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# Best Response Adaptation for Role Games

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The Adaptive Dynamics Network at IIASA fosters the development of new mathematical and conceptual techniques for understanding the evolution of complex adaptive systems.

Focusing on these long-term implications of adaptive processes in systems of limited growth, the Adaptive Dynamics Network brings together scientists and institutions from around the world with IIASA acting as the central node.

Scientific progress within the network is reported in the IIASA Studies in Adaptive Dynamics series.

## THE ADAPTIVE DYNAMICS NETWORK

The pivotal role of evolutionary theory in life sciences derives from its capability to provide causal explanations for phenomena that are highly improbable in the physicochemical sense. Yet, until recently, many facts in biology could not be accounted for in the light of evolution. Just as physicists for a long time ignored the presence of chaos, these phenomena were basically not perceived by biologists.

Two examples illustrate this assertion. Although Darwin's publication of "The Origin of Species" sparked off the whole evolutionary revolution, oddly enough, the population genetic framework underlying the modern synthesis holds no clues to speciation events. A second illustration is the more recently appreciated issue of jump increases in biological complexity that result from the aggregation of individuals into mutualistic wholes.

These and many more problems possess a common source: the interactions of individuals are bound to change the environments these individuals live in. By closing the feedback loop in the evolutionary explanation, a new mathematical theory of the evolution of complex adaptive systems arises. It is this general theoretical option that lies at the core of the emerging field of adaptive dynamics. In consequence a major promise of adaptive dynamics studies is to elucidate the long-term effects of the interactions between ecological and evolutionary processes.

A commitment to interfacing the theory with empirical applications is necessary both for validation and for management problems. For example, empirical evidence indicates that to control pests and diseases or to achieve sustainable harvesting of renewable resources evolutionary deliberation is already crucial on the time scale of two decades.

The Adaptive Dynamics Network has as its primary objective the development of mathematical tools for the analysis of adaptive systems inside and outside the biological realm.

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# Contents

# Abstract

Consider a large population of individuals which can be in one of two distinct roles. The role of an individual is switched every now and then, and interactions occur between randomly paired individuals in different roles. These interactions are represented by a bimatrix game and individuals are modelled as boundedly rational expected utility maximizers who choose their actions according to a myopic best response rule. The resulting dynamics of the population state is given by a system of differential equations and differential inclusions. If the bimatrix game is zero-sum, the population state converges to a fixed point set corresponding to the set of Nash equilibria of this game. Moreover, if the zero-sum game has a unique Nash equilibrium, the global attractor of the population state is a unique and explicitly computable fixed point, even if the set of fixed points is a continuum (which is the case, if the unique Nash equilibrium is completely mixed). This global attractor does neither depend on the rates of role switching and strategy reviewing, nor on the relative size of the two subpopulations of individuals in different roles.

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# Best Response Adaptation for Role Games

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## 1 Role Games and Their Dynamics

When dealing with (evolutionary) game dynamics, one normally considers either *intragroup* interactions between individuals of the same type – expressed by a symmetric game as e.g. the well known hawk-dove game (*hawk* and *dove* are two different strategies for *one* type of players) – or *intergroup* interactions between members of two distinct populations, modelled by a bimatrix-game as is e.g. the case for the Battle-of-the-Sexes game (details can be found in Hofbauer and Sigmund (1988)).

Role games are somehow a mixture of these two cases: In the following sections we consider a large population of individuals who are all of one type, but may be in one of two distinct *roles*, called $I$ and $II$. We assume, that $I$-players (i.e. individuals currently in role $I$) interact only with $II$-players. As in the asymmetric case, the *payoffs* – in evolutionary games usually interpreted as increase in fitness – received from such an interaction by each of the players are given by a bimatrix game. However, the role of an individual need not be fixed, but can change several times during lifetime.

