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# Estimation of Origin-Destination Matrices Using Traffic Counts – A Literature Survey

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# Abstract

The Origin-Destination (OD) matrix is important in describing transportation in a region. This matrix has information on the travel and transportation made between different zones of a region. The OD matrix can be estimated using traffic counts on links in the transport network and other available information. This information on the travel is often contained in a “target OD matrix. The target OD matrix may be an old (probably outdated) matrix or a result from a sample survey. From these two data sources various approaches to estimating an OD matrix has been developed and tested. This survey provides a survey of generic approaches and an annotated bibliography of some individual contributions. Some models and methods are in another paper tested in medium sized applications in a congested region.


Keywords: Estimation of OD matrix, OD matrix estimation using traffic counts.

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# Contents

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## 1. Introduction

The Origin-Destination (OD) matrix is important in transportation analysis. The matrix contains information on the number of travelers that commute or the amount of freight shipped between different zones of a region. The OD matrix is difficult and often costly to obtain by direct measurements/interviews or surveys, but by using traffic counts and other available information one may obtain a "reasonable" estimate. Various approaches to estimating the OD matrix using traffic counts have been developed and tested. The purpose of this paper is to provide a brief survey of generic approaches and an annotated bibliography of some important individual contributions. Following this survey some models and methods are selected for later tests in medium sized transport planning applications.
We consider a network representation of the transport system of the region. The transport system consists of a number of directed transport links that are connected to each other at nodes. The different parts of the region are represented as zones, with centroids in which traffic originates and terminates. The centroids constitute a subset of the nodes of the region.
Assigning an OD matrix to a transportation network means that the demand for traffic between every pair of zones is allocated to available routes connecting the zonal pairs. From the assignment of the OD matrix one thus obtain the traffic volume on each transportation link: the link flow according to the selected assignment procedure. One may pose the "inverse" of the assignment problem: find an OD matrix which, when assigned to the network, reproduces the observed traffic counts. Normally there is a large number of different OD matrices which reproduce the observed traffic counts. The related equation system is underspecified (or degenerate) and may have many possible solutions. The number of OD-pairs (elements in the OD matrix) will, normally, by far exceed the number of links for which traffic counts have been collected. One may thus ask for the most "likely" or "best" OD matrix causing the observed traffic counts. Some models solve this problem by postulating a general model of trip distribution, for example a gravity type model, while other models adopt statistical inference techniques.
This survey paper describes and compares alternative models and methods for OD matrix estimation using traffic counts. We focus on the static OD matrix estimation problem where link traffic count data only exist for one time period.
For practical evaluation of the performance of the transportation system, traffic counts on a subset of the links provide important information. Techniques for traffic count measurements are well developed and traffic volumes in different parts of the network are routinely and accurately collected.

The transport links of the network may exhibit congestion and this can be modelled by cost functions. Cost functions describe the relation between link volume and cost (or time) for travel over the link. The treatment of congestion effects is an important property distinguishing various models for OD matrix estimation. The models assume either that congestion can be treated exogenously (by proportional assignment) or endogenously (by equilibrium assignment). The exogenous case relies on a low level of congestion or on a good "prediction" of the level of congestion. The endogenous case should be normal in the presence of non-negligible congestion. Most models are based on separable cost-flow relations (where the link cost depends only on the flow on that link; more general cases where the flow volumes on neighbouring links affect the cost of a link have been handled by e.g. variational inequalities).
Virtually all models for OD matrix estimation use prior information on the OD matrix. The prior information might be expressed in terms of a "target" OD matrix and/or the number of travelers attracted to/originating in different zones. The target OD matrix can be obtained by a sample survey or from an old (probably outdated) matrix. The existence and typology of a target OD matrix is thus a second important characteristic distinguishing various models for OD matrix estimation.
The models considered in this survey can be plotted as functions of the way congestion and the target OD matrix are handled:

| | Target OD matrix | No target OD matrix |
|---|---|---|
| Congestion | LF 82, YSIA 92, Wil 84, Spi 90, DL 92, EJL 84 | Fis 88, Fis 89, JN 79, ENS 79, JN 83, KLH 92 |
| No congestion | Mah 83, Bel 91, Spi 87, BT 95, KS 87, TW 89 | ZW 80, Cas 84 |

From this plot one can observe that most models assume or require that a target OD matrix is available. It can also be seen that during recent years the emphasis has shifted towards models that are suitable for problems with congested networks.
The survey is organized as follows: in section 2 we provide a more precise problem specification and in section 3 a survey of different approaches proposed for the OD matrix estimation problem using traffic counts is given. Section 4 contains an annotated bibliography. Finally, section 5 contains conclusions and suggestions for comparative tests of selected models in transport planning applications.

