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Price Expectations, Cobwebs and Stability

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Preface

This new research project at IIASA is concerned with modeling technological and organisational change; the broader economic developments that are associated with technological change, both as cause and effect; the processes by which economic agents – first of all, business firms – acquire and develop the capabilities to generate, imitate and adopt technological and organisational innovations; and the aggregate dynamics – at the levels of single industries and whole economies – engendered by the interactions among agents which are heterogeneous in their innovative abilities, behavioural rules and expectations. The central purpose is to develop stronger theory and better modeling techniques. However, the basic philosophy is that such theoretical and modeling work is most fruitful when attention is paid to the known empirical details of the phenomena the work aims to address: therefore, a considerable effort is put into a better understanding of the 'stylized facts' concerning corporate organisation routines and strategy; industrial evolution and the 'demography' of firms; patterns of macroeconomic growth and trade.

From a modeling perspective, over the last decade considerable progress has been made on various techniques of dynamic modeling. Some of this work has employed ordinary differential and difference equations, and some of it stochastic equations. A number of efforts have taken advantage of the growing power of simulation techniques. Others have employed more traditional mathematics. As a result of this theoretical work, the toolkit for modeling technological and economic dynamics is significantly richer than it was a decade ago.

During the same period, there have been major advances in the empirical understanding. There are now many more detailed technological histories available. Much more is known about the similarities and differencers of technical advance in different fields and industries and there is some understanding of the key variables that lie behind those differences. A number of studies have provided rich information about how industry structure co-evolves with technology. In addition to empirical work at the technology or sector level, the last decade has also seen a great deal of empirical research on productivity growth and measured technical advance at the level of whole economies. A considerable body of empirical research now exists on the facts that seem associated with different rates of productivity growth across the range of nations, with the dynamics of convergence and divergence in the levels and rates of growth of income in different countries, with the diverse national institutional arrangements in which technological change is embedded.

As a result of this recent empirical work, the questions that successful theory and useful modeling techniques ought to address now are much more clearly defined. The theoretical work described above often has been undertaken in appreciation of certain stylized facts that needed to be explained. The list of these 'facts' is indeed very long, ranging from the microeconomic evidence concerning for example dynamic increasing returns in learning activities or the persistence of particular sets of problem-solving routines within business firms; the industry-level evidence on entry, exit and size-distributions – approximately log-normal; all the way to the evidence regarding the time-series properties of major economic aggregates. However, the connection between the theoretical work and the empirical phenomena has so far not been very close. The philosophy of this project is that the chances of developing powerful new theory and useful new analytical techniques can be greatly enhanced by performing the work in an environment where scholars who understand the empirical phenomena provide questions and challenges for the theorists and their work.

In particular, the project is meant to pursue an 'evolutionary' interpretation of technological and economic dynamics modeling, first, the processes by which individual agents and organisations learn, search, adapt; second, the economic analogues of 'natural selection' by which interactive environments - often markets - winnow out a population whose members have different attributes and behavioural traits; and, third, the collective emergence of statistical patterns, regularities and higher-level structures as the aggregate outcomes of the two former processes.

Together with a group of researchers located permanently at IIASA, the project coordinates multiple research efforts undertaken in several institutions around the world, organises workshops and provides a venue of scientific discussion among scholars working on evolutionary modeling, computer simulation and non-linear dynamical systems.

The research will focus upon the following three major areas:

1. Learning Processes and Organisational Competence. 2. Technological and Industrial Dynamics 3. Innovation, Competition and Macrodynamics

Price Expectations, Cobwebs, and Stability

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CONTENTS

ABSTRACT. There is given a market for several perishable goods, supplied under technological randomness and price uncertainty. We study whether and how producers eventually may learn rational price expectations. The model is of cobweb type. Its dynamics fit standard forms of stochastic approximation. Relying upon quite weak and natural assumptions we prove new convergence results.

Key words: cobweb models, adaptive learning, rational expectations, stochastic approximation, fixed point, stability.

