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Summary

This paper presents the results of investigation of nonlinear mathematical models of the behavior of closed lakes using the example of the Caspian Sea.

Forecasting the level of the Caspian Sea is crucial both for the economy of the region and for the region's environment. The Caspian Sea is a closed reservoir; it is well known that its level changes considerably due to a variety of factors including global climate change. A series of forecasts exists based on different methods and taking into account some of the following factors: the influence of the sun's activity; the atmospheric circulation; the changing shape of the world's ocean; geological phenomena; the river inflow; and the velocity of evaporation. All of these models were calculated based on the linearization of the relations considered.

For the last two decades, the most popular model has been the linear stochastic equation of water balance. This model was used as the base of the well known project of reversing the flow of the northward-flowing rivers. But the real behavior of the Caspian Sea contradicted the forecasting done using this model. One of the reasons of the failure was ignorance of the relations mentioned above. We are inclined to think however that the main reason for failure was that the forecast used a linear equation.

The goal of the present paper is to analyze and generalize, from the modern mathematical point of view, the forecasting methodology for the level of the Caspian Sea, including the nonlinear effects crucial influence on the dynamics of sea level. In particular, the mathematical problems concerning the nonlinear stochastic equations are considered. The results are partly published ([2],[3], [5]), and partly supported by a Grant from the Center for Scientific Research of Russia and by the "Dynamic Systems" project at the International Institute for Applied Systems Analysis (IIASA).

Nonlinear Stochastic Models for Water Level Dynamics in Closed Lakes

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1 Overview of present methods of forecasting the level of the Caspian Sea

The literature devoted to the study and forecasting of the level regime of the Caspian Sea amounts to hundreds of papers with a variety of approaches and points of view. Here we give a retrospective analysis of the typical methods, their classification, and possible approaches for generalization.

The Caspian Sea has a rather large-scale catchment. The level regime of the Caspian Sea is influenced by different factors: the influence of the local factors on the sea level is insignificant, but any more or less significant global climatic, hydrological and hydrogeological changes reflect in its behavior. For working out nonlinear models for forecasting the level of the Caspian Sea, the problem of accounting for the hydrological aspects of the long-term climate change is the most essential. At present, there is a considerable experience in the study of the Caspian Sea yielding a large experimental base and a great amount of theoretical research.

Let us briefly renew the basic data concerning the Caspian Sea and especially its level regime. The data from geological and historical studies show permanent and frequently essential changes in the level of the Caspian Sea. One can see in the papers of Appolov, Berg, Michajlovski, Klige, Nikolaieva and other investigators, the run of the level of the Caspian Sea over the last three and half centuries. The common issue of these papers is that raisings and droppings are typical for the level of the Caspian Sea, with the amplitude of the oscillation being about 5 meters. Appolov stated that, in some periods from the tenth to the sixteenth centuries, the level dropped to 12 meters below the present level.

Regular observations of the sea were organized by Peter the Great, who created the first measuring station in Baku. Reliable continuous observations seem to begin from the middle of the last century. Since that time up to now, the level of the sea oscillated around minus 26 meters of BS; from 1930 the sea began to fall dramatically and dropped more than 1 meter. After that, the level oscillated, with a visible decreasing trend until 1977 when the level overcame the least value minus 29 meters of BS. From that time, a phase of continuous rising began. Now the level has achieved the value of minus 27 meters of BS.

The level of the Caspian Sea is significant because the management of industry along the coast of the Sea and the value of the coastal industry is closely related to the level of the Sea. Unpredictable change of sea level costs much. For example, according to estimates by the specialists in ichtiology a decrease in sea level below minus 29.5 meters would lead to an unfavorable situation for fish (it should be noted that the present bad situation with the fish is due to an unsatisfactory ecological situation). On the other hand, a rise in sea level higher minus 27 meters leads to flooding of the coasts used by the population. All this shows that the long-term forecast of the level regime of the Caspian Sea has a great significance.

Let us now characterize and classify the typical approaches to the problem.

1.1 Approach related to studying and using Sun-Earth connections

This approach is based on the idea that changes in Sun activity have an influence on the climate. Chizhevskii seems to have been the first to put forward this point which has caused considerable objections. The first objection is the absence of visible physical connections between Sun activity and global climatic parameters.

Naturally, the direct correlation between Sun activity and the level regime of the Caspian Sea proves negligible because the behavior of the Caspian Sea depends on a lot of variables. A more proper conjecture might be a connection between the Sun activity and the average river outlet; but even in this aspect a direct correlation is too much to hope for. One should note that the modern climatological community is skeptical of the existence of a correlation between Sun activity and the hydrological process.

The second objection is that the energy from changes in Sun activity, when they finally reach Earth, would have little if any effect on atmospheric processes.