## 2. Problem specification

The region is divided into a number of zones represented by centroids. A set of directed links $A$ connect a set of nodes $N$ and the centroids makes up a subset of these nodes. For a subset, $\hat{A}$ of the links and for a specific time period (e.g. peak hour, average day), traffic count data are assumed to be available. The estimation problem is equivalent to finding a reasonable OD matrix $g$ which, when assigned to the transportation network, reproduces the observed traffic count data. In a practical application the reproduction might not be exactly achieved for all traffic counts. This may be explained by internal inconsistencies in the traffic counts due to traffic flow collection at different times or the aggregated transportation network representation. The static OD matrix estimation problem treated here is referring to observed traffic counts for one single time period.
A crucial point in the estimation of an OD matrix using traffic counts is the assignment technique used: what route(s) in the transport network do trips from zone $i$ to zone $j$ take. The matrix $P$ with elements $p_{ij}^a$ represents of the proportion of trips between zone $i$ and zone $j$ that uses link $a$,

$$0 \leq p_{ij}^a \leq 1$$

For a given link $a$, the sum of all $g_{ij}$ (OD-flows from $i$ to $j$) traversing this link is the link volume, $v_a$. The car occupancy factor $R$ (assumed to be 1.25) is included in this fundamental equations relating the link volumes and the OD-flows:

$$v_a = \frac{1}{R} \sum_{ij} p_{ij}^a g_{ij}, \quad a \epsilon A \tag{2.1}$$

where $A$ is the set of links in the transportation network. The matrix with elements $p_{ij}^a$ is often denoted the assignment matrix $P$. Depending on the treatment of congestion exogenous or endogenous determination of the assignment matrix is postulated:

*i)* *Proportional assignment.* In this case we assume independence between the traffic volumes and traffic proportions $p_{ij}^a$. The link volumes ($v_a$) are proportional to the OD-flows ($g_{ij}$). The proportion of travelers choosing a route will not depend on congestion in the network but only on traveler and route characteristics. The values of $p_{ij}^a$ can be determined before the estimation of the OD matrix is done and taken as exogenously given. The "all-or-nothing" assignment method can be used: all-or-nothing assignment of traffic is obtained when all traffic, for all OD pairs, is assigned to the cost minimizing route(s).

*ii)* *Equilibrium assignment.* Wherever congestion effects are important, equilibrium assignment is a more realistic approach. The cost for travelling on a link depends on the flow volume, through a cost-flow relation. Equilibrium assignment techniques try to satisfy Wardrop's first equilibrium principle (Wardrop, 1952): the traffic system is in "equilibrium" when no traveler (user) can achieve a lower travel cost by switching to another route. The value of $p_{ij}^a$ will depend on the volume on all links, i.e. $p_{ij}^a = p_{ij}^a(v)$, and cannot be determined independently of the trip matrix estimation process. The equations (2.1) become nonlinear in $v$.

A flow pattern with link flows $v_a$, $a \epsilon A$, and a related assignment matrix $P(v)$ that fulfills the