1. INTRODUCTION

Consider an isolated market with a fixed production lag of *several* commodities which cannot be stored. The market clears in each and every time period $t = 0, 1, ...$That is, when properly defined, supply S always equals demand D. Formally, the equation system

$$S(x^t, \xi^t) = D(p^t, \xi^t) \tag{1}$$

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holds for all t. Here $S(\cdot,\cdot)$, $D(\cdot,\cdot)$ denote supply and demand "curves" in R^N, N being the number of goods at hand. $x^t \in R^N$ stands for the price vector that producers recently *believed* would come about at time t, whereas

$$p^t =: p(x^t, \xi^t) \in R^N \tag{2}$$

equals the *realized* version which then ensures market clearing. The input ξ^t, featuring in (1) and (2), is stochastic with a fixed distribution. Most important: ξ^t is still unknown at the time when all production decisions underlying $S(x^t, \xi^t)$ are made, and unveiled only just before the market opens at time t. (In technical terms, x^t is measurable with respect to the sigma-field generated by $\xi^0, ..., \xi^{t-1}$.) For interpretation one may envisage a regional market for agricultural goods, letting x^t be price expectations held when crops are planted, and ξ^t represent, say, weather variations during the growing season. We emphasize that the notion of market clearing should not be taken too literally. For example, it could be that $D(p^t, \xi^t) = \mathcal{D}(p^t, \xi_1^t) + \xi_2^t$, where the second component $\xi_2^t \in R^N$ of $\xi^t = (\xi_1^t, \xi_2^t)$ denotes *excess supply*. Whatever the particular specification might be, we ask:

(*) *In general, can producers eventually learn to form rational price expectations? If so, how?*

These questions have a long history in economic theory, and are key issues structuring so-called *cobweb models* [1], [20], [22].[1] Those models all fit the following unifying form: Immediately after market closure in period t price expectations are updated by the rule

$$x^{t+1} := (1 - \lambda_t)x^t + \lambda_t p^t,$$

that is,

$$x^{t+1} := (1 - \lambda_t)x^t + \lambda_t p(x^t, \xi^t). \tag{3}$$

Since $x^{t+1} = x^t + \lambda_t(p^t - x^t)$, the parameter $\lambda_t \in]0, 1]$ will naturally be called a *stepsize*. It strikes a balance between the most recent opinion x^t and new evidence p^t at stage t.[2] Note that the "learning process" (3) is totally driven by the realizations ξ^t, and requires virtually no insight on the part of producers into the workings or the structure of the market. To wit, neither the demand curve $D(\cdot,\cdot)$ nor the random mechanism generating the time series $\{\xi^t\}$ need be known. Nonetheless, under reasonable conditions we shall see that convergence to a steady state obtains all the same.

Evidently, (3) defines recursively a random process. Our interest is with its long term evolution. Thus, for given initial price prediction x^0 let the stochastic set $\omega(x^0)$

[1] Such models were introduced in the thirties (see [10] for a historical account) and have been central in economic dynamics ever since.

[2] Clearly, chosing $\lambda_t = 0$ would be nonsensical.

contain every almost sure cluster point of $\{x^t\}$, and such points only. As usual, $\omega(x^0)$ is a closed set. If $\lambda_t \to 0$, it will also be connected. The key questions (*) above can now be rephrased more precisely:

(*) *Will the set $\omega(x^0)$ of accumulation points reduce to a collection of singletons? If so, can convergence be ascertained? And then: do accumulation points embody rational price expectations?*

Below we shall provide reasonable conditions allowing positive answers to these questions. Thus we are somewhat at variance with traditional cobweb models which frequently tend to offer a distinctly more gloomy perspective. Indeed, employing

$$\lambda_t \equiv 1, \tag{4}$$

or so-called adaptive versions [23]

$$\lambda_t \equiv \lambda \in \,]0, 1]\,, \tag{5}$$

those models often feature divergence. Clearly, the completely myopic regime (4) can make for great instability, but this may also happen under (5).