This idea of constructing a role game from a bimatrix game already appears (for zero-sum games) in the classical work of von Neumann and Morgenstern (1944). Selten (1980), Maynard Smith (1982), and Weibull (1995) (who calls the underlying bimatrix game the *base game* and the role game the *role-conditioned game*) used it in an evolutionary context. If there are $n$ (pure) actions available in role $I$, and $m$ actions in role $II$, a *strategy* for an individual must specify, what action to implement if in role $I$ or role $II$, hence there will be $nm$ pure strategies. For the case $n = m = 2$ the *replicator dynamics* has been analyzed by Gaunersdorfer, Hofbauer, and Sigmund (1991). This dynamics describes the evolution of strategy–frequencies across generations by a selection process. In this paper we study a quite different dynamics, which is based on best response adaptation. Thus we model a kind of learning process, which takes place *within* one generation. The basic idea is to assume, that individuals are playing a certain strategy for some time, but have every now and then the chance to review their strategy and switch to the current best response to the average opponent–strategy. This kind of strategy–adaptation may be motivated in two different ways: either as myopic utility–maximization of boundedly rational agents,

or as being derived from an "imitate the best" learning rule for individuals with rather restricted rationality. In the next section we will have a closer look at our model.

## 2  The Model

Imagine that couples of individuals in different roles are matched randomly at every point in time. The interactions are assumed to be represented by an $n \times m$ bimatrix game with payoff matrices $U^I$ and $U^{II}$. As mentioned above, the role an individual is in, need not be fixed. Suppose instead, that every now and then it switches to the other role. Any individual uses some pure strategy, represented by a pair $(i, j)$, where $i \in \{1, \dots, n\}$ is its action as a $I$-player, and $j \in \{1, \dots, m\}$ its $II$-action. Sometimes the individual has the chance of reviewing its strategy. An individual in role $I$ will every now and then review the $I$-action it is using and change to the present best response against the average action–mix among the $II$-players, and vice versa. We have to specify some other assumptions:

1) The share of individuals which are in role $I$ at time $t$ is assumed to be constantly equal to some fixed number $w$ between 0 and 1.

2) In order to keep $w$ constant, we assume, that the number of $II$-players, switching in some small time interval $dt$, equals the number of $I$-players switching in $dt$. The two groups need not be equally large, and hence the *switching rate* of an individual depends on the role it is in. If it is in the smaller group, it has a higher chance of being selected to switch its role. The individual switching rate of a $I$-player will thus be $(1-w)R$, while the switching rate of a $II$-player is $wR$, where $R > 0$ is some constant depending on the overall rate of role–switching.

3) The rate at which an individual reviews its currently used action, is supposed to be proportional to its interaction frequency. This seems reasonable, since an individual facing more encounters in some time interval than another individual, will gather more information about the opponents' action–mix, and will most likely review more often. The interaction frequency of an individual depends solely on its role: A member of the smaller group will engage in more encounters than a member of the larger group. The reviewing rates of a $I$-player and a $II$-player thus turn out to be $(1-w)r$ and $wr$, respectively, where $r$ again is some positive constant, which we set to 1 without loss of generality.

Now we introduce the following $2(n + m)$ variables:

$$z_i \quad \dots \quad \text{Proportion of } I\text{-players with strategy } (i, *)$$

$Z_i \quad \ldots \quad$ Proportion of $II$-players with strategy $(i, *)$

$y_j \quad \ldots \quad$ Proportion of $II$-players with strategy $(*, j)$

$Y_j \quad \ldots \quad$ Proportion of $I$-players with strategy $(*, j)$

Here, the $*$ denotes a wildcard for the available actions in the two roles. The total share of the population having strategy $(i, *)$, i.e., playing the pure action $i$, if in role $I$, is then given by $wz_i + (1-w)Z_i$, and $(1-w)y_j + wY_j$ is the share of $(*, j)$–individuals. For simplicity, we will identify the pure action $i$ with the $i$-th unit vector $\mathbf{e}_i$ in $S_n$ or $S_m$, respectively. In the following, bold face letters denote column vectors, for example $\mathbf{z} = (z_i)_{i=1,\ldots,n}$ .

# 3 A System of Differential Inclusions and Differential Equations

Let $\mathbf{B}^I(\mathbf{x})$ be the set of all best responses for a $I$-player confronted with a $II$-player–group with average action–mix $\mathbf{x}$, and $\mathbf{B}^{II}(\mathbf{x})$ its analogous counterpart. With these variables our proposed model yields the following system of *differential inclusions* in $S := (S_n \times S_m)^2$ :

$$
\begin{aligned}
\dot{\mathbf{z}} &\in (1-w)[\mathbf{B}^I(\mathbf{y}) - \mathbf{z} + R(\mathbf{Z} - \mathbf{z})] \\
\dot{\mathbf{y}} &\in w[\mathbf{B}^{II}(\mathbf{z}) - \mathbf{y} + R(\mathbf{Y} - \mathbf{y})] \\
\dot{\mathbf{Z}} &= -wR(\mathbf{Z} - \mathbf{z}) \\
\dot{\mathbf{Y}} &= -(1-w)R(\mathbf{Y} - \mathbf{y})
\end{aligned}
\tag{1}
$$

