \int_{\tau_{i-1}}^{\tau_i} A\xi^h(t)dt,$$

$$\xi_i^h = \xi^h(\tau_i), \quad \delta = \delta(h).$$

Let the partition Δ_h be such that

$$\delta(h) \to 0, \quad h\delta^{-1}(h) \to 0 \quad \text{as} \quad h \to 0.$$

Describe the desired algorithm. Before the initial time of the process, values h, k and the partition $\Delta = \Delta_h$ are fixed. The work of the algorithm starting at time t_0 is decomposed into m_h-1 steps. At the i-th step run out during a time interval $\delta_i = \delta_{h,i} = [\tau_{h,i}, \tau_{h,i+1})$, a control $v^h(t) = v^h(t; x_r(\cdot)) = v_i^h$ $(t \in \delta_{h,i})$, $i \geq 1$,

$$v_i^h = \begin{cases} \arg\min\{|u| : u \in V_{k,i}^{h,\delta}(p_i^h)\}, & \text{if} \quad V_{k,i}^{h,\delta}(p_i^h) \neq \emptyset \\ 0, & \text{in the opposite case,} \end{cases} \tag{2.3}$$

$$v_0^h = \arg\min\{|u| : u \in P\},$$

is calculated. The procedure stops at time ϑ.

Denote : X_T — the bundle of motion : $X_T = \{x(\cdot; t_0, x_0, u(\cdot)) : u(\cdot) \in U_T\}$; $u_*(\cdot; x(\cdot))$ — a control generating a motion $x(\cdot) \in X_T$; $\Xi(x_r(\cdot), h)$ — the set of all functions $\xi^h(\cdot) : T \to D(A_h)$ such that the inequalities (2.2) hold; $(\Xi(x_r(\cdot), h)$ — the set of all measurement results admissible for $x_r(\cdot))$;

$$\nu(h) = \sup\{\|v^h(\cdot; x(\cdot)) - u(\cdot; x(\cdot))\|_{L_2(T;H)}^2 :$$

$$x(\cdot) \in X_T, \quad \xi^h(\cdot) \in \Xi(x(\cdot), h)\}.$$

Theorem 2.1. There exists a $k_* \geq 0$ such that for every $k \in [k_*, +\infty)$

$$\|v^h(\cdot; x(\cdot)) - u_*(\cdot; x_r(\cdot))\|_{L_2(T;H)} \to 0 \quad \text{as} \quad h \to 0.$$

The constant k_* is found explicitly.

The proof of Theorem 2.1 is analogous to that of Theorem 1 [18].

So, if values h, $\delta(h)$ and $h\delta^{-1}(h)$ are "sufficiently small", then $v^h(\cdot) = v^h(\cdot; x_r(\cdot))$ is a "good" approximation to $u_*(\cdot) = u_*(\cdot; x(\cdot))$.

Theorem 2.2. Let $\mathrm{int} P \neq \emptyset$. Then for $h \to 0$ the following estimations for the convergence rate are true:

$$c_1(h\delta^{-1} + \delta)^2 \leq \nu(h) \leq c_2(h\delta^{-1} + \delta). \tag{2.4}$$

Note that some upper estimations for the convergence rate of dynamical algorithms of control reconstruction are obtained in [24] for systems described by ordinary differential equations, and in papers [14, 27] for systems described by partial differential equations.

Proof of Theorem 2.2. Let $x(\cdot) \in X_T$, $h(\cdot) \in (0,1)$, $\xi^h(\cdot) \in \Xi(x(\cdot), h)$. Using the local Lipschitz property of the mapping A_2, one can easily deduce the inequality

$$|\delta\psi(\tau_i, \tau_{i-1}, \xi^h_{\tau_{i-1}, \tau_i}(\cdot)) - \int_{\tau_{i-1}}^{\tau_i} \{\dot{x}(t) + Ax(t) + $$

$$+ A_2 x(t) - f(t)\}dt| \leq k_1(h + \delta^2). \tag{2.5}$$

Then for $k \geq k_1$, $i \in [1 : m-1]$, $m = m_h$

$$\delta^{-1} \int_{\tau_{i-1}}^{\tau_i} u(t; x(\cdot))dt \in V^{h,\delta}_{k,i}(p^h_i) \tag{2.6}$$

and, consequently, $V^{h,\delta}_{k,i}(p^h_i) \neq \emptyset$. Note that (2.3), (2.5) imply the inequality

$$|\int_0^t \{v^h(\tau; x(\cdot)) - u(\tau; x(\cdot))\}d\tau| \leq k_2(h\delta^{-1} + \delta). \tag{2.7}$$