There are counter-objections. As regards the first point, it is hardly surprising that there is no visible physical mechanism of the influence of changes in Sun activity on the atmosphere; these processes are too complicated to be described by any conceptual model. As for the second point, analogies from other sciences may be useful. In biophysical systems, there are some processes (neuron nets) that have low-energy levels. But it is not unusual that such processes have more impact on the biophysical system than high-energy processes do. Hence low energy, in itself, is not enough to disprove the phenomenon.

The weak correlation between Sun activity and the hydrological processes does not mean that Sun activity has no influence on these processes. Moreover, there are a lot of data showing that both processes are connected with each other. As a confirmation, one can look at the forecast of the level regime of the Caspian Sea according to M.S.Eigenson (1957). This forecast, done only on the base of connection with the Wolf numbers, proved to be quite accurate. Namely, in correspondence with the Eigenson forecast, the level of the Caspian Sea in the period from 1970 to 2000 should be high, and at the end of that period should achieve the value of 1930. Thus, the evolution of Wolf numbers reflects real features of the phenomenon in question.

1.2 The Gulf Stream and others processes

An approach similar to that based on the Sun-Earth linkage is an attempt to take into account the impact of the Moon and the effect of the changes of the Earth's rotation and the tidal waves. This approach was elaborated by I.V. Maximov and his school. The main idea is that the fluctuation of the Gulf Stream intensity has impacts on the European climate. This brings up the question: what are the causes for the fluctuation of the Gulf Stream intensity? It is supposed that one of them is the long-periodic components of the ocean's tides. Recall that the period of the Moon orbit is approximately 18.6 years. Another cause is the change of the centrifugal forces under the influence of the Earth axis rotation. The period of deviation of the deformational potential which is the result of the beating of a 14-month-long and a years-long waves is approximately 6-7 years. All these factors and corresponding periods are correlated with the changes in rivers flow. However, this approach was developed by Maximov's school only qualitatively.

An actual and difficult mathematical problem is to calculate fluctuations in the form of the World's Oceans under the action of above mentioned forces, and then to calculate the corresponding fluctuations of intensity of the main ocean currents which have considerable impact on the global climate.

1.3 Atmospheric processes

The impact of the World's Oceans on the climate is realized through the atmosphere. Many forecasts that take into consideration atmospheric processes exist. The most valuable contributions in this area are due to G.J. Vangengame, A.A. Girs, B.L. Dzerdzeevskii, and others who elaborated the theory of atmospheric circulation of the Northern Hemisphere and correlated changes of river flow and of the level of the Caspian Sea with these types of circulations. It was shown that the domination of the Eastern circulation leads to aridization, the meridian-type (and moreover the Western) circulation leads to humidization of the territory.

The book of K.I. Smirnova [30] illustrates vividly the essence of this method for long-term forecasts of the Caspian Sea level. Smirnova's approach is based on the establishment of a multidimensional correlation between the level of the Caspian Sea and the Barical fields of the atmosphere. To estimate the Barical field, the indices of Belinskij-Kalinin cyclonic activity were taken at the most influential points of the Northern Hemisphere (Azores as the maximum, Iceland as the minimum among others). In 1970, K.I. Smirnova made a forecast for 19 years, which justified with high accuracy: the prognosis and the real behavior of the Caspian Sea level differ for about 30 santimeter.

The high accuracy of this forecast shows that methods that take into account Barical fields in the atmosphere have a good perspective.

1.4 Periodicity of processes

It was mentioned above that there are periodical (or more accurately, nearly periodic) processes that have an impact on the level of the Caspian Sea. There exist works whose authors select periodic components that correspond to the changes in the Caspian Sea level (L.S. Berg, B.A. Shljamin, A.V. Shnitnicov among others). L.S. Berg and A.V. Shnitnicov use the a priori taken periods while B.A. Sljamin investigates the spectrum of the level and corresponds the frequencies obtained to periods of exogeneous forces. A forecast based on this approach, showed a good correspondence with the real level behavior.

It seems reasonable that the sea level kinetics (or, preferably, sea-volume kinetics) is an almost periodic function. However, it would be better to interpret its main frequencies during the process of modeling rather than after it. This approach needs more consideration.

1.5 Water-balance methods

One should notice that from the middle of the 70s sciences opinions in the hydrology began to be a subject of unification. All ideas and approaches described above were considered to be nonscientific and the water balance stochastic methods were dominant. The first papers on forecasting the level of the Caspian Sea that used stochastic methods were these of S.N.Kritsky and M.F.Menkel. These papers assumed the dynamics of the level of the Caspian Sea to be described by the linear difference equation of water balance

$$w(t+1) = w(t) + V(t) - e(t)F(w) - V_{wd} - V_{KBG}, \tag{1}$$

where $w(t)$ is the volume of the sea year t, $V(t)$ is the inflow, $e(t)$ is the (averaged over space) evaporation coefficient (the difference between the quantity of evaporated water and rainfall per a unit of square); $F(w)$ is the square of the water surface, which changes according to the bottom and coast reliefs; V_{wd} is the withdrawal from the inflow of the catchment of the Caspian Sea, V_{KBG} is the outflow to the Gulf of Kara-Bogaz-Gol. It was assumed that the $V(t)$ and the evaporation coefficient $e(t)$, being the main determinents of the dynamics of the level of the Caspian Sea are stationary stochastic processes.