equilibrium assignment requirements of *ii*) is recognized as being of the user-equilibrium type. The travellers are said to comply with a user-optimal behaviour. Alternatively, a stochastic equilibrium concept can be employed that takes into account that travelers have different perceptions of travel costs, e.g. due to individual variations. The result is that multiple routes with unequal travel costs are used between any pair of zones. As noted in Cascetta and Nguyen (1988) the computational complexity of an OD matrix estimation model depends largely on the assignment technique used. Building on an equilibrium assignment technique and explicit treatment of congestion effects is more burdensome than relying on proportional assignment.
Given observed link volumes $\hat{v}_a$ on a subset of the links $\hat{A}$ and traffic proportions $p^a_{ij}$, the OD matrix $g$ is determined by solving the equation system (2.1). This equation system is normally highly underspecified: there are many more elements in the OD matrix $g$ than links for which traffic counts are collected. Thus additional data (prior information) and/or assumptions about the travel behaviour are needed in order to find a unique OD matrix.
A central part of the prior information is the target OD matrix. In statistical approaches the target OD matrix is typically assumed to come from a sample survey and is regarded as an observation of the "true" OD matrix to be estimated. The true OD matrix is assumed to belong to some statistical distribution and may be obtained by estimating the parameters of the statistical distribution. In traffic model based approaches, the target OD matrix is normally assumed to be an old OD matrix and one may ask for adjustments of the target OD matrix to satisfy the traffic counts. The distance between the estimated OD matrix and the target OD matrix is minimized subject to the flow constraints. In common for both approaches, the problem of finding the OD matrix $g$, given the target OD matrix $\hat{g}$, is stated as minimizing a function $F_1(g, \hat{g})$. In the notation below all prior information on the OD matrix is contained in the target matrix $\hat{g}$.
Statistically, the observed set of traffic count data may also be assumed to be an observation of the "true" traffic count data to be estimated, related to and obtained as an assignment of the estimated OD matrix. Also for other reasons, touched upon above, deviations between estimated counts and observed counts may be accepted (this conception contrast with the alternative assumption of exact reproduction of the observed traffic volumes). Hence, an OD matrix is sought which produces "small" differences between the estimated link flows $v$ and the observed flows $\hat{v}$. This ambition can be expressed as a criterion $F_2(v, \hat{v})$ to be minimized subject to the assignment constraints.
Although the underlying motivations and assumptions of the OD estimation approaches differ the related optimization problems can be expressed in the following general form:
Determine the demand for traffic between zones of the region, i.e. the OD matrix $g$ solving the program:

$$\begin{aligned} \min \quad F(g, v) &= \gamma_1 F_1(g, \hat{g}) \; + \; \gamma_2 F_2(v, \hat{v}) \quad v, g \geq 0 \\ s.t. \quad v &= assign(g) \end{aligned} \tag{2.2}$$

where $\hat{g}$ is the target OD matrix and $\hat{v}$ the observed traffic counts with $F_1$ and $F_2$ being some distance measures. The assignment of $g$ to the transportation network is denoted *assign(g)*, leading to a split of the OD-flows $(g_{ij})$ over available routes with path flows $(h_{ijk})$.

If the target OD matrix is very reliable and accurate $\gamma_1$ should be large compared to $\gamma_2$ which would result in a $g$ close to $\hat{g}$. Then larger deviations between $v$ and $\hat{v}$ would be accepted. If, on the other hand the observed traffic counts are reliable compared to the information in $\hat{g}$ the magnitude of $\gamma_2$ should be large compared to $\gamma_1$. The second part of (2.2) would then guide the optimization, leading to estimated flows ($v$) that are close to the observed values ($\hat{v}$) while larger deviations between the estimated OD matrix ($g$) and the prior information ($\hat{g}$) would be accepted. The values of the weights ($\gamma_i$) are thus closely related to the conception of the modelling situation. General multicriterion models in the case of proportional assignment were discussed in Brenninger-Göthe et al. (1989).

The distance measures $F_1$ are mainly of the minimum-information type and $F_2$ is often taken to be a Euclidean distance measure. In principle, any other combination of measures is conceivable.

## 3. Survey of modelling approaches

In the survey below, the first category of approaches is based on traffic modelling concepts. These approaches include the "minimum information" ("entropy maximizing") model and combined models for traffic planning. An estimate of the OD matrix is obtained by direct solution ($F_1$ = entropy measure) or by estimating the parameters of the combined model. The second category, statistical inference approaches, includes the Maximum Likelihood (ML), Generalized Least Squares (GLS) and Bayesian Inference approaches. Here the traffic volumes and the target OD matrix are assumed to be generated by some probability distributions. An estimate of the OD matrix is obtained by estimating the parameters of the probability distributions. The target OD matrix $\hat{g}$ is in traffic modelling approaches normally obtained as an old (outdated) OD matrix, while statistical approaches rely on a target OD matrix obtained from sample surveys.

