EXPLORATIONS IN PARAMETER-SPACE

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To date most of the mental and graphic thinking that we have been doing has been with a PHASE-SPACE outlook. (Relative to what follows we would more accurately call it a STATE-SPACE.) In principle, if we knew precisely what was going on everywhere in state-space we would know all that we need to know. But since we don't we must draw our characteristic spirals and express the behaviour in qualitative terms. What I am proposing here is ~~not~~ to supplement (not replace) the state-space approach with another--PARAMETER-SPACE.

To me the difficulty with the phase-space outlook came in trying to invision what perturbations would do to system behaviour. Instantaneous changes in one or more of the state variables was obvious. Gradual changes in the state-variables could be imagined although the actual net result of such inputs is difficult to visualize. The effect of changes in parameters would require a set of phase plots for a large collection of parameter values.

Of course we could lump the parameters and driving variables together with our original state variables into a bigger system and look at trajectories in this new "super system". (Another viewpoint could have picked birth rate as a state variable and population size as a parameter.) But adding dimensions doesn't seem to be a good thing to do at this stage. So instead we can use the cross-section called state-space and another cross-section call parameter-space. The P-space would exibit the gross qualitative trajectory types. The "boundaries" that appear in the P-space are not the same as those we use in S-space. Nevertheless they are very real and easy to appreciate. As the parameters are more static than the state variables (by definition?) the P-space is more static and the "distance to a boundary" is easier to interpret.

An example of a simple one-dimensional P-space concerns fish exploitation(Silliman and Gutsell, 1958). As long a exploitation was below 60% of the adult population per generation the population presisted. When exploitation above 60% was imposed the population was wiped out.

Below are some examples of P-spaces (necessarily restricted to two or three dimensions) to show the types of qualitative information that can be shown. The first example (Stwart and Levin, 1973) concerns the coexistence of two species competing for a single resource in a seasonal environment.

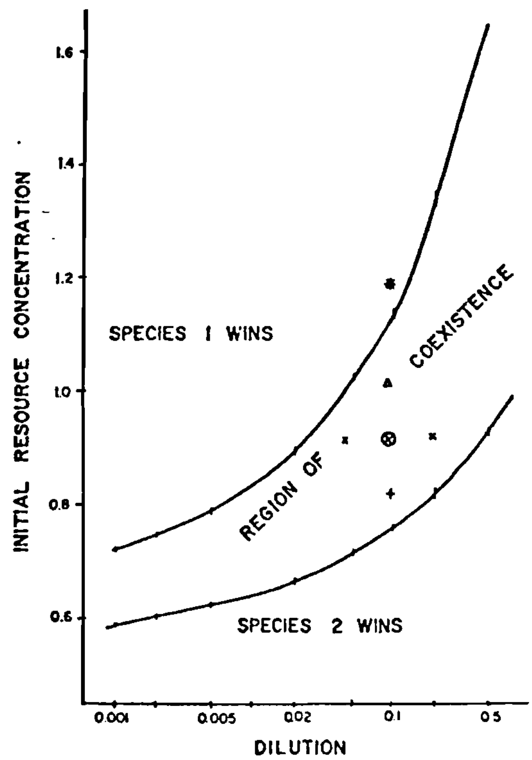


FIG. 6.—Coexistence for two species competing for a single resource. Areas of structural stability and instability. See text for an explanation of symbols used in the figure. Resource uptake and conversion efficiency parameters given in fig. 1.

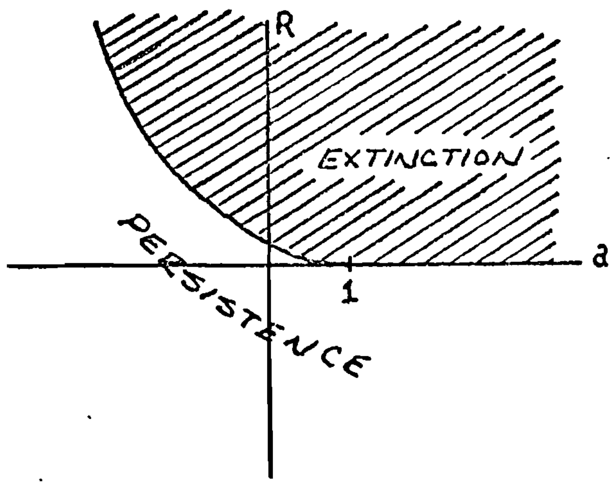
The one-dimensional logistic type equation can be presented in this P-space form. The basic equation is

