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Bounds for Stochastic Programs in Particular for Recourse Problems

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Abstract

In this paper, we shall discuss the bounds for the optimal value of recourse problems from the point of view of assumptions and of possible generalizations. We shall concentrate on bounds based on the first order moment conditions and to those based on sample information. We shall indicate when it is possible to remove the convexity assumptions, when there is a hope for extensions to multistage problems and we shall point out reflections of bounds and stability results.

1. Bounds for stochastic programs

The interest in bounding the optimal value of stochastic programs has been apparent from the very origin of stochastic programming, cf. Edmundson-Madansky inequality [41] in the fifties, minimax bounds [52] in the sixties, bounds based on the moment problem [11]–[13], [24] or bounds on the error due to the approximation [34], [51] in the seventies. The reasons come from incomplete information about the distribution and from numerical techniques: we construct and solve approximate problems using various algorithms. We need stopping rules and tests of optimality, an error analysis, strategies for refinement, conclusions concerning the results valid for *the true problem*, statements about stability and robustness of the output, etc. See e.g. [5], [36] for further discussions.

Bounds become often a part of a numerical procedure and we are naturally interested in numerically tractable bounding techniques. Generally speaking, it is easier to bound the objective function and its optimal value than to get bounds on optimal solutions and it is not easy to extend the results valid for two-stage stochastic programs to the multistage case. Different approaches require different assumptions, for instance, there are techniques applicable only under appropriate convexity or smoothness assumptions, for independent random variables, for problems of a special structure, etc. In case of an incomplete knowledge of the probability distribution, the design of bounds reflects the existing level of information; the bounds that correspond to sample information are different from those based on knowledge of moments of the underlying probability distribution.

To be more specific, let us consider a class of stochastic programs of the form

$$\text{minimize} \quad E_P f(\mathbf{x},\omega) \quad \text{on the set} \quad \mathcal{X} \tag{1}$$

where $\mathcal{X}$ is a given nonempty convex polyhedral set in a finite dimensional space, P is a probability distribution of ω on Ω, and $f : \mathcal{X} \times \Omega \to R^1$ is a given function. We shall assume that the expectation in (1) is finite for all $\mathbf{x} \in \mathcal{X}$ and that the optimal solution of (1) exists.

The above formulation covers the expected utility models and the two stage stochastic programs with relatively complete recourse. In the latter case, for each $\mathbf{x} \in \mathcal{X}$ and $\omega \in \Omega$, the value of the random objective is

$$f(\mathbf{x},\omega) = \mathbf{c}^\top \mathbf{x} + Q(\mathbf{x},\omega)$$

with

$$Q(\mathbf{x},\omega) = \min_{\mathbf{y}} \left\{ \mathbf{q}(\omega)^\top \mathbf{y} \mid \mathbf{W}(\omega)\mathbf{y} = \mathbf{h}(\omega) - \mathbf{T}(\omega)\mathbf{x}, \mathbf{y} \geq 0 \right\} \tag{2}$$

the optimal value of the second-stage program.

There are various natural ideas how to get bounds on the optimal value of (1): Any approximation of the objective function $E_P f(\mathbf{x},\omega)$ that is valid uniformly for all $\mathbf{x} \in \mathcal{X}$ provides an equally precise approximation of the optimal value. This idea was applied in the first papers of Kaňková, e.g. in [39], and appears for instance also in [53]. One can relax the constraints in definition $\mathcal{X}$ to get a lower bound or to add new constraints to get an upper bound, cf. [50]. It is possible to approximate the *random* objective function $f(\mathbf{x}, \omega)$ by another simpler or more convenient function; see the piecewise linear bounds [4], [7], [49].

Further techniques are based on different ideas that come from results on stability and sensitivity with respect to the probability distribution P (e.g. [47]) and are related to asymptotic properties of statistical estimators such as consistence, rate of convergence, asymptotic distribution, probabilistic bounds on large deviations [38]; see also [17] and [48, Chapter 6] and references therein. These results can be used to construct various asymptotic confidence intervals for the true optimal value and optimal solutions. Moreover for special types of perturbations, such as contamination, one can obtain global non-asymptotic bounds useful in postoptimality analysis; cf. [16], [17]. Error bounds for the optimal value can be often used also for construction of bounds for the optimal solutions provided that some additional assumptions (growths conditions, unique true optimal solutions, etc.) hold true; cf. [39], [47], [48].

We shall deal with bounds for the true optimal value of (1) that exploit in a simple way a sample based information (Section 2) and with bounds based on knowledge of moments (Section 3). We shall discuss them from the point of view of assumptions and of possible generalizations. Finally in Section 4, we shall concentrate on multistage stochastic linear programs with recourse and with random right-hand sides to indicate when it is possible to extend the well-known upper bounding technique based on the first order moment conditions to multistage problems.

2. Bounds based on sample information

Assume now that there is at disposal a sample information about the true probability distribution P that allows to construct an empirical distribution function based on the observed dates with the aim to draw conclusions about the optimal value $\varphi(P)$ of the true program (1) using the optimal value of its sample based counterpart.