It was postulated that within a considered interval of sea levels the dependence of a surface square from a level is linear; in other words, the function $F(w)$ was approximated by a linear function of variable w. For forecasting the stochastic inflow and evaporation coefficients in equation (1) were replaced by their mathematical expectations. This gave a solution of the linear difference equation with constant coefficients:

$$w(t+1) = w(t) + \bar{V} - \bar{e}F(w) - V_{wd} - V_{KBG}. \tag{2}$$

It was assumed that the solution of equation (2) is the mathematical expectation of the solution of equation (1). Thus, the method provided a forecast mathematical expectations of the sea level, and corresponding confidence intervals. So far as for constant $\bar{V}$ and $\bar{e}$ the equation (2) has the single stationary point, it is clear that the solutions of equation (2) are processes whose expectations asymptotically converge to the single equilibrium, being in a sense stable. Thus, due to the above method, if there is constant withdrawal of water for industry, the sea level tends to a single equilibrium level, which was called the attractor level. In the case of falling withdrawal, the method forecasts monotonical decreasing of the sea level.

It was proved that the above forecasting methodology is inaccurate. In reality, starting from 1977 the Caspian Sea has been rising, and at present its level has achieved the value minus 27 meters of BS.

It should be admitted that a purely statistical method based on a linear model of water balance failed. It seems that the main reason of failure was ignoring so-called hidden periodicities mentioned in sections 1.2, 1.3, 1.4, and the assumption of linearity of the water balance equation.

1.6 Nonlinear models

Recently, approaches connected with nonlinear behavior of the Caspian Sea have been used. The nonlinear conjecture was stated probably for the first time by M.I. Zelikin, I.F. Zelikina and J. Schultze, who noticed that rejection of linearity leads to the so-called quantification of asymptotics; that is the existence of several attractor levels. They conjectured that the behavior of the Caspian Sea can be treated as a random walk between these characteristic levels. This conjecture obtained a partial confirmation in our research of nonlinear stochastic equations. We showed that, in the case of the stochastic differential equation with cubic nonlinearity that describes the water balance of the Caspian Sea, there are no more than three stationary solutions, two of which are being stable (i.e. they are attractor levels). Any solution asymptotically converges to linear combinations of stable attractor levels.

Recently, Naidionov and Podsiechin showed that heat processes accompanying evaporation leads to nonlinearity in the water balance equation. The approach that takes into account nonlinear effects seems to be promising for forecasting the level of the Caspian Sea. Meanwhile, to specify nonlinear dynamics of evaporation in areas of low water, it would be desirable to investigate the phenomenon of multiple reflection of sunlight from the bottom and the surface, which from the qualitative point of view explains rapid heating of water at low depths and the corresponding increase in evaporation. The quantitative effects of this phenomenon are unexplored and need serious attention.

1.7 Morphometric problems

For forecasting global climate changes one should establish the causes and the dynamics of climate changes in previous ages. The placement of terraces along the coast line is used for reconstruction of the ages of climate. Such work has been done, for example, for Lake Victoria and partly for the Caspian Sea. In this respect it seems that the most important problem is to describe the interaction between the Sea and the coast. It seems clear that the value of the level when the square of shallows is maximal should be a stable state because the rise of the level beyond this value should cause a rapid increase in the square of evaporation, and a drop of the level below this value should cause the reduction of the square. On the other hand it is known that long-term standing of the Sea level at the same value also causes the creation of coastal terraces.

2 Some mathematical problems related to a nonlinear stochastic model of water regimes for a closed lake

2.1 A nonlinear stochastic water balance equation

A level regime of a closed lake is described by a water balance equation being a first-order differential equation for a function $w(t)$ whose value is a volume, or a level, of the basin at time t:

$$w'(t) = v(t) - a(t)F(w(t)). \tag{3}$$

Here v is a rate of the water inflow, F is a water surface square, and a is an evaporation coefficient. The functions v and a are assumed to be stationary stochastic processes, and F describes the dependence of a water surface square on w, according to the geometric structure of the coast of the basin. The most impressive example of a closed basin is Caspian Sea level of which has a great influence on the economic life of the region. In most prevalent models, the function F is as a rule assumed to be a linear function ([1]). But a simple analysis of data in the period of the last century shows that the assumption of linearity of the function F gives forecasts that do not reflect the actual situation ([2]). More careful analysis of the data shows that one should take into account a dependence of the evaporation coefficient a on the level w. Therefore, the second term of the water balance equation should be represented as the product of coefficient a and the so-called effective water surface square F^*. The water balance equation has, in general, the form