Part of our motivation stems from some dissatisfaction with commonplace cobweb models comprising merely one commodity and ignoring uncertainty. Also, we find that recent studies display quite some predilection with instability [3], even chaos [8], [9], and often invoke rather ad hoc functional forms [13], [16], [17], [19]. Additional motivation, admittedly subjective, derives from our slightly optimistic view on the strength of equilibrating economic forces. Reflecting this view, our approach differs in many respects from the received literature. To wit, we demonstrate that the chances of seeing stable - possibly path dependent - rational expectations are fairly good in the long run. Besides this feature the analysis accommodates more than one commodity and incorporates genuine randomness at every stage. Also, no assumptions are made about the functional forms $S(\cdot,\cdot)$ and $D(\cdot,\cdot)$. At places, when needed, we shall merely require that the realized price vector $p(x,\xi)$ be continuous, or continuouly differentiable in x, and measurable in ξ.

The paper is organized as follows. Section 2 collects hypotheses and comments so as to justify and explain procedure (3). Section 3, which is the heart of the paper, provides the convergence analysis, and Section 4 concludes.

2. ASSUMPTIONS AND PRELIMINARIES

It is convenient to collect most hypotheses in one place. (Impatient readers, looking for main results, can proceed directly to Section 3.) We begin with

2.1. Stepsizes. As indicated, (3) is meant to portray grosso modo a learning process. Clearly, it is not our intention to present a particularly deep or innovative story about adaptation of price forecasts. Nonetheless, as such our modelling should reflect accumulation of producers' experience somehow. That is, over time their market behavior will presumably "mature" by way of assigning increasing weight to opinions synthesized from numerous observations. To account for such features we naturally posit that

$$\lambda_t \to 0. \tag{6}$$

As said, regimes with constant stepsize (4), (5) may cause divergence. This can also happen when stepsizes decreases too slowly. Therefore, we shall strengthen (6) by imposing the more restrictive condition

$$\sum \lambda_t^2 < \infty. \tag{7}$$

Although the significance of (7) appears difficult to grasp, it should be intuitively clear that it contributes towards avoiding long-run compounding of errors, and helps reducing permanent exposure to risky, extreme outcomes of ξ^t.

In the other direction of (6) and (7), if stepsizes are too small, the rate of learning may become exceedingly slow. New price evidence, brought by the endogenous process $\{p^t\}$, should not be overlooked, but rather offered chances to have a reasonable impact. To ensure so let

$$\sum \lambda_t = \infty. \tag{8}$$

In essence, (8) guarantees that substantial progress can be maintained, even when far from rational expectations. Indeed, if $Ex^t \to x$, then for remote enough times $T < T'$ we have

$$Ex^T \approx Ex^{T'} \approx x,$$

so that

$$Ex^{T'} \approx Ex^T + \sum_{t=T}^{T'-1} \lambda_t \left[Ep(x, \xi) - x \right].$$

Under (8) the last approximate equality is difficult to uphold, or it indicates a contradiction, unless $Ep(x, \xi) = x$. So, broadly speaking, (7) and (8) turn (3) into a time inhomogeneous, adaptive, learning process which is most effective initially, which never stops, but will dampen out asymptotically.

An important example, verifying (7) and (8), comes with $\lambda_t = \frac{1}{t+2}$. Then (3) is simply a recursive version of the empirical mean

$$x^{t+1} = \frac{x^0 + p^0 + \ldots + p^t}{t+2},$$

aggregating initial beliefs and subsequent observations into the prediction formed at time $t + 1$.

2.2. Uncertainty. The random entities $\xi^0, \xi^1, ...$ are all defined on a common probability space, not made explicit here, and they take their values in a fixed measurable space Ξ. For example, $\xi^0, \xi^1, ...$ can be real-valued variables or vectors in some fixed Euclidean space. We shall not elaborate on their precise nature or origin, but we do assume that they are independent, with a common distribution μ induced on Ξ.