Further, from (2.6) we have

$$|v^h(\cdot; x(\cdot))|_{L_2(T;H)} \leq |u(\cdot; x(\cdot))|_{L_2(T;H)}. \tag{2.8}$$

By (2.7) and (2.8) we deduce

$$|v^h(\cdot; x(\cdot)) - u(\cdot; x(\cdot))|^2_{L_2(T;H)} \leq $$

$$\leq 2 \int_0^\vartheta (u(\tau; x(\cdot)), u(\tau; x(\cdot)) - v^h(\tau; x(\cdot)))d\tau$$

This and Lemma 2.1 imply for $k \geq k_1$, $h \in (0,1)$

$$|v^h(\cdot; x(\cdot)) - u(\cdot; x(\cdot))|^2_{L_2(T;H)} \leq c_2(h\delta^{-1} + \delta).$$

The upper bound (2.4) is proved. Obtain the lower bound. Let $v_0 \in \mathrm{int} P$, $|v_0| \neq 0$, $\beta > 0$ be such that

$$S(v_0; 2\beta) = \{v \in H : |v - v_0| \leq 2\beta\} \subset P. \tag{2.9}$$

Note that

$$u(\cdot; x(\cdot)) = v_0(\cdot) \tag{2.10}$$

if $v_0(t) = v_0 \; \forall t \in T$, $x(\cdot) = x(\cdot; x_0, v_0(\cdot))$. In order to prove the lower bound (2.4), it is sufficient to show that for small h, $\delta(h)$, $h\delta^{-1}(h)$ we have

$$|v^h(\cdot; x(\cdot)) - u(\cdot; x(\cdot))|_{L_2(T;H)} \geq k_3(h\delta^{-1} + \delta) \tag{2.11}$$

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uniformly with respect to all Δ_h, $\xi^h(\cdot) \in \Xi(x(\cdot), h)$. Here the constant k_3 does not depend on h, $\delta(h)$. Further, we have

$$|\delta^{-1}(x_{i+1} - x_i) + \delta^{-1} \int_{\tau_i}^{\tau_{i+1}} Ax(t)dt + A_2 x_i - f_i - v_0| \leq k_4 \delta$$

$$(x_i = x(\tau_i), \quad f_i = f(\tau_i)).$$

Consequently,

$$|\psi(\tau_{i-1}, \tau_i, \xi^h_{\tau_{i-1}, \tau_i}(\cdot)) - v_0| \leq k_5(h/\delta + \delta).$$

Let $k \geq k_* = 2\max\{k_1, k_5\}$. Then from (2.9) we have

$$S(v_0; d_1) \subset V^{h,\delta}_{k,i}(p^h_i)$$

if $d_1 = \min\{\beta, k_5(h\delta^{-1} + \delta)\}$. Since v^h_i is the minimum-norm element from $V^{h,\delta}_{k,i}(p^h_i)$, so $v^h_i \notin S(v_0; d_1/2)$. Therefore

$$|v^h_i - v_0| \geq d_1/2 \geq k_5(h\delta^{-1} + \delta)/2 \tag{2.12}$$

$$\forall\, i \in [0 : m - 2],$$

if δ and $h\delta^{-1}$ are sufficiently small, i.e. $h \in (0, h_*)$, where h_* is such that $k_5(h\delta^{-1}(h) + \delta(h)) \leq \beta$ for $h \in (0, h_*)$. From (2.10), (2.12) we deduce (2.11). The theorem is proved.
□

Let $P = \{u \in U : |u| \leq m\}$, $m \in (0, +\infty)$. Then by virtue of the equality (2.3) the element v^h_i is determined by the formula [28, Example 1.4.1.] :

$$v^h_i = \begin{cases} \beta_i - d \cdot \beta_i/|\beta_i|, & \text{if } |\beta_i| \geq d, \\ 0, & \text{in the opposite case,} \end{cases} \tag{2.13}$$

where

$$d = k(h\delta^{-1} + \delta),$$
$$\beta_i = (\xi^h_i - \xi^h_{i-1})\delta^{-1} + \psi^h_i + A_2\xi^h_{i-1} - f(\tau_{i-1}).$$