$$w'(t) = v(t) - a(t)F^*(w(t)). \tag{4}$$

The value $F^*(w)$ describes a more complicated dependence than that on the geometrical structure of the coast; it takes into account the dependence of the evaporation velocity on the basin bottom geometry, the temperature distribution in various water stratums, the structure of the distribution of moisture into the land in a neighborhood of basin etc.; all these are constant external conditions. There are physical arguments for considering the factors mentioned above ([4]). The analysis shows that the function F^* is essentially nonmonotonic. If F is a polynomial of an odd degree, one can estimate the number of stationary solutions and find the asymptotics of solutions ([3]). This would result in more explicit methods for forecasting level regimes.

2.2 Existence and structure theorems

One can consider a more general case. Let $X = Y \times I$, and $\varphi : Y \longrightarrow Y$ be a transformation preserving the measure μ on the space Y. Then one can consider a dynamical system on X, i.e. a uniform flow along the second coordinate $t \in I$ switching under the transformation φ from the top to the bottom of the cartesian product $Y \times I$. Then one can consider a stationary stochastic process as a function $f(y,t)$, $y \in Y$ such that

$$f(y, t+1) = f(\varphi(y), t).$$

Consider a differential equation

$$\frac{dw}{dt} = F(w,t), \tag{5}$$

where the function $F(w,t) = F(w,y,t)$ is a polynomial with respect to w, of an odd order n, belongs to $L_1(Y \times I)$ for each value of w, and is a stationary process for each w. Assume that the highest coefficient of polynomial F equals -1. Let $L_\infty(Y)$ be the space of essentially bounded measurable functions on a compactum Y with a measure μ, the norm of which is given by

$$\|f\|_\infty = \mathbf{esssup}_{y\in Y}|f(y)| \tag{6}$$

An $w(y,t) \in L_\infty$ is understood to be a solution of equation (5) if

$$w(y,t) = w(y,0) + \int_0^t F(w(y,\tau), y, \tau) d\tau \tag{7}$$

for almost all $y \in Y$ (with respect to the measure μ).

Theorem 1 *Equation (7) has at least one stationary solution.*

The structure of the set of stationary solutions w can be described in the following way. This set is linear, and ordered with the natural ordering: if $w_1(y,t)$, $w_2(y,t)$ are two stationary solutions, then for almost all $y \in Y$, one has

$$w_1(y,t) \le w_2(y,t), \text{ or } w_2(y,t) \le w_1(y,t). \tag{8}$$

There are two stationary solutions,

$$w_+(y,t) \ge w_-(y,t), \text{ for almost all } y \in Y,\ t \in I, \tag{9}$$

which are maximal and minimal respectively:

$$w_-(y,t) \le w(y,t) \le w_+(y,t), \text{ for almost all } y \in Y,\ t \in I. \tag{10}$$

Consider a stationary solution $w_0(y,t)$ of equation (7). There is a sufficient condition which guarantees that the solution $w_0(y,t)$ is isolated. One can write this condition in the following way:

$$\int_{X=Y\times I} \frac{d}{dw} F(w_0(y,t),t) d\mu dt \neq 0. \qquad (11)$$

For the maximal and minimal solutions $w_{\pm}(y,t)$, the condition (11) is supplemented by the following one:

$$\int_{X=Y\times I} \frac{d}{dw} F(w_{\pm}(y,t),t) d\mu dt \leq 0. \qquad (12)$$

In case $n=3$ one can say more.

Theorem 2 *In the case where $n=3$ the number of stationary solutions is no bigger than three.*

Two numbers

$$A_+ = \int_{X=Y\times I} \frac{d}{dw} F(w_+(y,t),t) d\mu dt \leq 0 \qquad (13)$$

and

$$A_- = \int_{X=Y\times I} \frac{d}{dw} F(w_-(y,t),t) d\mu dt \leq 0 \qquad (14)$$

gives information on the real number of stationary solutions. In the case where both of these numbers are negative, one has three different solutions. If one number is negative and other equals zero, then there are two stationary solutions.

In the case where there exists a single stationary solution, then these numbers equal zero.