2.3. Prices. We take it that predicted prices x^t as well as their realized outcomes p^t all belong to a compact convex subset X of the Euclidean space R^N, the integer N denoting the number of considered goods. Typically, $X := \Pi_{n=1}^N X_n$ where $X_n := [0, \overline{x}_n]$ for given upper price levels $\overline{x}_n > 0$. The compactness assumption is crucial in the analysis below. We regard it as innocuous though. For one thing, one might let each upper bound $\overline{x}_n$ equal the total abundance there is of money or numeraire. For another, compactness will naturally come about if for all goods $\int p_n(x, \xi)\mu(d\xi) < x_n$ whenever x_n is sufficiently large.

The initial belief $x^0 \in X$ is arbitrary, being determined either by accident or by historical factors not discussed here.

We tacitly assume that the market clearing condition $S(x,p) = D(p,\xi)$ yields a unique solution $p(x,\xi) \in X$ for every pair $(x,\xi) \in X \times \Xi$. This assumption is mathematical in nature, but note that $p(x,\xi)$ is simply provided by the market, acting as a black box, possibly with hidden or unknown mode of operation. Anyway, with stepsizes in $[0,1]$, selecting $x^0 \in X$ implies $x^t \in X$ for all t.

2.4. The Problem. Above all we seek to arrive at a *fixed point* $x = f(x)$ of the mapping

$$f(x) := \int p(x,\xi)\mu(d\xi), \tag{9}$$

which furnishes the *average realized price.* More precisely, we want almost sure convergence of $\{x^t\}$ towards a fixed point x of this function f. Any such a point x embodies *rational expectations* in the sense that

$$x = Ep(x, \xi).$$

To make our wish realistic we shall assume, in most of the analysis, that f is a well defined, continuous self-mapping on X. Indeed, it will be so provided the integrand $p(x,\xi) \in X$ is continuous in x and Borel measurable in ξ. Then, as is commonly known, there exists at least one fixed point.

2.5. The Differential Equation. To gain intuition about convergence properties of (3) it helps, for the sake of the argument, to assume temporarily that there is no uncertainty, or, alternatively, that (3) has been replaced by its deterministic version

$$x^{t+1} = (1 - \lambda_t)x^t + \lambda_t Ep^t. \tag{10}$$

To grasp the essence of (10) introduce the function

$$h(x) := f(x) - x \tag{11}$$

which indicates the expected direction $Ep(x, \xi) - x$ of movement in price beliefs. Change the time scale

$$\tau_T := \sum_{t=0}^{T-1} \lambda_t, \quad x(\tau_T) := x_T$$

to see that recursion (10) takes on the form

$$\frac{x_{t+1} - x_t}{\lambda_t} = f(x_t) - x_t = h(x_t),$$

that is,

$$\frac{x(\tau_{t+1}) - x(\tau_t)}{\tau_{t+1} - \tau_t} = h(x(\tau_t)).$$

Thus, lurking behind the stochastic difference system (3) there is the ordinary differential equation

$$\frac{dx(\tau)}{d\tau} = h(x(\tau)). \tag{12}$$

Now the leading idea, central in the theory *stochastic approximation* [6], is that stability of the idealized, continuous-time system (12) should help, and at best suffice, to have convergence of its more realistic, discrete-time counterpart (3). To pursue this idea, suppose the vector field $h(\cdot)$, representing expected price dynamics (11), has unique *integral curves*

$$\Phi(\tau, x^0) := x(\tau) := x^0 + \int_0^\tau h(x(\cdot)).$$

Note, by the way, that such curves are *viable*:

Proposition 1. *For any initial point x^0 a solution of (12) will remain inside the feasible set X.*

Proof. Simply observe that $h(x)$ belongs to the tangent cone of X at any point $x \in X$. Therefore any solution trajectory (integral curve) $x(\cdot)$ of (12) starting within X will remain there forever, see [4]. □

The important thing here is that quite a few difficulties are likely to arise when (10) is used in place of its stochastic analogue (3). Namely, the integral (9) is often hard or costly to evaluate. Even worse, the distribution μ or the function $p(\cdot, \cdot)$ may very well be unknown. If so, the *stochastic mean-value process* (3) becomes a more natural object than (10). The first presumes merely the ability to simulate or record the discrete-time stochastic process of independent identically distributed vectors $\xi^t \sim \xi$, and the possibility to observe the associated time series $p^t = p(x^t, \xi^t)$ as it evolves.