The best response correspondences $\mathbf{x} \mapsto \mathbf{B}^I(\mathbf{x})$ and $\mathbf{x} \mapsto \mathbf{B}^{II}(\mathbf{x})$ are upper–semicontinuous with closed and convex values. Hence the existence of at least one solution through each initial value, which is Lipschitz continuous and defined for all positive times, is guaranteed, see e.g. Aubin and Cellina (1984). Every such solution can be written as a function $t \mapsto (\mathbf{z}(t), \mathbf{y}(t), \mathbf{Z}(t), \mathbf{Y}(t))$, satisfying

$$
\begin{aligned}
\dot{\mathbf{z}}(t) &= (1-w)[\mathbf{b_y}(t) - \mathbf{z}(t) + R(\mathbf{Z}(t) - \mathbf{z}(t))] \\
\dot{\mathbf{y}}(t) &= w[\mathbf{b_z}(t) - \mathbf{y}(t) + R(\mathbf{Y}(t) - \mathbf{y}(t))] \\
\dot{\mathbf{Z}}(t) &= -wR(\mathbf{Z}(t) - \mathbf{z}(t)) \\
\dot{\mathbf{Y}}(t) &= -(1-w)R(\mathbf{Y}(t) - \mathbf{y}(t)) \,,
\end{aligned}
\tag{2}
$$

for almost all $t \geq 0$, where $t \mapsto \mathbf{b_y}(t) \in \mathbf{B}^I(\mathbf{y}(t))$ and $t \mapsto \mathbf{b_z}(t) \in \mathbf{B}^{II}(\mathbf{z}(t))$ are measurable functions.

As can be seen, the equation for $\dot{\mathbf{z}}$ consists of a "reviewing–term" $(1 - w)[\mathbf{b_y}(t) - \mathbf{z}(t)]$ and a "switching–term" $R[\mathbf{Z}(t) - \mathbf{z}(t)]$. The variable $Z_i$ however, the share of $II$-players playing $(i, *)$, is only affected by switching of the roles, since the $I$-action of a strategy does not matter, as long as the strategy is used by a $II$-player. The same is of course true for $\dot{\mathbf{y}}$ and $\dot{\mathbf{Y}}$. The only asymmetry is caused by the different rates, at which the groups adapt, according to their different size.

Looking at the equations for $\dot{\mathbf{Z}}$ and $\dot{\mathbf{Y}}$, it is evident, that $\mathbf{Z} = \mathbf{z}$, $\mathbf{Y} = \mathbf{y}$ is a necessary condition for a fixed point of the system. Then $\dot{\mathbf{z}} \ni \mathbf{0}$, $\dot{\mathbf{y}} \ni \mathbf{0}$ holds, if $\mathbf{z} \in \mathbf{B}^I(\mathbf{y})$ and $\mathbf{y} \in \mathbf{B}^{II}(\mathbf{z})$. Thus any fixed point of the system has the form $\mathbf{z} = \mathbf{Z} = \mathbf{z}^*$, $\mathbf{y} = \mathbf{Y} = \mathbf{y}^*$, where $(\mathbf{z}^*, \mathbf{y}^*)$ is a Nash equilibrium of the underlying game $(U^I, U^{II})$.

# 4   A Ljapunov Function

A special case is given, if $U^{II} = -(U^I)'$ (the negative transpose of $U^I$), i.e., if the game is a zero-sum game. For such games we will prove, that the set of Nash equilibria is globally asymptotically stable under (1). (It is then an easy implication that this result also holds for games, which are equivalent to a zero-sum game.) In the simplest case $n = m = 2$, a game is equivalent to a zero-sum game if and only if it has a cyclic best response structure. A typical orbit of $(\mathbf{z}(t), \mathbf{y}(t))$ and $(\mathbf{Z}(t), \mathbf{Y}(t))$ for this case is shown in figure 1.

To prove the stability of the Nash equilibrium set we construct a function $V$, which acts as a Ljapunov–function does for differential equations.