Let $\mathcal{S}$ be the available sample of size n, say $\omega^1, \ldots, \omega^n$, from the distribution P and let us denote the value of the objective function based on this sample $\mathcal{S}$ of size n at a point $\mathbf{x} \in \mathcal{X}$ as

$$E_{\mathcal{S}} f(\mathbf{x}, \omega) := \frac{1}{n} \sum_{i=1}^{n} f(\mathbf{x}, \omega^i)$$

The commonly accepted procedure is to approximate the optimal solution of (1) and its optimal value $\varphi(P)$ by an optimal solution $\mathbf{x}_{\mathcal{S}}$ and the optimal value $\varphi_{\mathcal{S}}$ of the sample based program

$$\text{(3)} \qquad \text{minimize} \quad E_{\mathcal{S}} f(\mathbf{x}, \omega) := \frac{1}{n} \sum_{i=1}^{n} f(\mathbf{x}, \omega^i) \quad \text{on the set} \quad \mathcal{X}$$

Indeed, the optimal solutions and the optimal value of (3) are consistent estimates of the true optimal solution $\mathbf{x}(P)$ and of the true optimal value $\varphi(P)$ of (3) under relatively modest assumptions – see e.g. [20]. Asymptotic normality of these estimates, however, holds true only under rather stringent assumptions. Therefore we shall base the bounds on the optimal value of (1) on direct exploitation of the standard central limit theorem; see also [45, Chapter 15] and references ibid.

Under assumptions of existence of a finite true expectation $E_P f(\mathbf{x},\omega)$ and variance $\mathrm{var}_P f(\mathbf{x},\omega)$, the central limit theorem allows to construct approximate confidence intervals for the values of the true objective function $E_P f(\mathbf{x},\omega)$ at individual points $\mathbf{x} \in \mathcal{X}$. The approximate $1-\alpha$ confidence interval is

$$E_{\mathcal{S}} f(\mathbf{x},\omega) \pm \frac{t_\alpha}{\sqrt{n}} \left(\mathrm{var}_{\mathcal{S}} f(\mathbf{x},\omega)\right)^{1/2} \tag{4}$$

where

$$\mathrm{var}_{\mathcal{S}} f(\mathbf{x},\omega) = \frac{1}{n-1} \sum_{i=1}^{n} \left[f(\mathbf{x},\omega^i) - E_{\mathcal{S}} f(\mathbf{x},\omega)\right]^2$$

and t_α denotes the $(1-\alpha/2)$ quantile of $\mathcal{N}(0,1)$. (For moderate sample sizes n, t_α may be replaced by the $1-\alpha/2$ quantile of Student's distribution with $n-1$ degrees of freedom.)

For each $\omega \in \Omega$ separately, we can also get the value

$$\varphi(\omega) = \min_{\mathbf{x}\in\mathcal{X}} f(\mathbf{x},\omega) \tag{5}$$

and quite similar arguments allow to derive an approximate $1-\alpha$ confidence interval based on the sample $\mathcal{S}$ for the true expectation $E_P\varphi(\omega)$, i. e., for the expected value of the population wait - and - see problem, provided that the true expectation and variance $\mathrm{var}_P\varphi(\omega)$ are finite:

$$\frac{1}{n}\sum_{i=1}^{n} \varphi(\omega^i) \pm \frac{t_\alpha}{\sqrt{n}} \left(\mathrm{var}_{\mathcal{S}}\varphi(\omega)\right)^{1/2} \tag{6}$$

where

$$\mathrm{var}_{\mathcal{S}}\varphi(\omega) = \frac{1}{n-1}\sum_{i=1}^{n}\left[\varphi(\omega^i) - \frac{1}{n}\sum_{i=1}^{n}\varphi(\omega^i)\right]^2$$

If the normal approximation used in construction of the confidence interval (4) is precise enough, the confidence interval (4) with $\mathbf{x} = \mathbf{x}_{\mathcal{S}}$ covers approximately with probability $1-\alpha$ the value of the true objective function $E_P f(\mathbf{x},\omega)$ *at the point* $\mathbf{x}_{\mathcal{S}}$. Together with the obvious inequality $\varphi(P) \le E_P f(\mathbf{x}_{\mathcal{S}},\omega)$ it implies that

$$\varphi_{\mathcal{S}} + \frac{t_\alpha}{\sqrt{n}} \left(\mathrm{var}_{\mathcal{S}} f(\mathbf{x}_{\mathcal{S}},\omega)\right)^{1/2} \tag{7}$$

is an *approximate probabilistic upper bound for the true optimal value* $\varphi(P)$. Such an upper bound can be obviously based on any feasible solution $\mathbf{x} \in \mathcal{X}$. Due to the mentioned results on consistence of the sample based optimal solutions, there is a good reason to use $\mathbf{x}_{\mathcal{S}}$.

To get a *sample based lower bound* for $\varphi(P)$, we use (6):

$$\varphi(P) \geq E_P\varphi(\omega) \geq \frac{1}{n}\sum_i \varphi(\omega^i) - \frac{t_\alpha}{\sqrt{n}}\left(\mathrm{var}_S\varphi(\omega)\right)^{1/2} \tag{8}$$

The whole procedure of constructing bounds for the true optimal value $\varphi(P)$ consists of two steps that allow for exploitation of parallel techniques:

(i) Solution of the sample based program (3) to get an optimal solution $\mathbf{x}_S$ and the optimal value φ_S and evaluation of the random objectives $f(\mathbf{x}_S, \omega^i)$ at the optimal solution $\mathbf{x}_S$ for all considered sample values ω^i. The average and variance of the obtained values $f(\mathbf{x}_S, \omega^i)$ are used in the upper bound (7).