2.3 A system of differential equations

In a similar way, one can consider the system of finite numbers of differential equations

$$\frac{d\vec{w}}{dt} = F(\vec{w},t),\ \vec{w} \in \mathbf{R}^m. \qquad (15)$$

The conditions for the polynomial function F is similar:

$$(\vec{F}(\vec{w},y,t),\vec{w}) \leq -C|w|^{n+1},\ C \text{ is a positive number.} \qquad (16)$$

Theorem 3 *There exist a space X' with a measure μ', a dynamic system $g'_t : X' \longrightarrow X'$, and a projection $X' \longrightarrow X$ commuting with the actions of the one-parameter group of transformations such that there exists at least one stationary solution $\vec{w}(x',t)$, $x' \in X'$ of the equation (16).*

2.4 Conjecture on a nonlinear mapping to be Fredholm: a verification for an example of a stochastic differential equation

We showed ([3]) that the stochastic equation

$$\frac{dw}{dt} = v(t) + a(t)F(w) \qquad (17)$$

for F being a polynomial of the third order, may have no more than three stationary solutions.

The two questions arise. The first is to clarify in what cases the equation (5) has exactly k (where $k = 1, 2, 3$) stationary solutions.

The second is, to estimate the number of stationary solutions of the equation (5) for $F(w)$ being a polynomial of an odd order with separate roots. Attempts to use the technique of equation [3] failed. Here we suggest to use methods of the theory of Fredholm nonlinear mappings (see [36], [37], [38], [39]).

Equation (16) determines the mapping $G : L_\infty \longrightarrow L_\infty(Y)$ in the following way. Let $w_0(y) \in L(Y)$ and $w(y,t)$ be the solution of equation (16) with initial condition $w(y,0) = w_0(y)$. Then put

$$G(w_0(y)) = w(y,1).$$

Denote by

$$U : L_\infty \longrightarrow L_\infty$$

the linear operator defined by

$$(Uf)(y) = f(\varphi(y)), \tag{18}$$

where $\varphi : Y \longrightarrow Y$ is the ergodic automorphism present in the definition of the one parametric transformation group

$$g_t : X = Y \times I \longrightarrow Y \times I = X.$$

Let

$$F(w) = G(w) - U(w). \tag{19}$$

In these terms, the problem can be formulated as that of estimating the cardinality of the set $F^{-1}(0)$. The aim is to study whether the mapping

$$F : L_\infty(Y) \longrightarrow L_\infty(Y)$$

is a Fredholm mapping using the methods of the degree theory of nonlinear Fredholm mappings of Banach manifolds.

In some special cases we are be able to establish that the nonlinear mapping (19) is Fredholm. For example the equation

$$\frac{dw(y,t)}{dt} = (-2 + \cos 2\pi t)(w^3(y,t) - w(y,t)) \tag{20}$$

has the Fredholm Frechet derivative at a stationary solution.

2.5 The dimension of the kernel of the differential of nonlinear mapping of equation (19) equals to 1.

In a common case the following statement is true.

Theorem 4 *The dimension of the Frechet derivative of (19) is no bigger than* 1.

3 Optimal stochastic cubature formulas and applications to a problem of estimation of the level of closed lakes

In forecasting the Caspian Sea level, level observation results play the most important role. In view of the great size of the reservoir, the results of the measurements made simultaneously at distant points differ significantly (up to 20-30 sm, which is very big if one takes into account the immense area of the Caspian Sea). Such large deviations could be partly explained by discrepancies in geodesic data. But the main fact is that the surface of the Caspian Sea is curved due to the action of winds and seishian waves. The standard method of estimating the Caspian Sea level is to taking the arithmetic mean of data from all (or the four basic) stations.

That leads to the following questions:

-Does this practice give a satisfactory result?

-Is it possible to use available data in a better way?

-What are optimal methods to estimate the level (or more accurately - the volume) of the Caspian Sea?

It is easy to see that all these questions relate to the abstract mathematical problem of finding a best cubature formula for a given number of knots. Unlike the standard theory of best cubature formulas, we interpret the term "best" in a stochastic sense, namely in the sense of minimal mean square deviation with respect to a certain Gaussian measure in a space of functions associated with a relevant stochastic process. This section is devoted to an exploration of this problem.

Let X be a manifold with a measure dx, and (F, H) be a pair of spaces which define a Gaussian measure $d\mu$ on the space of functions on the manifold X. The integral

$$I = \int_X f(x)dx \tag{21}$$

defines a linear functional on H . Consider the linear functionals

$$\delta(x - a_i) = f(a_i) \tag{22}$$

where a_i are fixed points of the manifold X. Choose k functionals of the form (22) (the number k is fixed) and take all their linear combinations.

$$\sum_{i=1}^{k} \lambda_i \delta(x - a_i) \tag{23}$$

These functionals generate a continual basis of a space H^*, each element of H^* can be represented as a continual linear combination of elements of the form (22) (see the theorem on the expansion by δ-functions, Shilov [31]).

The problem is to choose points a_i and coefficients λ_i so as to reach the best approximation to the functional (21). The standard approach of the theory of cubature formulas is to find formulas which are exact on the first m polynomials of a certain orthogonal system. That gives the possibility to estimate the maximal deviation of the functional (21) on the unit sphere in space H (in the sense of the norm in the conjugate space) from the finite-dimensional set (23) (see Babenko [32], Sobolev [33], Mysovskikh [34] et al.)