So let $U^{II} = -(U^I)'$, take any solution, and consider the following continuous function on $S$:

$$V(t) := (1 - w)[\max_h(U^I \mathbf{y}(t))_h + \max_h(U^{II} \mathbf{Z}(t))_h] + w[\max_h(U^I \mathbf{Y}(t))_h + \max_h(U^{II} \mathbf{z}(t))_h] \,.$$

Note, that

$$V \geq (1 - w)[\mathbf{Z}' U^I \mathbf{y} + \mathbf{y}' U^{II} \mathbf{Z}] + w[\mathbf{z}' U^I \mathbf{Y} + \mathbf{Y}' U^{II} \mathbf{z}] = 0 \,,$$

since $U^{II} = -(U^I)'$, and equality holds if and only if

$$\mathbf{Z} \in \mathbf{B}^I(\mathbf{y}) \text{ and } \mathbf{y} \in \mathbf{B}^{II}(\mathbf{Z}) \text{ and } \mathbf{z} \in \mathbf{B}^I(\mathbf{Y}) \text{ and } \mathbf{Y} \in \mathbf{B}^{II}(\mathbf{z}) \,,$$
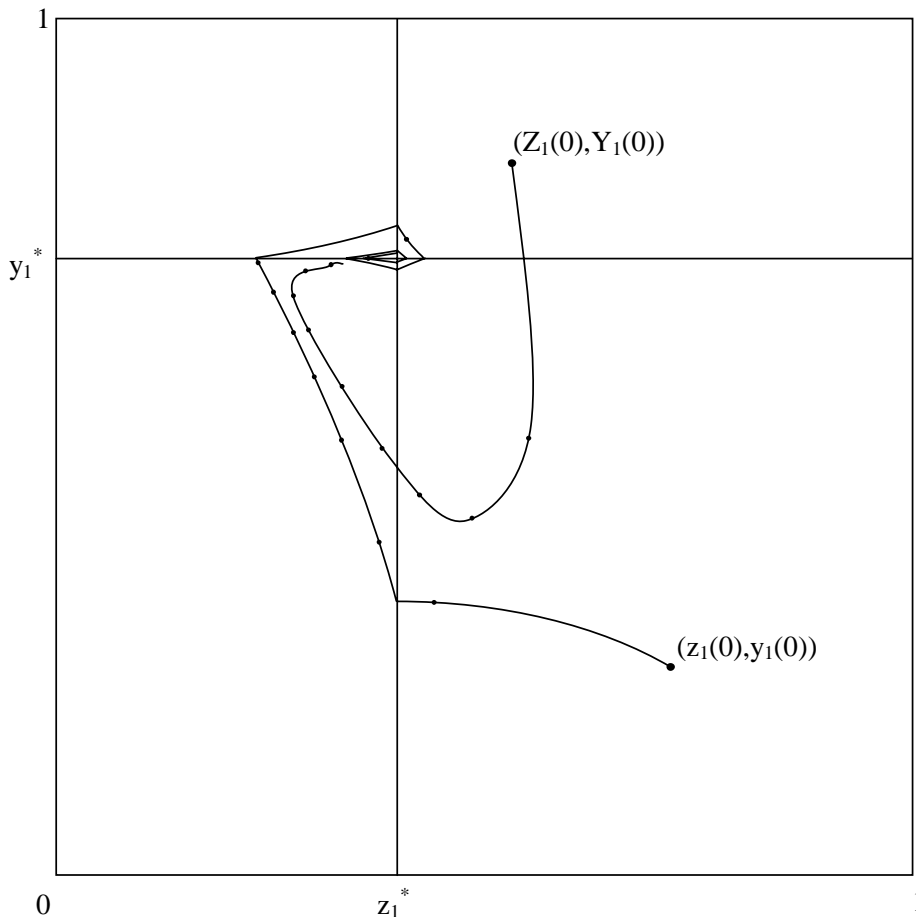
Figure 1: *Orbits $(z_1(t), y_1(t))$ and $(Z_1(t), Y_1(t))$ for the case $n = m = 2$ and cyclic best response structure. The dots mark equidistant time steps. Here, $R = 2$ and $w = \frac{1}{3}$.*

i.e., if and only if $(\mathbf{Z}, \mathbf{y})$ and $(\mathbf{z}, \mathbf{Y})$ are Nash equilibria of the zero-sum game. In a zero-sum game, Nash equilibria are interchangeable, and thus the above condition is equivalent to the condition that $(\mathbf{z}, \mathbf{y})$ and $(\mathbf{Z}, \mathbf{Y})$ are Nash equilibria.