(ii) Solution of the n individual scenario problems is needed to get the optimal values $\varphi(\omega^i)$ for all considered sample points ω^i and the average and variance of these "sample" optimal values provide the necessary entries for construction of the lower bound (8).

An alternative procedure can be based on minimization of the upper bound of the confidence interval (4) on the set $\mathcal{X}$. It resembles the form of the *robust optimization* objective function (cf. [43]) and this upper bound is more tight than (7). It means that the problem

$$\min_{\mathbf{x}\in\mathcal{X}} \quad E_S f(\mathbf{x}, \omega) + \frac{t_\alpha}{\sqrt{n}}\left(\mathrm{var}_S f(\mathbf{x}, \omega)\right)^{1/2}$$

has to be solved instead of (3) in the first step (i) of the above bounding procedure and the obtained optimal value provides the upper bound. The second step (ii) applies without any change.

Similar results can be obtained for the case of sampling from a large *finite population*, say, $\Omega = [\omega^1, \ldots, \omega^N]$ and for distribution P that assigns equal probability $1/N$ to all elements of Ω. Except for the finite population factor $1 - n/N$, there is no difference between the bounds based on sampling from finite population and the former ones. For to get a tighter lower bound, one can always try to use various variance reduction sampling techniques.

Conclusions. The approximate confidence intervals are *distribution free*, i.e., they do not depend on the assumed form of the true probability distribution P. No assumptions about convexity or smoothness of the objective function are needed and these are the main advantages of the introduced approximate probabilistic bounds. On the other hand, the precision of the bounds depends on the precision of the approximation by the central limit theorem, on the sample size, etc., and this may be one of stumbling blocks. Possible applications of these bounds for construction of *stopping rules* depend on the algorithm concerned; for instance, upper bounds of the type (7) appear in [10], [31], [32], [42] and [44]. Even when some stochastic dependence can be incorporated (cf. [42]), to extend bounds (7) to multistage stochastic programs with *interstage dependent* random coefficients does not seem to be straightforward.

In case of sampling from a *continuous distribution* we can in addition construct rough confidence intervals for the optimal value using the following results of [8], [21]:

If ξ is a continuous random variable with an unknown unimodal density then for any fixed a and $t > 1$, the interval with endpoints

$$\xi \pm t|\xi - a| \tag{9}$$

based on one observation of ξ covers the unknown mode θ of the distribution with probability at least $1 - \frac{2}{t+1}$. With $t = 19$, one gets thus an at least 0.9 confidence interval.

The expert "guess" a has to be fixed *prior* to the random experiment that provides the realization ξ and it essentially influences the length of the confidence interval (9). The assumption of continuous distribution cannot be relaxed; on the other hand, some improvements can be obtained under more stringent assumptions about the distribution; for instance under additional assumption of symmetry, the confidence level for interval (9) increases to $1 - \frac{1}{t+1}$ and to $1 - \frac{.484}{t+1}$ for normal distribution of ξ.

A similar result can be derived also for confidence intervals based on several independent observations [8] in which case, no prior expert guess is needed and the confidence interval takes on the common form based on the sample mean and the sample standard deviation of the observations; compare (4). For two independent observations ξ^1, ξ^2 the interval is

$$1/2(\xi^1 + \xi^2) \pm t/2|\xi^1 - \xi^2| \tag{10}$$

These results were used in [18] for stochastic linear programs with individual probabilistic constraints and random right-hand sides. For their application to stochastic programming problems with recourse, we consider a *fixed number*, say n, of i.i.d. scenarios sampled from the given continuous distribution. One sample of size n can be taken as the random experiment that leads to the observed value φ of the optimal value function. The confidence intervals (9), (10) will cover the modus of the distribution of optimal values computed from n independent scenarios at least with the probabilities $1 - \frac{2}{t+1}$, resp. $1 - \frac{1}{t+1}$ provided that the distribution is continuous and unimodal. The value a needed in (9) can be for instance chosen as the optimal value of the expected value problem or the value of an approximate solution.

3. Bounds based on moment conditions

Whenever the knowledge of the probability distribution P in (1) reduces to an information about its support and about values of some moments we can use results known from the moment problem (e.g., [6], [12], [15], [16], [35]) to construct bounds for the optimal value $\varphi(P) := \min_{\mathbf{x}\in\mathcal{X}} E_P f(\mathbf{x}, \omega)$. It is also possible to exploit a qualitative information about P such as its unimodality ([12], [15]) or, in case of a discrete probability distribution, the existence of an incomplete ordering of probabilities [9]. Sometimes the moment conditions stem from the intrinsic features of the solved problem [19], e.g., from a low level of information. The moment bounds can be also constructed in the course of an algorithmic solution [37] or considered just for needs of stability considerations, for the worst case analysis and EVPI evaluation. There is a host of papers devoted to these bounds in the context of stochastic programming, to their refinement, to extensions to noncompact supports, etc. We refer to [35] and [45] and references ibid. The common idea of bounding techniques based on the moment problem is to replace the complete knowledge of P in (1) by knowledge of a set $\mathcal{P}$ of probability distributions that is supposed to contain P and is defined, inter alia, by moment conditions. We assume that $\mathcal{P}$ does not depend on the first-stage decision $\mathbf{x}$ and we assume the existence of the optimal value $\varphi(P)$ of (1) for all $P \in \mathcal{P}$.