Here another approach using notations of the probability theory is suggested. Consider a Gaussian measure $d\mu$ on the functional space of H. We estimate the deviation between functionals using the integral (sometimes called continual, or Feynman) over the infinite-dimensional space H equipped this measure.

$$\int_H \left(\sum_{i=1}^{k} \lambda_i f(a_i) - \int_X f(x)dx \right) f(a_i) d\mu \to \min \qquad (24)$$

Problem 1 *Minimize the functional (3) with respect to $\lambda_i, a_i, (i = 1, ...k)$.*

Here we give the key definitions and the formulations of the basic assertions resulting our investigations of this problem.

Suppose first that points $a_1, ...a_k$ are found. Then to find constants $\lambda_i, (i = 1, ...k)$ which minimize the functional (24), it is sufficient to differentiate it in $\lambda_i, (i = 1, ...k)$.

$$\sum_{i=1}^{k} \lambda_i \int_H f(a_i) f(a_j) d\mu = \int_H f(a_j) \int_X f(x) dx d\mu$$

This system of linear equations is uniquely solvable with respect to λ because the matrix of this system is nondegenerate (having the determinant of Gramm for the functionals $\delta(x - a_i)$). The solution of this system gives the minimum to the functional (3) since the latter is convex in λ. Hence the problem is reduced to finding of points $a_1, ...a_k$.

Let us consider the case where the manifold X is a compact Riemanian homogeneous space with the group of isometries G.

Lemma 1 *The measure $d\mu$ is invariant with respect to the action of the group G*

Definition 1 *A net S of points of the manifold X is called homogeneous if the subgroup $K \subset G$ of the isometries of the space X acts transitively on this net.*

A connection of the notion of a homogeneous net to the Clifford-Klein problem of space forms for the homogeneous space X is evident (see Wolf [35]).

Definition 2 *The freedom coefficient of a knot a_i of a net S is defined to be the minimum of the distances of this knot to the set of all other knots of S.*

Definition 3 *A variation of the net S is called releasing if under this variation, the freedom coefficients of all knots do not increase.*

Definition 4 *A net S is called maximal if no one of its freedom coefficients strongly increases under sufficiently small releasing variations of S.*

Conjecture 1 *A solution of problem 1 determines a maximal net on X.*

Theorem 5 *If S is a homogeneous maximal net, then $\lambda_1 = \lambda_2 = ... = \lambda_k$.*

Example 1 *An equidistant set of points in a circle is a homogeneous maximal net.*

Example 2 *Vertexes of Platonic solids provide homogeneous maximal nets on spheres*

An important point is that sets of the roots of Chebyshev's polynomials are the projections of the equidistant nets in the circle symmetrical in the sense that each root of a Chebyshev's polynomial is the projection of exactly two points in the net. Consider the Gaussian measure $d\nu$ on the space of functions on the interval [-1,1] being the projection of a Gaussian measure in the space of functions on the unit circle.

Theorem 6 *The roots of Chebyshev's polynomials provide a solution of the Problem 1 of finding a best stochastic cubature formula for the interval [-1,1] and the measure $d\nu$.*

This approach opens up possibilities to find the best stochastic cubature formulas for various manifolds with a boundary

1. Let us consider the orthogonal projection π of a circle in R^3 onto the ellipse $x^2/a^2 + y^2/b^2 = 1$ in the plane O_{xy}. Let $d\lambda$ be the projection of the Gaussian measure on the space of function on the circle onto that on the ellipse which is induced by π.

Theorem 7 *The image of an equidistant net on the circle give a solution of the Problem 1 of finding a best stochastic cubature formula for the ellipse and the measure $d\lambda$*

2. Consider the sphere S^2 and a maximal net Q on it symmetric with respect to the orthogonal projection p on the plane O_{xy}. That means that each point of pQ is the projection of two points from Q. Let $d\kappa$ be the Gaussian measure on the disk induced by p.

Theorem 8 *The net pQ gives the solution of the problem 1 of finding a best stochastic cubature formula for the disk and the measure $d\kappa$.*

3. Consider the orthogonal projection q of the sphere S^2 onto a domain $U \subset R^2$ bounded by an ellipse. Consider a maximal net T on the sphere symmetric with respect to this projection in the previous sense. Let $d\sigma$ be the Gaussian measure on U induced by q.

Theorem 9 *The net qT gives a solution of the problem 1 of finding a best stochastic cubature formula for the domain U and the measure $d\sigma$.*

Using this method, optimal stochastic cubature formulas for domains represented as projections of homogeneous spaces can be obtained.