3. Example.

In this example we illustrate the algorithm of Section 2. Let

$$\begin{aligned}
\Omega &= (0,1) \times (0,1), \\
T &= [0, 0.5], \\
P &= \{u \in H = L_2(\Omega) : |u(\eta)|_{L_\infty(\Omega)} \leq 10\},
\end{aligned}$$

$$\beta(r) = \begin{cases} 2.15(r - 0.2), & \text{if } r > 0.2, \\ 0, & \text{if } r \in [-0.2, 0.2] \\ 1.71(r + 0.2), & \text{if } r < -0.2, \end{cases}$$

$$\begin{aligned}
u(t, \eta) &= -40t \sin(10t)\nu(\eta), \\
\eta &= \{\eta_1, \eta_2\} \in \Omega, \\
n &= 2, \\
\nu(\eta) &= (1 - \eta_1)(1 - \eta_2)\eta_1\eta_2, \\
x_0(\eta) &= 0, \\
x(t, \eta) &= 40t\nu(\eta), \\
f(t, \eta) &= 80t((1 - \eta_1)(1 - \eta_2)\eta_1\eta_2 + (1 - \eta_2)\eta_2) + \\
&\quad + \beta(x(t, \eta)) + (1 - \sin(10t))\nu(\eta), \quad \eta \in \Omega.
\end{aligned}$$

For these data, an approximation $v^h(t, \eta)$ to $u(t, \eta)$ was computed by a net method with the usage of formula (2.3), (2.4). The domain Ω was replaced by the net with step $1/N$, $N = 15$. The space H and the sets $V_{k,i}^{h,\delta}(p_i^h(\cdot))$ were replaced by the net space H_N and the sets $V_{k,i,N}^{h,\delta}(p_i^h(\cdot))$ respectively :

$$H_N = \{\nu = \{\nu_{jr}\}_{i,r=1}^N \in \mathbf{R}^{N \times N} : \sum_{j,r=1}^N \nu_{jr}^2 jr < \infty\},$$

$$|\nu|_{H_N} = (\sum_{j,r=1}^N \nu_{jr}^2 jr N^{-2})^{1/2},$$

$$V_{k,i,N}^{h,\delta}(p_i^h(\cdot)) = \{\nu = \{\nu_{jr}\}_{i,r=1}^N \in \mathbf{R}^{N \times N} : |\nu_{jr}| \leq 10,$$

$$j, r \in [1 : N], |\psi^N(\tau_i, \tau_{i-1}, p_i^h(\cdot)) - \nu|_{H_N} \leq k(\delta^{1/2} + \varepsilon\delta^{-1})\},$$

$$\psi^N(\tau_i, \tau_{i-1}, p_i^h(\cdot))_{jr} = \delta^{-1}(\xi^h(\tau_i, \eta_{jr}) - \xi^h(\tau_{i-1}, \eta_{jr})) -$$

$$-\delta \sum_{j=1}^{i-1} \Delta\xi^h(\tau_{j-1}, \eta_{jr}) + \beta(\xi^h(\tau_i, \eta_{jr})) - f(\tau_i, \eta_{jr}),$$

$$\eta_{jr} = jr N^{-2}, \qquad j, r \in [1 : N].$$

Here Δ is the Laplace operator.