We want to show that $V$ is decreasing along any solution of (1) outside the equilibrium set. Let us first write $V = (1-w)(V_1 + V_2) + w(V_3 + V_4)$, where the functions $V_m$ are given by $V_1(t) = \max_h (U^I \mathbf{y}(t))_h, \ldots, V_4(t) = \max_h (U^{II} \mathbf{z}(t))_h$. Note that, along any solution of (1), $V_m(t)$ is absolutely continuous as the maximum of absolutely continuous functions. Thus $V_m(t)$ is differentiable almost everywhere ($m = 1, 2, 3, 4$), and so is $V$ itself. We know, that this also holds for the functions $\mathbf{z}(t), \mathbf{y}(t), \mathbf{Z}(t), \mathbf{Y}(t)$. Let $t_0$ be a point of differentiability of these four functions as well as the four functions $V_m$. Now consider e.g. $V_1$. For any sequence $t_n \to t_0$, $t_n \neq t_0$ there is a sequence of pure actions $\mathbf{e}_{i_n} \in \mathbf{B}^I(\mathbf{y}(t_n))$. This sequence always has a constant subsequence, since the number of indices is finite. Assume this is the constant sequence $(\mathbf{e}_i)$; then we know that $\mathbf{e}_i \in \mathbf{B}^I(\mathbf{y}(t_0))$. It follows

that

$$
\begin{aligned}
\dot{V}_1(t_0) &= \lim_{n\to\infty} \frac{V_1(t_n) - V_1(t_0)}{t_n - t_0} = \\
&= \lim_{n\to\infty} \frac{\mathbf{e}_i' U^I \mathbf{y}(t_n) - \mathbf{e}_i' U^I \mathbf{y}(t_0)}{t_n - t_0} = \\
&= \mathbf{e}_i' U^I \lim_{n\to\infty} \frac{\mathbf{y}(t_n) - \mathbf{y}(t_0)}{t_n - t_0} = \\
&= \mathbf{e}_i' U^I \dot{\mathbf{y}}(t_0) \,.
\end{aligned}
$$

If $\mathbf{B}^I(\mathbf{y}(t_0))$ is not a singleton, and there is another subsequence with a different limit $\mathbf{e}_k \in \mathbf{B}^I(\mathbf{y}(t_0))$, then the existence of $\dot{V}_1$ implies $\mathbf{e}_k' U^I \dot{\mathbf{y}}(t_0) = \mathbf{e}_i' U^I \dot{\mathbf{y}}(t_0)$. But what, if for some index $k$ with $\mathbf{e}_k \in \mathbf{B}^I(\mathbf{y}(t_0))$ there is no sequence $t_n \to t_0$ with $\mathbf{e}_k \in \mathbf{B}^I(\mathbf{y}(t_n))$ for all $n$? In this case there is a neighborhood $N$ of $t_0$, such that $\mathbf{e}_k$ is not a best response to $\mathbf{y}(t)$ for $t \in N - \{t_0\}$. This in turn implies, that

$$
(\mathbf{e}_i - \mathbf{e}_k)' U^I \dot{\mathbf{y}}(t_0) = 0 \,,
$$

since otherwise there are always points $t_n$ arbitrarily close to $t_0$, with

$$
(\mathbf{e}_k - \mathbf{e}_i)' U^I \mathbf{y}(t_n) > (\mathbf{e}_k - \mathbf{e}_i)' U^I \mathbf{y}(t_0) = 0 \,,
$$

contradicting the fact, that $\mathbf{e}_k$ is not a best response for such $t_n$.

We have proved $\dot{V}_1(t_0) = \mathbf{e}_i' U^I \dot{\mathbf{y}}(t_0)$ for all $\mathbf{e}_i \in \mathbf{B}^I(\mathbf{y}(t_0))$. Hence also $\dot{V}_1(t_0) = \mathbf{b_y}(t_0)' U^I \dot{\mathbf{y}}(t_0)$. Analogous considerations for $\dot{V}_2$, $\dot{V}_3$, and $\dot{V}_4$ yield