4 Some proposals on scientific themes

4.1 Background and Significance

The Caspian Sea is isolated from the world's oceans and hence can be treated as a closed lake. The amplitude of fluctuation in its level ranges up to several meters which results in over hundreds of kilometers in width. The change of level has a great impact on the precaspian region: land use, transport, etc. From 1930 to 1940 the level of the Caspian Sea suddenly dropped 1.5 meters, and after that slowly fell with some fluctuations until 1979. From the beginning of 1979 the level has been raising steadily, and now it has increased some 2 meters.

This brings up the question: Is climate change the cause of the rising sea level, or is it a regional phenomenon? The rising of the Caspian Sea level hurts the economy of the precaspian region, which has been declared a disaster. In the seventies (the period of decrease in the level of the Caspian Sea) the Plenum of the Communist Party (KPSU) ratified the program of irrigation of the southern part of the USSR. The idea was that this program would stimulate the process of decreasing the level of the Caspian Sea. Moreover, the forecast based on linear models predicted that the Caspian Sea level would fall catastrophically. The project of turning northward-flowing rivers back to the south was aimed (among other goals) at feeding the Caspian Sea. The realization of this project would be extremely dangerous for the northern part of Russia. That is why representatives of the departments of culture and science were against this project. However, rejecting the solution of the KPSU Plenum was a difficult task.

In 1983 a group of mathematicians (including the authors of this proposal) proved that the forecasting of a decrease in the Caspian Sea level was erroneous. The subsequent rise in the sea level reinforced our opinion. The group conclusion caused that the project of turning northward-flowing rivers back to the south was cancelled in 1986.

Since that time our group has worked on applications of fundamental mathematics to environmental problems (Leningrad Dam, Volga - Chogray channel, and others). We proposed and justified a new nonlinear model of the level dynamics of closed lakes.

The former linear model of the level dynamics of closed lakes was a linear stochastic differential equation with stationary coefficients. The equation has the single stationary solution, which defines the asymptotic behavior of all other solutions. The mathematical expectation of the stationary solution was called the attracting level. Our nonlinear model is described by a nonlinear stochastic differential equation [2, 3]. We proved that this equation has several stationary solutions that define the asymptotic behavior of all other solutions. Hence we obtain several attracting levels (the phenomenon of quantization of levels), which explains the process of fluctuation as transfers from one attracting level to another without resort to substantial structural change in environment.

In general, we believe that at any rate the massive part of so-called structural changes in environment is due to a nonlinear nature of the processes involved.

This general view requires treatment of the problem at a highly abstract level. Modern mathematics is developing towards explicit solutions of substantially nonlinear problems arising in such areas as infinite dimensional completely integrable Hamiltonian systems, partial differential equations of Korteweg-de Vries type, global analysis, quantum groups, etc.

We intend to use the modern mathematical technique (and to elaborate a new one) for solving nonlinear models, and identification of its unexpected (from the point of view of linear theory) asymptotic behaviors.

We believe that similar methods can be used for forecasting the level for big closed lakes (the Dead Sea, Victoria Lake, Salt Lake, and others).

4.2 Research design

Usually in linear models the evaporation rate is proportional to the level of a lake. Our statistical estimates show a nonlinear dependence of the evaporation rate of the Caspian Sea on its level.

To take this fact into account, we consider a differential equation with a polynomial right-hand side. Let the degree of the polynomial be n, and its coefficients be stationary stochastic processes. We consider the specific but relatively common case where these processes are generated by an ergodic transformation of a smooth manifold to itself. This assumption gives an opportunity to using geometrical methods.

The stochastic differential equation defines a nonlinear operator on the infinite-dimensional space of distribution functions.

Conjecture: The derivative of this operator is a Fredholm linear operator of index 0, and the degree of the operator equals n.

This conjecture was proved for $n = 1$ and $n = 3$ (n must be odd to be physically admissible). It was also proved that in the case of $n = 3$ there exist at most 3 stationary solutions and the asymptotic behavior of an arbitrary solution is described as a linear combination of stationary ones.

Big lakes as a rule are nonhomogeneous. A strange effect is stratification in depth, temperature, and salinity. Various strata have different physical and chemical characteristics such as evaporation rate, density, etc. A river flowing into a lake contributes an additional stratification, because, as it is known, its flow continues in the lake as an internal flow reaching tens and hundreds kilometers (for instance, the flow of the Volga inside the Caspian Sea is observed up to 270 km). The turbulent exchange between strata leads to a consideration that a big lake is in fact a system of reservoirs (each stratum being a separate object). One obtains a system of nonlinear stochastic differential equations with stationary coefficients (in some cases, a distributed parameter system) that can serve as a model for closed and open lakes, or a system of reservoirs.

We conjecture that this system has at least one stationary solution which determines the asymptotic behavior of all other solutions.

This is supposed to extend the investigation of this and related mathematical problems. For instance, it is desirable to find the exact asymptotics for some integrals of stationary processes. This investigation may reveal new qualitative phenomena in the level dynamics of lakes and environmental processes.