Given the set $\mathcal{P}$ we want to construct bounds

$$L(\mathbf{x}) = \inf_{P \in \mathcal{P}} E_P f(\mathbf{x}, \omega) \tag{11}$$

$$U(\mathbf{x}) = \sup_{P \in \mathcal{P}} E_P f(\mathbf{x}, \omega) \tag{12}$$

for the objective function or bounds

$$L = \min_{\mathbf{x} \in \mathcal{X}} \inf_{P \in \mathcal{P}} E_P f(\mathbf{x}, \omega) \tag{13}$$

$$U = \min_{\mathbf{x} \in \mathcal{X}} \sup_{P \in \mathcal{P}} E_P f(\mathbf{x}, \omega) \tag{14}$$

for the optimal value $\varphi(P)$ by means of the moment problem.

The lower bounds (11), (13) reduce to Jensen's inequality [33]

$$L(\mathbf{x}) = \min_{P \in \mathcal{P}} E_P f(\mathbf{x}, \omega) = f(\mathbf{x}, E\omega) \tag{15}$$

provided that the probability distributions $P \in \mathcal{P}$ are characterized, inter alia, by a fixed mean value $E\omega$ and that the function $f(\mathbf{x}, \omega)$ is *convex* in ω. This bound is attained for the degenerated distribution concentrated in the mean value $E\omega$ *independently on* $\mathbf{x} \in \mathcal{X}$; hence, the lower bound for $\varphi(P)$ is the optimal value of the convex deterministic *expected value program*

$$L = \min_{\mathbf{x} \in \mathcal{X}} f(\mathbf{x}, E\omega) \tag{16}$$

Similarly for convex functions $f(\mathbf{x}, \bullet)$, the upper bound for expectation $E_P f(\mathbf{x}, \omega)$ with P belonging to the set of distributions carried by a fixed convex polyhedron Ω and with prescribed mean value - a fixed interior point of Ω - is attained and reduces to the Edmundson - Madansky bound [41]. It is easily computable (i.e, it reduces to one-dimensional moment problems and/or the extremal distribution is independent on $\mathbf{x}$) only under special circumstances, for instance, when Ω is a rectangle and $f(\mathbf{x}, \bullet)$ is separable in components of ω or the random variables are independent, or when Ω is a simplex. (See [5], [35] for a detailed discussion.) Otherwise, for $\Omega = \operatorname{conv}\{\omega_1, \ldots, \omega_H\}$, $U(\mathbf{x})$ is the optimal value of the linear program

$$U(\mathbf{x}) = \min_{\mathbf{p}} \left\{ \sum_{h=1}^{H} p_h f(\mathbf{x}, \omega_h) \;\middle|\; \sum_{h=1}^{H} p_h \omega_h = E\omega, \sum_{h=1}^{H} p_h = 1, p_h \geq 0 \quad \forall h \right\} \tag{17}$$

(see e. g. [11], [12], [24, Chapter II], [29], [45, Chapter 5]). If $f(\mathbf{x}, \omega)$ is convex separable with respect to individual components of ω, (17) splits to moment problems with respect to one-dimensional random variables carried by closed intervals. The corresponding (marginal) distributions are uniquely determined by the first order moment conditions. This is the case when the extermal distribution does not depend on $\mathbf{x}$ and can be given explicitly: It is carried by the vertices of the cartesian product of the one-dimensional intervals and the probabilities of these *upper bounding scenarios* are products of the corresponding probabilities that come from the marginal extremal distributions. This is the most welcome situation when the upper bound for $\varphi(P)$ follows by solution of the stochastic program for the obtained discrete extremal distribution, without any reference to the *inner* optimization problem (17).

In general, however, to get the upper bound (14) for the optimal value $\varphi(P)$ means to use a procedure suitable for solving the minimax problem

$$\min_{\mathbf{x} \in \mathcal{X}} \max_{P \in \mathcal{P}} E_P f(\mathbf{x}, \omega) = \min_{\mathbf{x} \in \mathcal{X}} U(\mathbf{x}) \tag{18}$$

cf. [25], [28].

The assumption of convexity of the random objective $f(\mathbf{x}, \omega)$ with respect to ω means, except for very special cases, the restriction to two-stage stochastic programs with *fixed recourse, fixed coefficients* $\mathbf{q}$ in the second-stage objective function and with $\mathbf{h}, \mathbf{T}$ *linear* in ω. Inclusion of random coefficients $\mathbf{q}$ requires developing parallel results for saddle functions that are convex with respect to a group of random parameters (typically, the right-hand sides) and concave with respect to the remaining random parameters (typically, the random parameters of the second-stage objective function); this was done, e.g., in [23], [26].