Figs. 1–5 show the cross-sections of the surfaces $u(t, \eta)$ and $v^h(t, \eta)$ for $\xi^h(t) = x(t) + ht$.
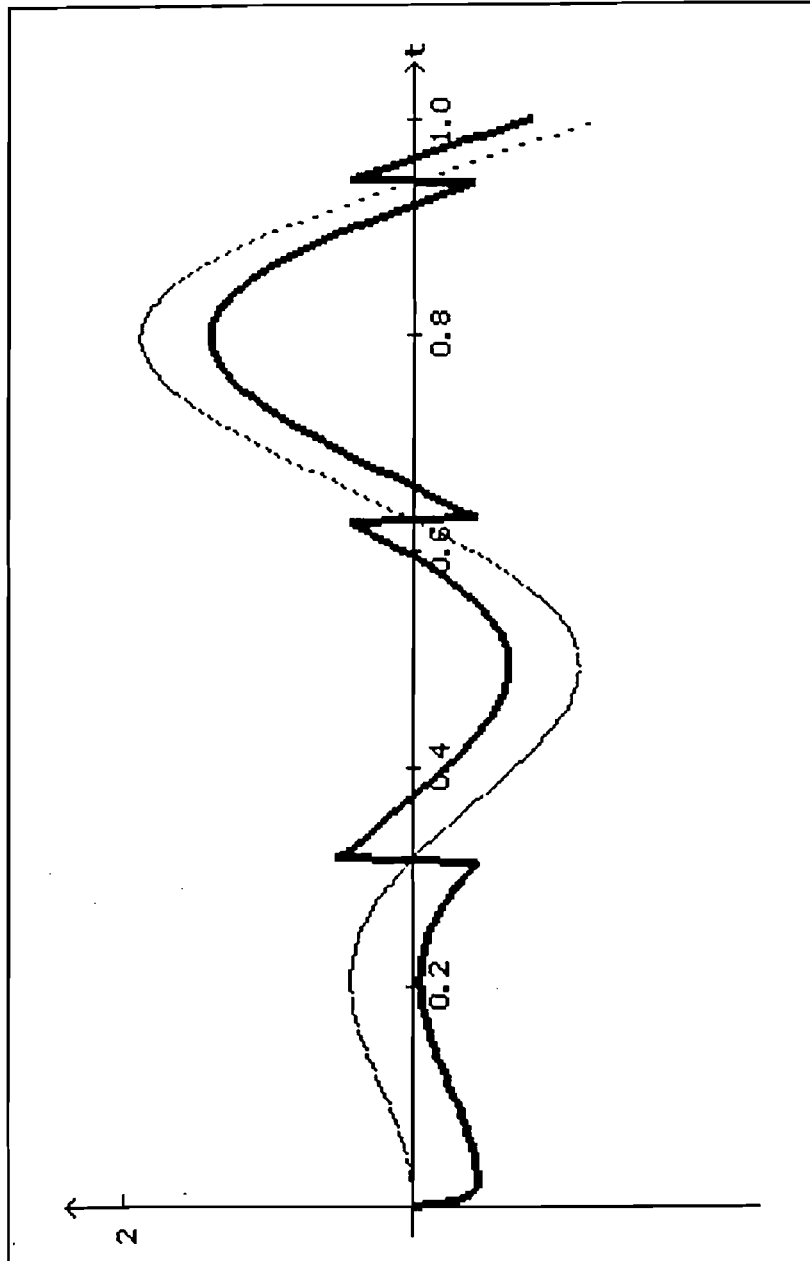
Structure of noise in the model : eps*t

Fig. 1
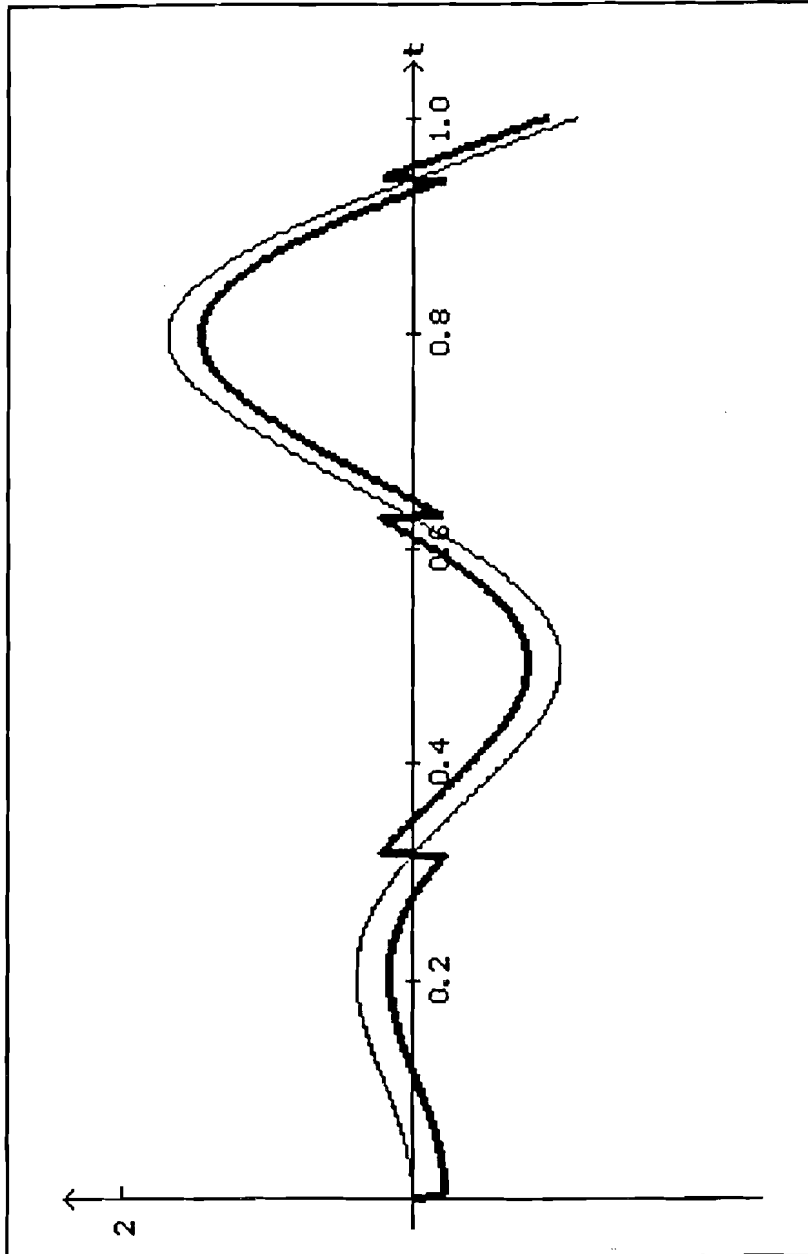
Structure of noise in the model : eps*t