Next we take up the crucial issue of limiting sets of (3):

Definition 1. *A closed subset $Y \subseteq X$ is called internally chain recurrent if for every point $y \in Y$ and every $\delta > 0$, $\tau > 0$ there exists an integer T, points $y_t \in Y$, and time instants $\tau_t \geq \tau$, for $t = 0, ..., T-1$ such that*

$$|y - y_0| < \delta; \ |\Phi(\tau_t, y_t) - y_{t+1}| < \delta \text{ for } t = 0, ..., T-1; \ y = y_T. \ \square$$

Intuitively, all points in such a set Y are periodic up to any accuracy δ. A set $\mathfrak{I} \subseteq X$ is called *invariant* (under h, i.e., under (12)) if

$$x \in \mathfrak{I} \Rightarrow \Phi(\tau, x) \in \mathfrak{I} \text{ for all } \tau \geq 0.$$

A set $\mathfrak{I} \subseteq X$ which is internally chain recurrent, invariant, closed, and connected will simply be called a *limit set.* A point $\bar{x} \in X$ is named *asymptotically stable* if there exists an open vicinity V of $\bar{x}$ (relative to X) such that

$$\lim_{\tau \to \infty} \Phi(\tau, x) = \bar{x}$$

uniformly in $x \in V$.

3. CONVERGENCE

After these preparations we are ready to explore the long term behavior of (3). We shall demonstrate somewhat surprising assertions, saying that convergence of (3) often obtains without assumptions about global asymptotic stability of (12). To isolate key arguments we begin with a fairly abstract result that postulates desirable aymptotic stability of the expected dynamics (12):

Theorem 1. *(Cobweb convergence with any finite number of commodities)*

Suppose that every limit set of (12) reduces to an isolated singleton. Then $\{x^t\}$ generated by (3) converges almost surely to a fixed point of f.

Proof. We shall draw heavily on arguments due to Benäim [5]. Observe that (3) can be rewritten equivalently as

$$x^{t+1} - x^t = \lambda_t \left[p(x^t, \xi^t) - x^t\right] = \lambda_t \left[h(x^t) + e^t\right]$$

where $e^t := p(x^t, \xi^t) - f(x^t)$. Note that $E[e^t | \xi^0, ..., \xi^{t-1}] = 0$. Thus (3) can be seen as a discrete-time, stochastic version of the ordinary differential equation (12). In a different jargon, (3) is a *stochastic approximation* of (12).

Let the stochastic set $\mathfrak{I} := \omega(x^0)$ denote the so-called *omega limit set* of the sequence $\{x^t\}$ generated by (3). It consists of all accumulation points of $\{x^t\}$. Invoking Proposition 2.1 and Theorem 1.2 of [5], in that order, we obtain that $\mathfrak{I}$ must be an internally chain recurrent, invariant, closed, connected set. In other words, $\mathfrak{I}$ is what we just called a limit set. Since, by assumption, that set reduces to an isolated singleton, it must, by invariance, be a fixed point of f, and almost sure convergence is immediate. □

Not all fixed points of f appear equally attractive as ultimate outcomes of (3). The asymptotically stable ones seem being the better candidates. Indeed, granted non-degenerate randomness in (3) one would intuitively guess that *unstable fixed points cannot show up as asymptotic limits.* More precisely, if $\overline{x}$ is a *linearly unstable* singular point (steady state) of (12), then, for arbitrary $x^0 \in X$, it presumably holds with probability one that $x^t \nrightarrow \overline{x}$. The following result, taken from Lemma 2 of [2], or from Theorem 1 of [7], substantiates this insight:

Theorem 2. *(Non-attainability of linearly unstable points)*

Let $\overline{x} \in X$ be a fixed point of f near which this function is continuously differentiable. Suppose $\overline{x}$ is linearly unstable, that is, the matrix $\nabla h(\overline{x}) = \nabla f(\overline{x}) - I$ has at least one eigen-value with positive real part. Then for any initial point $x^0 \in X$ it holds with probability one that $x^t \nrightarrow \overline{x}$ provided

(i) $0 < \liminf_{t\to\infty} t^{\gamma} \lambda_t \le \limsup_{t\to\infty} t^{\gamma} \lambda_t < \infty$ for some $\gamma \in \left]\frac{1}{2}, 1\right]$; and

(ii) the covariance matrix $Cov(x) := Ep(x,\xi)p(x,\xi)^*$ is continuous near $\overline{x}$ with $\det Cov(\overline{x}) \neq 0$.[3] □

Admittedly, the hypothesis of Theorem 1 is hard to verify a priori. Therefore we offer next some sufficient conditions.

Corollary 1. *Suppose (12) admits a Lyapunov function with finitely many terminal points, then the conclusion of Theorem 1 holds. In particular, if f is continuously differentiable with*

$$\frac{\partial f_i(x)}{\partial x_j} = \frac{\partial f_j(x)}{\partial x_i}$$

for all $i, j = 1, ..., N$, then h is the gradient of some Lyapunov function naturally called a potential. □

The next result goes in the same direction but has a better, more explicit economic motivation:

[3] Here * designates transposition.

Corollary 2. *(Convergence under strong monotonicity)*

Suppose there is a fixed point $\overline{x} = f(\overline{x}) \in X$ and a number $s \in [0, 1[$ such that the following "monotonicity condition" holds

$$\langle f(x) - \overline{x}, x - \overline{x} \rangle \leq s \left\| x - \overline{x} \right\|^2 \text{ for all } x \in X. \tag{13}$$

Then $\overline{x}$ is the unique fixed point of f, and $\{x^t\}$ converges almost surely to $\overline{x}$. In particular, (13) is satisfied if f is Lipschitz continuous with modulus $s < 1$ on X, or if f is monotone decreasing there ($s = 0$).

Proof. Let $L(x) := \frac{1}{2} \left\| x - \overline{x} \right\|^2$, and observe that along any solution $x(\cdot)$ of (12) it holds that

$$\begin{aligned} \frac{dL(x(\tau))}{d\tau} &= \langle x(\tau) - \overline{x}, h(x(\tau)) \rangle = \langle x(\tau) - \overline{x}, f(x(\tau)) - \overline{x} - [x(\tau) - \overline{x}] \rangle \\ &\leq (s-1) \left\| x(\tau) - \overline{x} \right\|^2 . \end{aligned}$$

Thus, under (13), $L(\cdot)$ is a *Lyapunov function* for (12), whence the latter is globally asymptotically stable, with limit set reduced to $\overline{x}$. The conclusion now derives from Theorem 1. □

An argument offering partial support to (13) goes as follows: The supply $s_n = S_n(x, \xi)$ of any good n is most likely to increase if the own price expectation x_n becomes greater. At the same time the market clearing condition $D_n(p, \xi) = s_n$ typically yields a lower own price p_n as more supply s_n is brought onto the market place. This reasoning indicates a monotone mapping $x \to Ep(x, \xi) = f(x)$, i.e., a stronger condition than (13), namely

$$\langle f(x) - f(x'), x - x \rangle \leq 0 \text{ for all } x, x' \in X.$$

Admittedly, this argument ignores cross effects stemming from substitution between various goods. It is therefore incomplete and only suggestive in nature. Surprisingly, it serves us perfectly well when there are only two goods $n = 1, 2$. Assume then that

$$\frac{\partial}{\partial x_n} f_n(x) := \frac{\partial Ep_n(x, \xi)}{\partial x_n} < 1 \text{ for } n = 1, 2. \tag{14}$$