5 Supplement 1

Definition of the nonlinearity of the dependence of the evaporation on the level of the Caspian Sea

Research on the behavior of the dependence of evaporation from the surface of the Caspian Sea on the sea level was provided using statistical tools and is based on data of the sea level, the river inflow, precipitation and the square of the sea surface, taken from a single source where these data are given for each month from 1925 to 1969 [30]. Taking into account the need for a large sampling and the fact that measurements were done only during a 100 period, it seems the best to deal with the monthly data. It should be noticed that in this case one should take into account some non-homogeneity of the data.

For determining evaporation, the water balance equation was written in the form

$$\frac{dw}{dt} = a(t) + c(t) - b(t, h), \tag{25}$$

where $w(t)$ is the volume of water in the Caspian Sea, $a(t)$ is the inflow, $c(t)$ is precipitation, $b(t, h)$ is evaporation from the sea surface, h is the sea level. Taking into account the non-homogeneity from month to month of all data, the additional parameter $j = 1, \dots, 12$ (the number of month) was introduced. Evaporation was modeled as the product

$$b(t, h) = y(j)x(h), \tag{26}$$

where $y(j)$ corresponds to the monthly average evaporation at month j and $x(h)$ does not depend on month. Finding evaporation from the water balance equation is stipulated by the fact that the accuracy of measurement of evaporation is considerably lower than that of other components of the equation. Our aim was to estimate the behavior of the parameter $x(h)$. For quantification of the problem, the domain of the sea level was decomposed into equal intervals, indexed by $k, k = 1, \dots, K$. In our case $K = 15$ and the interval were equal to 20 cm. The evaporation function was written as an array $B(i, j)$

or $B(i,j,k)$, depending on a year ($i, i = 1,\dots,45$, $i = 1$ corresponding to 1925, $i = 45$ corresponding to 1969), a month (j), and a sea level k (implicitly). Different numbers i and j correspond to different values of k . For averaging, we used the method of the least squares, i.e. the minimization problem of the function

$$\sum_{i,j,k}(B(i,j,k) - x(k)y(j))^2 \tag{27}$$

was solved; here $\{x(k)\}$ is the vector corresponding to the evaporation coefficient $x(h)$ and summation is taken over those i and j for which the sea level value h lies in the kth interval. The solution of this problem provides the system of the nonlinear equations

$$\begin{array}{rcl} \sum_{i,j}(B(I,j,k) - x(k)y(j))y(j) & = & 0 \\ \sum_{i,k}(B(i,j,k) - x(k)y(j))x(k) & = & 0. \end{array} \tag{28}$$

Let

$$b(j,k) = \sum_{i,j,k} B(i,j,k),$$

$n(j,k)$ be the number of different years $\{i\}$ for which $B(i,j,k)$ are defined, $B = (b(j,k))$, $X = (x(k))$, $Y = (y(j))$. Put

$$|X|_j^2 = \sum_k n(j,k)x(k)^2,$$

$$|Y|_j^2 = \sum_k n(j,k)y(j)_j^2;$$

$$D_X = \mathrm{Diag}\{|X|_1^2,\dots,|X|_{12}^2\},$$

$$D_Y = \mathrm{Diag}\{|Y|_1^2,\dots,|Y|_K^2\}.$$

In these notations the system (28) can be written as

$$BX = D_X Y, \quad B^*Y = D_Y X. \tag{29}$$

So far as the statistical sampling is sufficiently large, the matrices D_X and D_Y are invertible for $X \neq 0$, $Y \neq 0$, and consequently

$$D_Y^{-1}B^*Y = X, \quad D_X^{-1}BX = Y. \tag{30}$$

In the case $n(j,k) \neq 1$ there is no method of finding all solutions, but the method of tterations gives a solution which is in a sense stable.

The method was realized as a program in Basic. The initial value $Y = Y^{(1)} = (y_j^{(1)})$. We normalize it by

$$\sum_{j=1}^{12} y_j^{12} = 12.$$

The iterations for $X^{(n)}$ and $Y^{(n)}$ are

$$\begin{array}{c} X^{(n+1)} = D_{Y^{(n)}}^{-1}B^*Y^{(n)} \\ Y^{(n+1)} = D_{X^{(n+1)}}^{-1}B^*X^{(n+1)} \\ n = 1,2,\dots \end{array} \tag{31}$$

leading to

$$x_k^{(n+1)} = \frac{\sum_{i,j} B(i,j,k)y_j^{(n)}}{\sum_{i,j}(y_j^{(n)})^2}, \quad y_j^{(n+1)} = \frac{\sum_{i,j} B(i,j,k)x_k^{(n+1)}}{\sum_{i,j}(x_k^{(n+1)})^2}, \tag{32}$$

After each step one normalizes $Y^{(n+1)}$. The iterations quickly converge.

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