$$\dot{N} = N - N^2$$

The two exploitation parameters a and R give a form

$$\dot{N} = N - N^2 - aN - R$$

and a P-space



The simplest two state variable system is the linear form

$$\dot{\underline{X}} = \underline{\underline{A}}\,\underline{X}$$

where

$$\underline{\underline{A}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

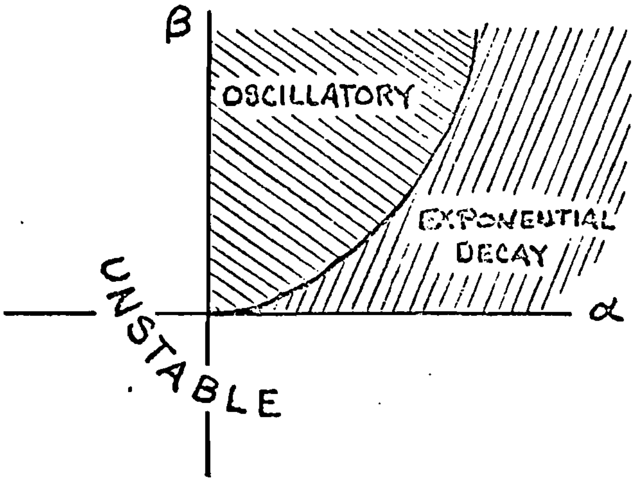
The characteristic equation is

$$\lambda^2 - \mathrm{tr}A\lambda + A = 0$$

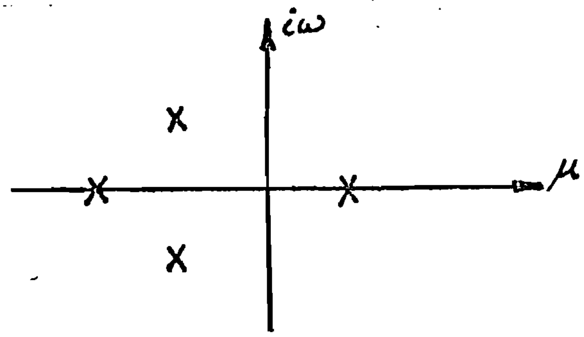
or

$$\lambda^2 + \alpha\lambda + \beta = 0 \quad ; \alpha = -a-d,\ \beta = ad-bc$$

The condition for a stable equlibrium is for $\alpha > 0$ and $\beta > 0$. If $\alpha^2 < 4\beta$ the λ's are complex and we have oscillatory behaviour. The P-space is shown below. In this example the four original parameters have been grouped into two; α and β. We could conceivably construct a four-dinensional space for a, b, c and d. Or given a and d constant, a plot for the cross terms b and c.



A more common approach used in engineering problems is the "frequency domain". Here instead of α and β the λ's are ploted in a complex plane with $\lambda = \mu + i\omega$ ($i = \sqrt{-1}$).



λ's on the right (μ>0) are unstable; λ's with an ω component appear in pairs (±iω) and represent oscillatory behaviour. With practice this type of plot makes qualitative behaviour readily apparent.

The following three plots from Hubbell,(1973) have similarities with the previous example. Stripping has been added to display oscillatory and unstable regions. The three plots are

A. One species with time delay.

B. Predator prey system. Globally stable; stripped regions are non oscillatory.

C. Prey with a predator/competitor. Oscillatory and unstable regions are stripped.

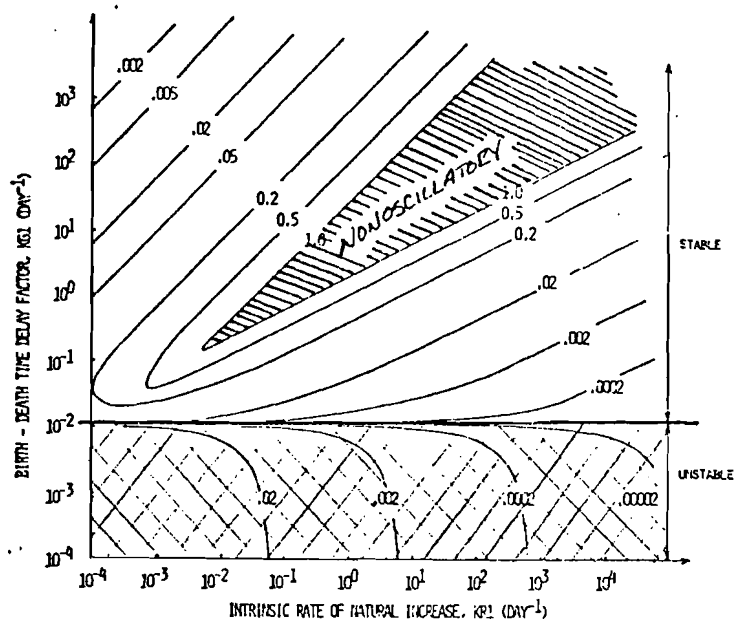


FIG. 14.—Parameter-space study of the stability and tendency to oscillate of the population with a birth-death time lag, as a function of the intrinsic rate of natural increase and the birth-death time delay factor. Isoclines are lines of equal damping ratios. Other parameters are set at 0.1.