For to get a valid lower bound (15), convexity assumption can be evidently relaxed if there exists a lower supporting linear function for $f(\mathbf{x}, \bullet)$ at the point $E\omega$. Similarly, [39] points out that Edmundson-Madansky upper bound holds true also for some nonconvex functions, for instance, for $f(\mathbf{x}, \bullet)$ defined on a multidimensional compact interval Ω and convex separately in each of components of ω or multi-chord-dominated on Ω. Whereas Jensen's lower bound (15) also extends to the related classes of convex *multistage* stochastic programs both with stage independent right-hand sides [30] and for their dependence [22], lack of convexity seems to be the main stumbling-block for designing a computable upper bound of the Edmundson-Madansky type. See Section 4 for details.

Theoretically, the moment problem provides bounds for the expectation $E_P f(\mathbf{x}, \omega)$ also for *nonconvex functions* $f(\mathbf{x}, \bullet)$ and under *higher moment conditions*. For convex compact set $\mathcal{P}$ of probability distributions, the expectation (a linear functional in P) attains both its maximal and minimal value at extremal points of $\mathcal{P}$. The corresponding distributions are discrete ones concentrated at a modest number of points, however, extremal distributions *independent of the form of* f (and thus independent of the first-stage decisions $\mathbf{x}$) appear only exceptionally. For a *fixed* $\mathbf{x}$, they can be generated and the bounds can be obtained as the minimal or maximal value of a *generalized linear program* [5], [24], [45]:

With fixed $\mathbf{x}$ and with the set $\mathcal{P}$ defined by a given *compact* support Ω and by moment conditions

$$E_P g_k(\omega) \le \alpha_k, k = 1, \dots, K$$

it is sufficient to select $K+1$ elements ω_k of Ω and assign them probabilities $p_k \geq 0, \sum_k p_k = 1$ so that the moment conditions are fulfilled and the expected value $\sum_k p_k f(\mathbf{x}, \omega_k)$ is maximal (minimal).

Duality arguments provide decision rules needed for replacement of individual points ω_k by other elements of Ω within the generalized revised simplex method; cf. [25]. Sometimes, it is possible to indicate a priori a finite set of elements from Ω, i. e., the scenarios that are of concern from the point of view of the worst case analysis; this is the case of convex $f(\mathbf{x}, \bullet)$, bounded convex polyhedral support and the first order moment information on ω, see (9) and its generalization to *piecewise convex* function $f(\mathbf{x}, \bullet)$ in [11]. Again, the *inner* optimization problems that give bounds $L(\mathbf{x}), U(\mathbf{x})$ have to be incorporated into the optimization problem with respect to $\mathbf{x}$. This was applied for the first and second order moment information, see e.g. [13], [35].

A completely different approach for bounding expectations can be based on Korovkin type inequalities, see [2, Chapter 7]. These inequalities provide for instance estimates of the difference between the expected value of a function and its value at the expectation of the random variable. They do not necessarily assume convexity and some of them are independent on the explicit form of the function. As an example we shall introduce the following simple result (see Corollary 7.4.1 of [2]):

Theorem. Let w, σ be given positive numbers, Ω a nonempty fixed compact convex set in R^k, $\tilde{\omega} \in \Omega$ an arbitrary fixed element and P a probability distribution on Ω such that

$$E_P\omega = \tilde{\omega}, \quad E_P\|\omega - \tilde{\omega}\|^2 = \sigma^2$$

Let $h \in C_B^1(\Omega)$ with the modulus of continuity of its partial derivatives $h_i \forall i$

$$m(h_i, 1/2\sigma) \leq w \forall i$$

Then

$$|E_P h(\omega) - h(\tilde{\omega})| \leq 1.5625 w\sigma \tag{19}$$

There exist more complicated results for $h \in C_B^n(\Omega)$, for higher=order moment conditions and also upper bounds on the difference in (19) that use the assumed fixed value $E_P\omega = \tilde{\omega}$. Results of this type can be helpful for estimating the EVPI when the random objective $f(\mathbf{x}, \bullet)$ is not convex. Instead of convexity, smoothness of gradients is required; for a given compact set Q and a positive constant ϵ the modulus of continuity of a continuous function g is defined as

$$m(g, \epsilon) := \sup \{|g(\omega_1) - g(\omega_2)| : \omega_1, \omega_2 \in Q, \|\omega_1 - \omega_2\| \leq \epsilon\}$$

with $\| \bullet \|$ the l_1 norm.

Differentiability properties of the random objective function for the two-stage stochastic linear program cannot be expected (recall the form of the second-stage program (2)) but it is not the only type of stochastic programming model. There are examples of smooth penalties for discrepances whose choice comes from a detailed analysis of the real-life problem without any reference to the second-stage program (2) and the piecewise linear - quadratic stochastic programs, see e.g. [46], enjoy both smoothness and convexity properties.

To conclude this Section let us mention another problem related to bounds based on moment problem for classes of probability distributions defined by prescribed values of some moments. This input information is not always completely known, it is based on a sample or past information, on expert's opinion, etc. Accordingly, we face uncertainty again, on a new level. There are scattered results concerning stability with respect to the prescribed values of moments based on parametric programming [15], complemented by statistical analysis [14] and discussed also in the context of a real life application [1].