We regard condition (14) as quite reasonable and innocuous. Indeed, the above reasoning indicated that

$$\frac{\partial p_n(x, \xi)}{\partial x_n} \leq 0,$$

whence, under continuous differentiability of $p_n(x, \xi)$ with respect to x_n,

$$\frac{\partial Ep_n(x, \xi)}{\partial x_n} \leq 0,$$

for each commodity. Evidently, this implies (14), and will suffice for stability as brought out next:

Theorem 3. *(Cobweb convergence with two goods)*
Suppose there are only two goods $n = 1, 2$ and that (14) holds with $\frac{\partial}{\partial x_n} f_n(x)$ continuous on X. Also suppose that (12) has only isolated equilibria. Then $\{x^t\}$ generated by (3) converges almost surely to a fixed point of f.

Proof. Because of (14) the *divergence* of the vector field $h(\cdot)$ (11) is of constant sign within X :

$$\begin{aligned} divh(x) &= \frac{\partial}{\partial x_1} h_1(x) + \frac{\partial}{\partial x_2} h_2(x) \\ &= \frac{\partial}{\partial x_1} \{f_1(x) - x_1\} + \frac{\partial}{\partial x_2} \{f_2(x) - x_2\} \\ &= \frac{\partial}{\partial x_1} Ep_1(x, \xi) + \frac{\partial}{\partial x_2} Ep_2(x, \xi) - 2 < 0. \end{aligned}$$

Hence via the Bendixon-Poincare theory for two-dimensional systems [14] we get that any limit set of (12) consists of stationary points. The conclusion now derives directly from the preceding theorem. □

Finally, we concentrate on the traditional, well studied instance comprising only one good.[4] Here we shall go so far as to drop the assumption that f must be continuous. Of course, then there is no gurantee any longer that f has a fixed point. Consequently we must accept to work with a generalized notion: Let

$$Genfix(f) := \left\{ x \in X : \lim \inf_{X \ni x' \to x} f(x') \leq x \leq \lim \sup_{X \ni x' \to x} f(x') \right\}$$

denote the set of *generalized fixed points* of f. Also, we need an additional concept: $x \in X$ is said to be an *oscillation point of* h (11) if for every $\varepsilon > 0$ one may find within one of the intervals $]x - \varepsilon, x[$ and $]x, x + \varepsilon[$ two points x^- and x^+ in X such that $h(x^-)h(x^+) < 0$. The following generalizes a recent result of Flåm and Horvath [11]:

Theorem 4. *(Cobweb convergence with only one good)*
Suppose there is only one good and that the oscillation points of h (11) form a nowhere dense set. Then $\{x^t\}$ generated by (3) converges almost surely to a generalized fixed point of f.

[4] If the setting were deterministic like (10), one could rely on "classical" theorems of Mann [21], Krasnoselski [18] and Franks and Marzec [12].

Proof. This follows from arguments given in [7]. □

Note that this theorem implicitely implies that there is at least one generalized fixed point. Indeed, otherwise the set mentioned in the theorem would be empty, and so we would arrive at the contradiction $\lim_{t\to\infty} x^t \in Genfix(f) = \emptyset$.

4. CONCLUDING REMARKS

Commodity prices often fluctuate in ways that cannot be explained simply by changes in exogenous variables or market structure [15]. Much variation tends, of course, to be purely random. Thus, even producers who entertain rational price expectations must confront ups and downs of markets. It appears though that adaptive learning, of the sort found and advocated in stochastic approximation theory [5], [6], will largely contribute to stabilize markets with production lags. In particular, our results indicate that this is likely to happen in simple settings comprising only one or two goods. For more than two commodities there are possibilities for periodic, even chaotic price dynamics. However, as brought out here, favorable instances, with expected price dynamics having merely isolated singletons as attractors, yield almost sure convergence to rational expectations in the long run.