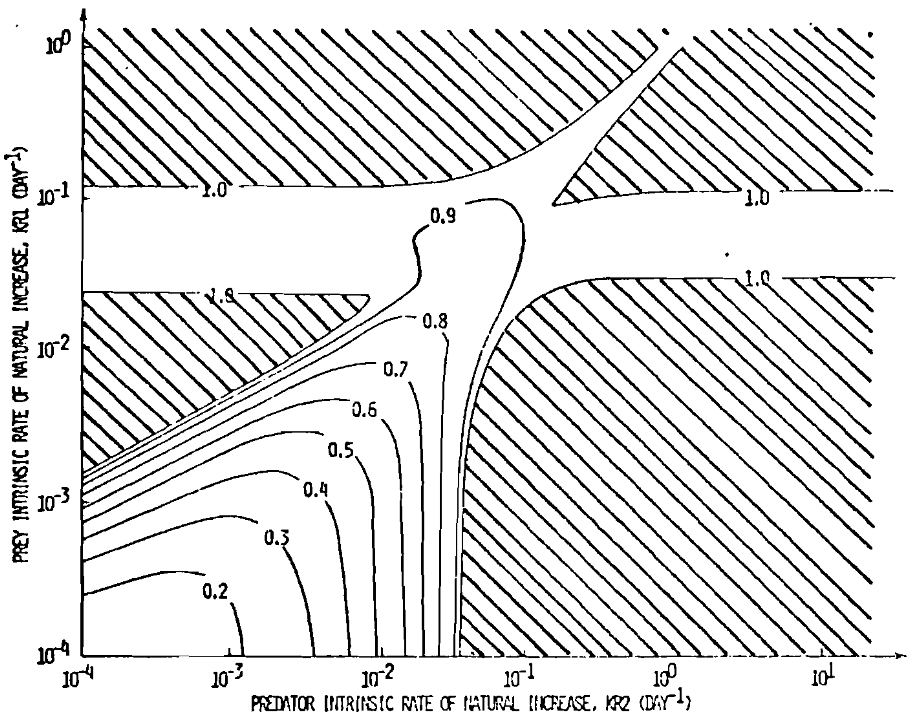


FIG. 2.—Parameter-space study of the influence of the prey and predator intrinsic rates of natural increase on the oscillatory condition of the two-species system. Lines are isoclines of equal damping ratios. Other system parameters are set at 0.1 for this plot.