4. Extensions to multistage SLP

For the purposes of this Section, it will be expedient to change slightly the notation: in the subscripts of expectations we shall replace the probability distribution P by the relevant components of ω. We shall deal with the following three stage stochastic linear program with recourse with random right hand sides

minimize

$$\mathbf{c}_1^\top \mathbf{x}_1 + E_{\omega_1}\{\varphi_1(\mathbf{x}_1, \omega_1)\} \tag{20}$$

subject to

$$\mathbf{A}_1\mathbf{x}_1 = \mathbf{b}_1 \tag{21}$$

$$\mathbf{l}_1 \leq \mathbf{x}_1 \leq \mathbf{u}_1,$$

where the function φ_1 is defined as

$$\varphi_1(\mathbf{x}_1, \omega_1) = \inf_{\mathbf{x}_2}\left[\mathbf{c}_2^\top \mathbf{x}_2 + E_{\omega_2|\omega_1}\varphi_2(\mathbf{x}_2, \omega_2)\right] \tag{22}$$

subject to

$$\mathbf{B}_2\mathbf{x}_1 + \mathbf{A}_2\mathbf{x}_2 = \mathbf{b}_2(\omega_1) \tag{23}$$

$$\mathbf{l}_2 \leq \mathbf{x}_2 \leq \mathbf{u}_2$$

and

$$\varphi_2(\mathbf{x}_2, \omega_2) = \inf_{\mathbf{x}_3} \mathbf{c}_3^\top \mathbf{x}_3 \tag{24}$$

subject to

$$\mathbf{B}_3\mathbf{x}_2 + \mathbf{A}_3\mathbf{x}_3 = \mathbf{b}_3(\omega_2)$$

$$\mathbf{l}_3 \leq \mathbf{x}_3 \leq \mathbf{u}_3$$

The subvectors ω_1, ω_2 of the random vector ω on $[\Omega, P]$ generate the right hand sides. We assume that the right-hand sides are *linear* in ω_1 and in ω_2, that there is an optimal solution for an arbitrary realization of right hand sides and that the expectations are finite. We want to construct bounds for the optimal value $\varphi(P)$ of (20) subject to subsequent constraints and recursive definitions, using just the first order moment information about ω. However, even under these rather simplifying assumptions, convexity of the recourse costs $\varphi_1(\mathbf{x}_1, \omega_1)$ with respect to ω_1 follows only under particular circumstances such as independence of ω_1, ω_2 or a special form of the conditional distribution function $P_{\omega_2|\omega_1}$ needed for evaluation of the conditional expectation $E_{\omega_2|\omega_1}\varphi_2(\mathbf{x}_2, \omega_2)$ in (22), for instance,

$$(25) \qquad P_{\omega_2|\omega_1}(\mathbf{z}) = Q(\mathbf{z} - \mathbf{H}\omega_1)$$

where Q is a probability distribution function and $\mathbf{H}$ is a fixed matrix of proper dimension.

Case 1. To simplify the presentation, assume first that ω_1, ω_2 are *independent random variables*, for instance the short term interest rates. Their marginal distributions P_1, P_2 are independent of the decision variables $\mathbf{x}_1, \mathbf{x}_2$ and are supposed to fulfil the following conditions:

$$(26) \qquad P_t\{r_t \le \omega_t \le R_t\} = 1, \quad \text{and} \quad E\omega_t = \mu_t \quad t = 1,2$$

The sets of marginal distributions that fulfil (26) will be denoted $\mathcal{P}_1, \mathcal{P}_2$ and assumed independent of $\mathbf{x}_1, \mathbf{x}_2$.

For this form of program, joint convexity of functions $\varphi_t, t = 1,2$ with respect to $\mathbf{x}_t, \omega_t$ holds true and the lower bound follows from Jensen's inequality [33]. It means, that the lover bound can be computed as the optimal value of the expected value program

minimize

$$(27) \qquad \mathbf{c}_1^\top \mathbf{x}_1 + \mathbf{c}_2^\top \mathbf{x}_2 + \mathbf{c}_3^\top \mathbf{x}_3$$

subject to

$$(28) \qquad \begin{aligned} \mathbf{A}_1\mathbf{x}_1 \qquad\qquad\qquad &= \mathbf{b}_1 \\ \mathbf{B}_2\mathbf{x}_1 + \mathbf{A}_2\mathbf{x}_2 \qquad\quad &= \mathbf{b}_2(\mu_1) \\ \mathbf{B}_3\mathbf{x}_2 + \mathbf{A}_3\mathbf{x}_3 &= \mathbf{b}_3(\mu_2) \end{aligned}$$

$$\mathbf{l}_t \le \mathbf{x}_t \le \mathbf{u}_t, \quad t = 1,2,3$$

The upper bound for $E_{\omega_2}\varphi_2(\mathbf{x}_2, \omega_2)$ follows from Edmundson-Madansky inequality [41]: For all distributions $P_2 \in \mathcal{P}_2$, the upper bound is attained for the distribution $P_2^* \in \mathcal{P}_2$ concentrated at the points r_2, R_2 with probabilities $\lambda_2 = \frac{R_2 - \mu_2}{R_2 - r_2}$ and $1 - \lambda_2$:

$$(29) \qquad E_{\omega_2}\varphi_2(\mathbf{x}_2, \omega_2) \le \lambda_2\varphi_2(\mathbf{x}_2, r_2) + (1 - \lambda_2)\varphi_2(\mathbf{x}_2, R_2) := E^*_{\omega_2}\varphi_2(\mathbf{x}_2, \omega_2)$$