$$
\begin{aligned}
\dot{V}_1(t_0) &= \mathbf{b_y}(t_0)' U^I \dot{\mathbf{y}}(t_0) \,, \\
\dot{V}_2(t_0) &= \mathbf{b_Z}(t_0)' U^{II} \dot{\mathbf{Z}}(t_0) \,, \\
\dot{V}_3(t_0) &= \mathbf{b_Y}(t_0)' U^I \dot{\mathbf{Y}}(t_0) \,, \\
\dot{V}_4(t_0) &= \mathbf{b_z}(t_0)' U^{II} \dot{\mathbf{z}}(t_0) \,,
\end{aligned}
$$

and we can finally calculate

$$
\begin{aligned}
(w(1-w))^{-1} \dot{V} &= w^{-1}(\dot{V}_1 + \dot{V}_2) + (1-w)^{-1}(\dot{V}_3 + \dot{V}_4) \\
&= \mathbf{b_y}' U^I \mathbf{b_z} + \mathbf{b_z}' U^{II} \mathbf{b_y} - (\mathbf{b_y}' U^I \mathbf{y} + \mathbf{b_z}' U^{II} \mathbf{z}) + \\
&\quad + R[(\mathbf{b_y} - \mathbf{b_Y})' U^I (\mathbf{Y} - \mathbf{y})] + R[(\mathbf{b_z} - \mathbf{b_Z})' U^{II} (\mathbf{Z} - \mathbf{z})] \,.
\end{aligned}
$$

We have $\mathbf{b_y}' U^I \mathbf{b_z} + \mathbf{b_z}' U^{II} \mathbf{b_y} = 0$, as well as

$$
\mathbf{b_y}' U^I \mathbf{y} = \max_h (U^I \mathbf{y})_h \geq \mathbf{b_Y}' U^I \mathbf{y} \tag{3}
$$

and

$$\mathbf{b}_{\mathbf{y}}' U^I \mathbf{Y} \leq \max_h (U^I \mathbf{Y})_h = \mathbf{b}_{\mathbf{Y}}' U^I \mathbf{Y}\,, \tag{4}$$

yielding $(\mathbf{b}_{\mathbf{y}} - \mathbf{b}_{\mathbf{Y}})' U^I (\mathbf{Y} - \mathbf{y}) \leq 0$ and analogously $(\mathbf{b}_{\mathbf{z}} - \mathbf{b}_{\mathbf{Z}})' U^{II} (\mathbf{Z} - \mathbf{z}) \leq 0\,.$

We also have

$$\mathbf{b}_{\mathbf{y}}' U^I \mathbf{y} + \mathbf{b}_{\mathbf{z}}' U^{II} \mathbf{z} \geq \mathbf{z}' U^I \mathbf{y} + \mathbf{y}' U^{II} \mathbf{z} = 0\,. \tag{5}$$

Together these inequalities yield $\dot{V} \leq 0$ for almost all $t > 0$.

Equality holds in (3), iff $\mathbf{B}^I(\mathbf{Y}) \subset \mathbf{B}^I(\mathbf{y})$ and in (4), iff $\mathbf{B}^I(\mathbf{y}) \subset \mathbf{B}^I(\mathbf{Y})$, from which $\mathbf{B}^I(\mathbf{y}) = \mathbf{B}^I(\mathbf{Y})$ follows.

In (5) equality holds, iff $\mathbf{z} \in \mathbf{B}^I(\mathbf{y})$ and $\mathbf{y} \in \mathbf{B}^{II}(\mathbf{z})$, implying that $\dot{V} = 0$, iff $(\mathbf{z}, \mathbf{y})$ and $(\mathbf{Z}, \mathbf{Y})$ are Nash equilibria of the game $(U^I, U^{II})$.

Thus $V$ is strictly positive except at Nash equilibria, and $V$ is strictly decreasing along any orbit in $S$, except along orbits contained in the Nash equilibrium set. It follows that $V$ is a Ljapunov–function for (2). This proves

**Theorem 1** *For zero-sum games, the set of fixed points of (1), corresponding to Nash equilibria of the game, is globally asymptotically stable.*

## 5 The Main Result

We will now go one step further and analyze the behavior of the frequencies

$$x_{ij}^I \qquad \ldots \qquad \text{Proportion of } I\text{-players with strategy } (i,j)$$

$$x_{ij}^{II} \qquad \ldots \qquad \text{Proportion of } II\text{-players with strategy } (i,j)$$