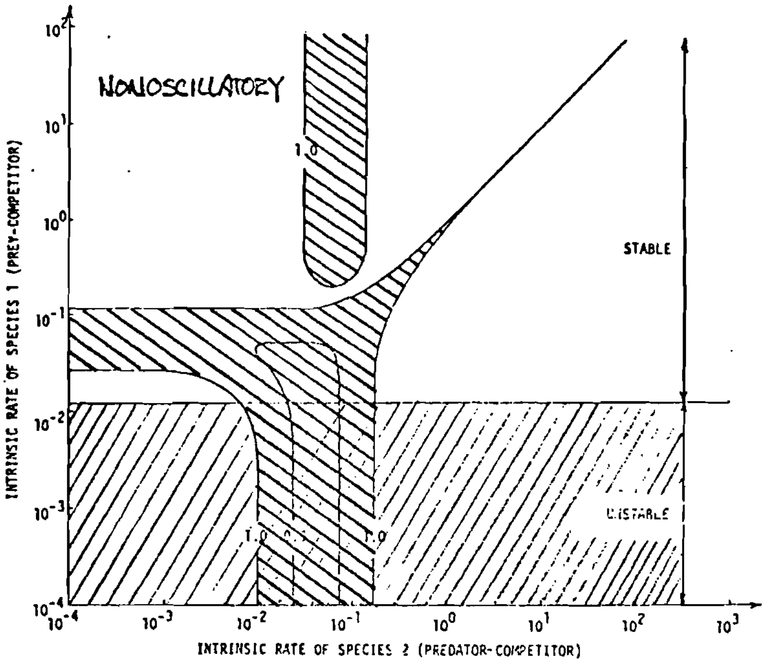
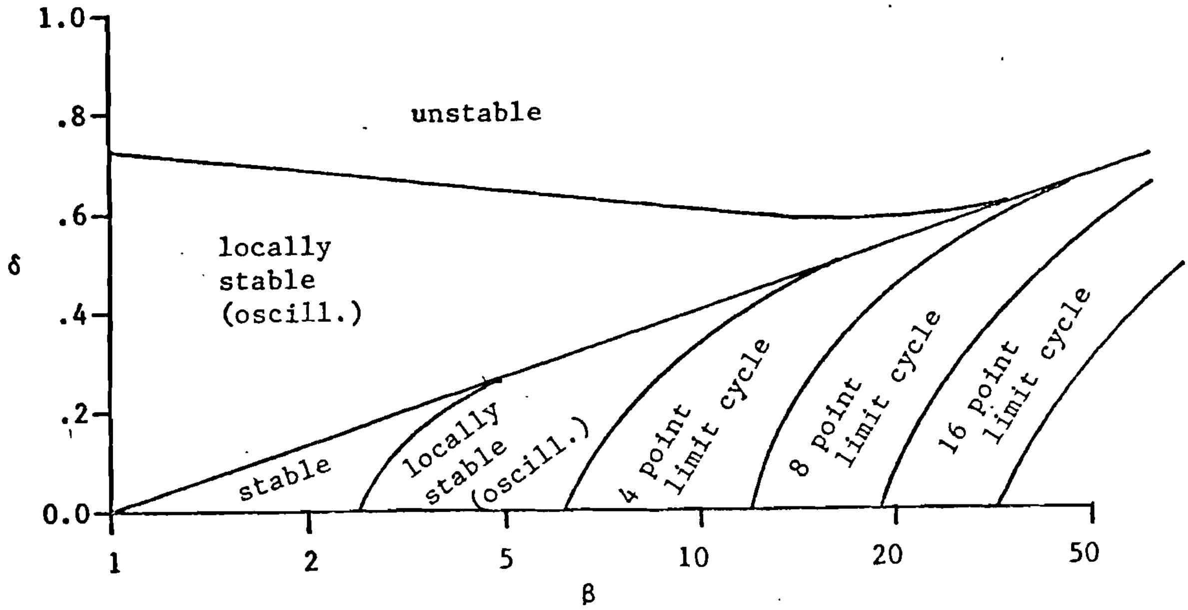


Fig. 14.—Parameter-space study of the influence of the intrinsic rates of natural increase of the species in the combined competition, predator-prey system, on the stability and oscillatory condition of the system. The lines are isoclines of equal damping ratios. All other parameters of the system are set at 0.1 in this plot.

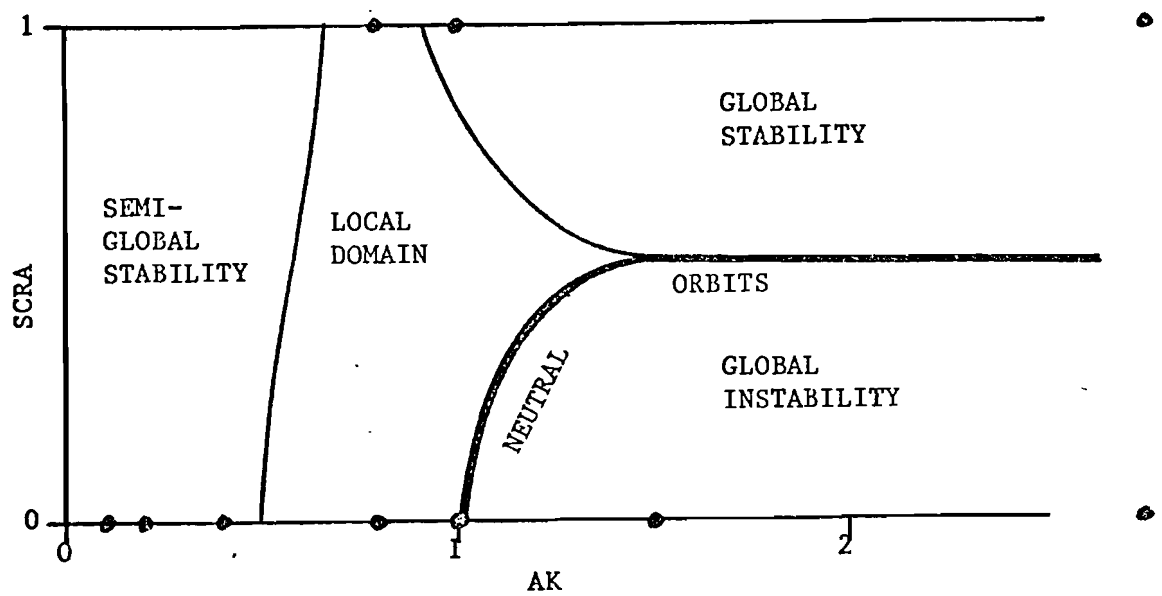
I have taken the periodic cicadas/fungus system that May and Lloyd have developed and liberally synthesized some of its properties into a parameter-space plot.