For a fixed $\mathbf{x}_2$, this bound can be obtained by solving the corresponding program of the third stage (24) for two scenarios $\omega_2 = r_2$ and $\omega_2 = R_2$; moreover, $E^*_{\omega_2}\varphi_2(\mathbf{x}_2, \omega_2)$ is evidently convex in $\mathbf{x}_2$. This gives an upper bound for $\varphi_1(\mathbf{x}_1, \omega_1)$:

$$(30) \qquad \varphi_1(\mathbf{x}_1,\omega_1) \le \min_{\mathbf{x}_2} \left[\mathbf{c}_2^\top \mathbf{x}_2 + E^*_{\omega_2}\varphi_2(\mathbf{x}_2,\omega_2) \quad \text{subject to} \quad (23)\right] := \varphi_1^*(\mathbf{x}_1,\omega_1)$$

The resulting upper bound $\varphi_1^*(\mathbf{x}_1,\omega_1)$ is jointly convex in $\mathbf{x}_1,\omega_1$ and the Edmundson-Madansky bound can be applied once more to get an upper bound for its expectation over the set of distributions $\mathcal{P}_1$. We get thus an upper bound for the expected recourse costs $\varphi_1(\mathbf{x}_1,\omega_1)$ in (20): For all marginal distributions $P_1 \in \mathcal{P}_1, P_2 \in \mathcal{P}_2$,

$$(31) \qquad E_{\omega_1}\varphi_1(\mathbf{x}_1,\omega_1) \le \lambda_1\varphi_1^*(\mathbf{x}_1,r_1) + (1-\lambda_1)\varphi_1^*(\mathbf{x}_1,R_1) := E^*_\omega\varphi_1^*(\mathbf{x}_1,\omega_1)$$

where $\lambda_1 = \frac{R_1-\mu_1}{R_1-r_1}$.

The upper bound (31) for the optimal value of (20), (21) equals thus the optimal value of the stochastic program based on scenarios $[r_1,r_2],[r_1,R_2],[R_1,r_2],[R_1,R_2]$ with probabilities $\lambda_1\lambda_2, \lambda_1(1-\lambda_2), (1-\lambda_1)\lambda_2, (1-\lambda_1)(1-\lambda_2)$ and it can be obtained as the optimal value of the corresponding linear program.

There is an obvious generalization to T-stage stochastic linear programs with random right hand sides - linear functions of stage independent random variables whose distributions belong to sets $\mathcal{P}_t$ described by fixed compact convex supports and by fixed mean values. There are 2^{T-1} upper bound scenarios identified by sequences of endpoints e_t of intervals $[r_t,R_t]$ for $t=1,\dots,T-1$; compare with [22]. It is also possible to generalize the results to right hand sides that are *linear transforms of interstage independent random vectors* ω_t whose supports are given simplices and the mean values are fixed interior points of these simplices. Further generalizations concern *nonlinear convex stochastic programs with stage independent random right-hand sides* and it is again possible to include another group of stage independent random parameters, say, η_t into the objective functions. The basic requirement is the saddle property of the optimal value functions φ_t with respect to decision variables and ω on one side and to η on the other side (cf. [23], [26]). If we continue to restrict our studies to random right - hand sides only, the crucial problem is to extend the upperbounding technique to interstage dependence.

Case 2. To illustrate the limitations we continue to discuss the three stage program (20) – (24) under assumption that the set of the considered distributions $\mathcal{P}_2(\omega_1)$ of ω_2 conditional on ω_1 is determined by the support $[r_2(\omega_1), R_2(\omega_1)]$ and by the conditional mean value $\mu_2(\omega_1)$. Given ω_1, the upper bound on $E_{\omega_2}\varphi_2(\mathbf{x}_2,\omega_2)$ is

$$(32) \qquad \lambda_2(\omega_1)\varphi_2(\mathbf{x}_2,r_2(\omega_1)) + (1-\lambda_2(\omega_1))\varphi_2(\mathbf{x}_2,R_2(\omega_1)) := E^*_{\omega_2|\omega_1}\varphi_2(\mathbf{x}_2,\omega_2)$$

To proceed further this upper bound has to be treated as a function of $\mathbf{x}_2$ *and* ω_1. Let this function be $U_2(\mathbf{x}_2,\omega_1)$. The next step involves minimization of

$$(33) \qquad \mathbf{c}_2^\top \mathbf{x}_2 + U_2(\mathbf{x}_2,\omega_1)$$

with respect to constraints (23) on $\mathbf{x}_2$. Denote again the resulting optimal value by $\varphi_1^*(\mathbf{x}_1,\omega_1)$. To get it convex in ω_1, for the sake of subsequent use of Edmundson-Madansky upper bound on its expectation, one needs $U_2(\mathbf{x}_2, \omega_1)$ *jointly convex* in $\mathbf{x}_2, \omega_1$. To this purpose, it is not enough to assume r_2, R_2 *linear in* ω_1 (recall that $\lambda_2(\omega_1) = \frac{R_2(\omega_1) - \mu_2(\omega_1)}{R_2(\omega_1) - r_2(\omega_1)}$). One possible set of additional assumptions concerning definition of $\mathcal{P}_2(\omega_1)$ reads:

A1 r_2, R_2 are linear in ω_1 and λ_2 is a fixed number.