For these frequencies, we will prove

**Theorem 2** *For zero-sum games with a unique Nash equilibrium $(\mathbf{z}^*, \mathbf{y}^*)$, the frequencies $x_{ij}^I$, as well as $x_{ij}^{II}$, converge to $z_i^* y_j^*$ for every $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$.*

*Proof:* Let us for example have a look at the frequency $x^I_{i_0 j_0}$. During some small time increment $\mathrm{d}t$ there will be a "switching–caused" flow $(1-w)R(x^{II}_{i_0 j_0} - x^I_{i_0 j_0})\mathrm{d}t$ into $x^I_{i_0 j_0}$, and a "reviewing–caused" flow $-(1-w)x^I_{i_0 j_0}\mathrm{d}t$ for $b_{\mathbf{y} i_0} = 0$ and $(1-w)\sum_{i \neq i_0} x^I_{i j_0}\mathrm{d}t$ for $b_{\mathbf{y} i_0} = 1$, changing linearly between these values. The analogous calculations for $x^{II}_{i_0 j_0}$ yield the system

$$
\begin{aligned}
\dot{x}^I_{ij} &= (1-w)[R(x^{II}_{ij} - x^I_{ij}) + Y_j b_{\mathbf{y} i} - x^I_{ij}] \\
\dot{x}^{II}_{ij} &= w[R(x^I_{ij} - x^{II}_{ij}) + Z_i b_{\mathbf{z} j} - x^{II}_{ij}]
\end{aligned}
\tag{6}
$$

for $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$, with

$$
z_i = \sum_k x^I_{ik}, \; Z_i = \sum_k x^{II}_{ik}, \; y_j = \sum_k x^{II}_{kj}, \; Y_j = \sum_k x^I_{kj}.
\tag{7}
$$

Note, that (6) and (7) imply (2), i.e., (6) is a refinement of (2). In fact, (2) results from (6) after a suitable projection (the one suggested by (7)), and the fixed point of the original system (1) gives rise to an invariant linear manifold in the phase space of (6). The constant solution of (6) is given by

$$
(\mathbf{z}(t), \mathbf{y}(t), \mathbf{Z}(t), \mathbf{Y}(t)) = (\mathbf{z}^*, \mathbf{y}^*, \mathbf{z}^*, \mathbf{y}^*) \qquad \text{for all } t \geq 0.
$$

The fixed point conditions $\dot{\mathbf{z}} = \mathbf{0}$ and $\dot{\mathbf{y}} = \mathbf{0}$ require

$$
\mathbf{b_y}(t) = \mathbf{z}(t) = \mathbf{z}^* \quad \text{and} \quad \mathbf{b_z}(t) = \mathbf{y}(t) = \mathbf{y}^* \quad \text{for } t \geq 0.
$$

Thus, restricted to the invariant manifold of (6) which corresponds to the fixed point of (1), the system (6) reduces to

$$
\begin{aligned}
\dot{x}^I_{ij} &= (1-w)[R(x^{II}_{ij} - x^I_{ij}) + z^*_i y^*_j - x^I_{ij}] \\
\dot{x}^{II}_{ij} &= w[R(x^I_{ij} - x^{II}_{ij}) + z^*_i y^*_j - x^{II}_{ij}]
\end{aligned}
\tag{8}
$$

and there it admits a unique fixed point, given by

$$
(x^I_{ij}, x^{II}_{ij}) = (z^*_i y^*_j, z^*_i y^*_j).
\tag{9}
$$

Restricted to the invariant manifold, this fixed point is a sink, as can be seen from the Jacobian