REFERENCES

[1] G. Akerman, The cobweb theorem: A reconsideration, *Quaterly Journal of Economics* 71, 151-160 (1957).

[2] W. B. Arthur, Y. M. Ermoliev, and Y. M. Kaniovski, Nonlinear adaptive processes of growth with general increments: Attainable and unattainable components of terminal set, *Working Paper* WP-88-86, International Institute for Applied Systems Analysis, Laxenburg, Austria.

[3] Z. Artstein, Irregular cobweb dynamics, *Economic Letters* 11, 15-17 (1983).

[4] J-P. Aubin and H. Frankowska, *Set-Valued Analysis,* Birkhäuser, Basel (1990).

[5] M. Benäim, A dynamical system approach to stochastic approximations, *SIAM J. Control and Optimization* 34, 2, 437-472 (1996).

[6] A. Benveniste, M. Métivier, and P. Priouret, *Stochastic Approximations and Adaptive Algorithms,* Springer-Verlag, Berlin (1987).

[7] G. R. Blum and M. Brennan, On the strong law of large numbers for dependent random variables, *Israel Journal of Mathematics* 37, 241-245 (1980).

[8] C. Chiarella, The cobweb model: its instability and the onset of chaos, *Economic Modelling* 5, 377-384 (1088).

[9] R. H. Day and K. A. Hanson, Cobweb chaos, in T. K. Kaul and J. K. Sengupta (eds.),*Economic Models, Estimation, and Socioeconomic Systems, Essays in honor of K. A. Fox,* North-Holland, Amsterdam (1991).

[10] M. Ezekiel, The cobweb theorem, *Quarterly Journal of Economics* 52, 255-280 (1938).

[11] S. D. Flåm and C. Horvath, Stochastic mean-values, rational expectations, and price movements, *Manuscript* (1997).

[12] R. L. Franks and R. P. Marzec, A theorem on mean-value iterations, *Proc. Americ. Math. Soc.* 30, 324-326 (1971).

[13] J. A. C. Gallas and H. E. Nusse, Periodicity versus chaos in the dynamics of cobweb models, *Journal of Economic Behavior and Organization* 29, 447-464 (1996).

[14] W. Hahn, *Stability of Motion,* Springer-Verlag, Berlin (1967).

[15] C. Holzer and M. Precht, Der chaotische Schweinezyklus, *Agrarwirtschaft* 42, 276-283 (1993).

[16] C. H. Hommes, Dynamics of the cobweb model with adaptive expectations and nonlinear supply and demand, *Journal of Economic Behavior and Organization* 24, 315-335 (1994).

[17] R. V. Jensen and R. Urban, Chaotic price behavior in a non-linear cobweb model, *Economic Letters* 15, 235-240 (1984).

[18] M. A. Krasnoselski, Two observations about the method of successive approximations, *Usp. Math. Nauk* 10, 123-127 (1955).

[19] A. J. Lichtenberg and A. Ujihara, Application of nonlinear mapping theory to commodity price fluctuations, *Journal of Economic Dynamics and Control* 13, 225-246 (1989).

[20] R. E. Lucas and T. J. Sargent, *Rational Expectations and Econometric Practice,* George Allen & Unwin, London (1981).

[21] W. R. Mann, Mean value methods in iteration, *Proc. Americ. Math. Soc.* 4, 506-510 (1953).

[22] J. F. Muth, Rational expectations and the theory of price movements, *Econometrica* 29, no. 6, 315-335 (1961).

[23] M. Nerlove, Adaptive expectations and cobweb phenomena, *Quarterly Journal of Economics* 72, 227-240 (1958).

[24] R. Pemantle, Non-convergence to unstable points in urn models and stochastic approximations, *Ann. Prob.* 18, 698-712 (1990).