Fig. 2

Fig. 3

Fig. 4
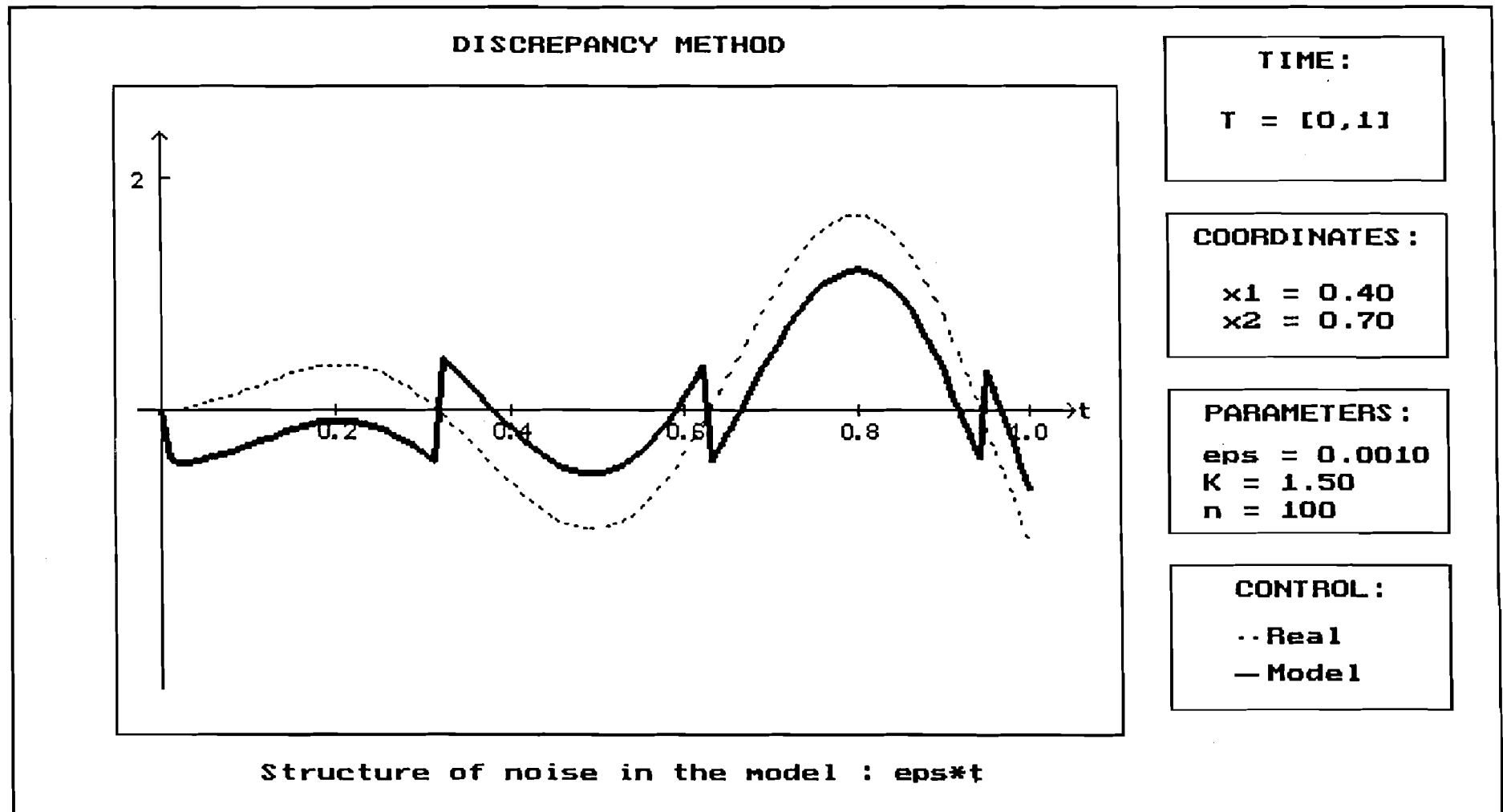
Fig. 5

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