$$
J = \begin{bmatrix} -(1-w)(1+R) & (1-w)R \\ wR & -w(1+R) \end{bmatrix},
$$

of (8), which has eigenvalues

$$
\lambda_{1,2} = -\frac{1}{2}\left(1 + R \pm \sqrt{(1+R)^2 - 4w(1-w)(1+2R)}\right) < 0.
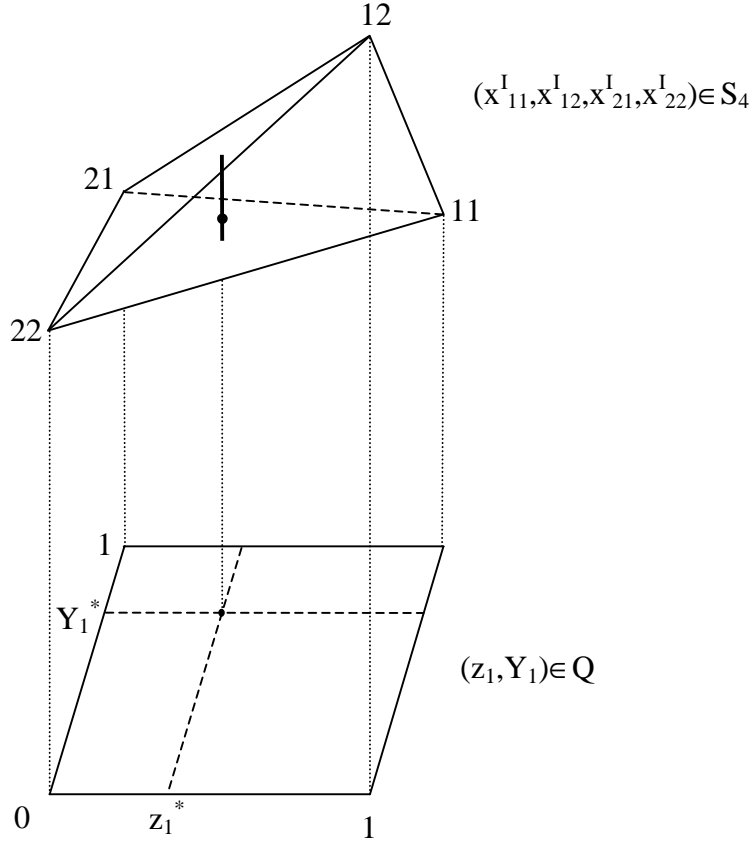$$

Figure 2: *In the $n = m = 2$ case the state space of $(x_{ij}^I)$ is the $S_4$. The projection onto the state space $Q$ of $(z_1, Y_1)$ is illustrated by the dotted lines. The bold vertical line is the invariant linear manifold corresponding to the equilibrium $(z_1^*, Y_1^*)$, and the dot on this line is the unique fixed point which is given by the first component in (9).*

Since the invariant manifold is globally attracting by the last theorem, we conclude, that every solution of (6) converges to the fixed point (9). *q.e.d.*

Figure 2 illustrates the case $n = m = 2$ with cyclic best response structure again. The state space of $(x_{ij}^I)_{i,j=1,2}$ is the 3-dimensional simplex $S_4$. The projection given by (7) yields $z_1 = x_{11}^I + x_{12}^I$ and $Y_1 = x_{11}^I + x_{21}^I$ and the state space of $(z_1, Y_1)$ is the unit square $Q := S_2 \times S_2$.

## 6 Discussion

By theorem 2, the state of the population converges to a strategy mix, where the actions in the two roles are independent. E.g. for $n = m = 2$ the attracting equilibrium satisfies

$$x_{11}^I x_{22}^I = x_{12}^I x_{21}^I . \tag{10}$$

(Analogously for role $II$.)

In population genetics, (10) appears in the context of the two-locus, two-alleles equation (see Crow and Kimura (1970)). There, $x_{11}, \ldots, x_{22}$ are the frequencies of the four gametes, and the set of frequency distributions satisfying (10) is called the *Wright manifold*. If $\mathbf{x}$ lies on this manifold, then the allele pairs are said to be in *linkage equilibrium*. Gaunersdorfer, Hofbauer, and Sigmund (1991) have analyzed the $n = m = 2$ case under *replicator dynamics*. Interestingly, the equilibrium $\mathbf{x}^*$ obeying (10) plays an important role for the behavior of replicator orbits, but a quite different one as it does for our dynamics. The equilibrium $\mathbf{x}^*$ on the Wright manifold divides the equilibrium line into two parts. Under replicator dynamics, if the role game is *not* equivalent to a zero-sum game, one of these parts is asymptotically stable, the other part is unstable, and $\mathbf{x}^*$ itself is surrounded by a continuum of closed orbits, which are contained in the Wright manifold. However, if the role game is a zero-sum game, then the replicator equation becomes a Hamiltonian dynamical system, all orbits cycle around the equilibrium line on closed curves, and the point $\mathbf{x}^*$ loses its exceptional status, in accordance with the classical, static approach where all the Nash equilibria of a zero-sum game are regarded as equivalent. As opposed to this, our results ensure convergence to $\mathbf{x}^*$ even in the case of a zero-sum role game. Thus the process of best response adaptation we introduced selects a particular equilibrium among a continuum of equivalent ones.

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