Assumption **A1** implies μ_2 linear in ω_1 and under assumption **A1**, the upper bound

$$U_2(\mathbf{x}_2, \omega_1) = \lambda_2 \varphi_2(\mathbf{x}_2, r_2(\omega_1)) + (1 - \lambda)\varphi_2(\mathbf{x}_2, R_2(\omega_1))$$

is jointly convex in $\mathbf{x}_2, \omega_1$.

For U_2 jointly convex in $\mathbf{x}_2, \omega_1$, minimization of (33) provides an upper bound, say, $\varphi_1^*(\mathbf{x}_1, \omega_1)$ for $\varphi_1(\mathbf{x}_1, \omega_1)$ that is convex in $\mathbf{x}_1$ and ω_1 so that the upper bound for expectation $E_{\omega_1}\varphi_1(\mathbf{x}_1, \omega_1)$ follows from Edmundson-Madansky inequality applied to the expectation of $\varphi_1^*(\mathbf{x}_1, \omega_1)$. Accordingly, under assumption **A1** for all distributions $P_1 \in \mathcal{P}_1$ of ω_1 and conditional distributions $P_2 \in \mathcal{P}_2(\omega_1)$ of ω_2, the upper bound for the objective function in (20) is

$$\mathbf{c}_1^\top \mathbf{x}_1 + \lambda_1 \varphi_1^*(\mathbf{x}_1, r_1) + (1 - \lambda_1)\varphi_1^*(\mathbf{x}_1, R_1)$$

and the upper bound for the optimal value of (20), (21) can be again obtained via four scenarios, namely, $[r_1, r_2(r_1)], [r_1, R_2(r_1)], [R_1, r_2(R_1)], [R_1, R_2(R_1)]$ with probabilities $\lambda_1\lambda_2$, $\lambda_1(1 - \lambda_2), (1 - \lambda_1)\lambda_2, (1 - \lambda_1)(1 - \lambda_2)$.

Generalization to T-stage problem means assuming a fixed position of the conditional mean values $\mu_t(\omega_1, \ldots, \omega_{t-1})$ (described by fixed values $\lambda_t \in (0,1)$) within the intervals $[r_t(\omega_1, \ldots, \omega_{t-1}), R_t(\omega_1, \ldots, \omega_{t-1})]$ whose endpoints are linear in $\omega_1, \ldots, \omega_{t-1}$. This type of assumptions can be used to model the increasing uncertainty by growing range of the variables around some trend described by the conditional mean values. The upperbounding scenarios are sequences

$$\rho_1, \rho_2(\rho_1), \ldots, \rho_t(\rho_1, \rho_2(\rho_1), \ldots), \rho_{T-1}(\rho_1, \rho_2(\rho_1), \ldots)$$

with r_1 or R_1 substituted for ρ_1 and $r_t(\rho_1, \ldots, \rho_{t-1})$ or $R_t(\rho_1, \ldots, \rho_{t-1})$ substituted for $\rho_t, t = 2, \ldots, T-1$; compare with [22].

An extension to random vectors ω_t whose distributions are carried by simplices is possible again. Assumption of fixed values of λ_t independent of past observations translates to fixed barycentric coordinates of the conditional mean values $\mu(\omega_1, \ldots, \omega_{t-1})$. The general bounding technique based on barycentric scenarios, see [27], follows, inter alia, from the *assumed convexity* or saddle property of the objective functions for all stages, for instance, convexity of the function $\varphi(\mathbf{x}_1, \omega_1)$ defined by (22). The same assumption is needed also for the multistage extension of the upperbounding technique in [22]. Our discussions imply that this type of assumptions corresponds, besides the interstage independence of random right-hand sides, to rather special form of interstage dependent right-hand sides so that the conditional distributions fulfil **A1** or possess a *Markovian property*, e.g.,

(34) $$\omega_2 = \mathbf{H}\omega_1 + \omega'$$

with ω' independent of ω_1 and $\mathbf{H}$ a fixed transition matrix. For T-stage models, the transition matrices $\mathbf{H}$ can be stage dependent what gives

$$\omega_t = \sum_{\tau=1}^{t-1} \mathbf{H}_\tau \omega_\tau + \omega'_t \quad \forall t \tag{35}$$

with ω'_t independent of $\omega_1, \ldots, \omega_{t-1}$. It means that the random parameters ω_t in stage t can be represented as a *sum of interstage independent random summands* related only to stages $1, \ldots, t$. Notice that (34), (35) correspond to the mentioned special form of conditional distributions, see (25).

Conclusions. The upperbounding techniques based on the first order moment information carry over to multistage stochastic linear programs with complete recourse and with random right-hand sides that are linear in random parameters ω only in special cases, e. g., when one of the following conditions holds true:

- right-hand sides are interstage independent;
- for all stages, the right-hand sides can be expressed in the form of a sum of interstage independent random vectors related to preceding stages and to the given stage, see (35);
- for all stages, the conditional distributions of random parameters ω_t are carried by simplices whose extremal points are linear in past values of $\omega_1, \ldots, \omega_{t-1}$ whereas the barycentric coordinates of the conditional mean values do not depend on this history; see **A1**.

Parallel conclusions can be derived for multistage convex stochastic programs with random right-hand sides and also for the convex-concave case with random right-hand sides and recourse costs.

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