β = Net Reproductivity of Cicadas

δ = Death Rate of Spores

Finally, some examples closer to home. The Predator/Prey model with its several dozen parameters would be difficult to put completely into this framework. However, I have looked at a couple of parameters that have been explored (AK and SCRA) for which the model shows some sensitivity. The number of points are few (the dots) so the form is highly speculative.

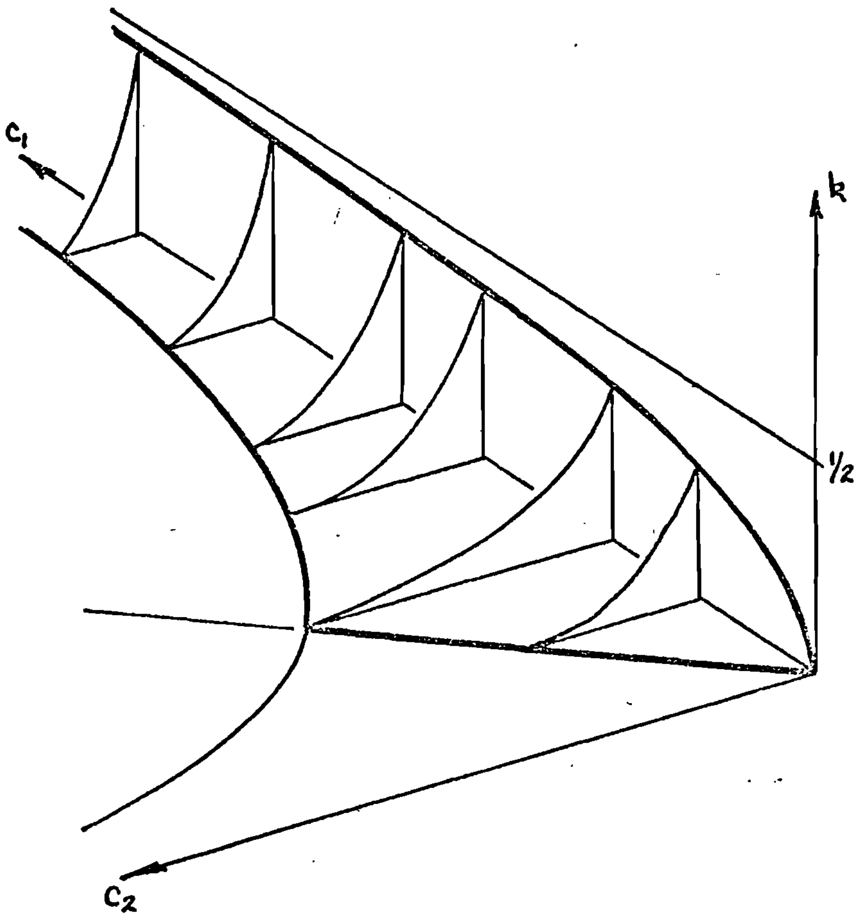


The substitute analytic model that was selected to give a stable limit cycle (and thus a local domain of attraction) has three adjustable parameters. The set of equations is

$$\dot{x} = \left\{ 1 - kx - \frac{y}{1+c_1x} \right\} x$$

$$\dot{y} = \left\{ \quad - 1 + \frac{x}{1+c_2y} \right\} y$$

The conditions for a stable equilibrium point are that $k < 1$ and the parameters not fall within the shaded region along the c_1 axis.



The last example comes from Larkin (1971). The results of the simulations for the Adams River Sockeye involve four parameter sets (Table 2).

s - Smolt buffering coefficient

Q - Scaling factor for climatic effects

a_1- Compensation coefficient

a_3- Rate of fry depensation

Selecting a_1=2.5 and a_3=3 the s,Q parameter-space is shown below. The cycle year for dominance for each point is indicated.