Gradient based solution techniques have been proposed, that solve optimization problems obtained from traffic modelling or statistical inference based approaches. This technique is important because an efficient solver is provided which can and has been applied to large-scale estimation problems based on equilibrium assignment.

Earlier reviews have been published by Cascetta and Nguyen (1988) and Nguyen (1984). Cascetta and Nguyen focus on statistical inference techniques while Nguyen concentrates on minimum information or entropy maximizing approaches. The survey of Cascetta and Nguyen does not promote any particular approach but provides a general framework for the problem of estimating an OD matrix from traffic counts. Nguyen (1984) only considers car traffic while Cascetta and Nguyen (1988) also consider transit traffic. In Fisk (1989) three entropy maximizing models are compared and under certain conditions shown to result in equivalent estimates. Some issues of implementation and computation are touched upon in this section. More comments related to these issues can be found in the annotated bibliography of section 4. This survey differs from the other reviews by its explicit intention to identify candidates for medium scale applications.

The categorisation used is not the only relevant classification. Referring to the figure in the introduction another classification is related to the treatment of congestion. In the characterisation used by Florian (1986) the models are classed into three groups; network equilibrium approaches, gravity-entropy models and combined distribution-assignment models. Most equilibrium based methods have not been applied to networks of sizes relevant in practical planning situations.

### 3.1. Traffic modelling based approaches

Because the information provided by the traffic counts on some links is insufficient to determine a unique OD matrix, it is possible to argue that one should choose a "minimum information" OD matrix. This is an OD matrix that adds as little information as possible to the information in the target OD matrix, while taking the equations relating the observed traffic counts with the estimated OD volumes into account. The estimated, minimum information matrix is obtained from minimizing the function $I$ that corresponds to $F_1$ in equation (2.2) of section 2 (see e.g. Snickars and Weibull (1977) for a combinatorial derivation):

$$I = \sum_{ij} g_{ij}\, ln(\frac{g_{ij}}{\hat{g}_{ij}}) \tag{3.1}$$

This is the minimum information or entropy maximizing ($\hat{g}_{ij} = 1$) function. An OD matrix minimizing (3.1) while reproducing the traffic count constraints and taking the information contained in the target OD matrix $\hat{g}$ into account may be derived as:

$$g_{ij} = \hat{g}_{ij}\, e^{\lambda_1 p^1_{ij} + \lambda_2 p^2_{ij} + \ldots + \lambda_{\hat{A}} p^{\hat{A}}_{ij}} \tag{3.2}$$

where each $\lambda_i$ is a Lagrange multiplier associated with the constraint that relates the link flow with the travel matrix, equations (2.1). The expression (3.2) for $g$ relies on the assumption of proportional assignment, i.e. constant $p^a_{ij}$.
Van Zuylen and Willumsen (1980) proposed two important models of this type. An OD matrix is found that when assigned to the network reproduces the observed traffic counts which are required to be consistent. Traffic modelling approaches directly or indirectly assume that the trip making behaviour is represented by a certain trip distribution model. The models of Van Zuylen and Willumsen are based on minimum information and entropy maximizing principles leading to trip distribution models of the gravity type. In the theoretical part, a target matrix $\hat{g}$ was assumed but in all applications presented, all origin-destination combinations were assumed equally probable, i.e. $\hat{g}_{ij} = 1$.

Fisk (1988) extended the entropy model of Van Zuylen and Willumsen to the congested case by introducing the user-equilibrium conditions as constraints. Smith (1979) has showed that the model of user-optimal behaviour can be expressed as variational inequalities, as stated below. The proposed model has a bilevel structure that maximizes the entropy on the upper level and solves a user-equilibrium problem on the lower choice level:

$$\min_{g_{ij},h_{ijk}} \sum_{ij} g_{ij} \cdot (\ln(g_{ij}) - 1) \tag{$P_{Fi88}$}$$

$$s.t. \quad \begin{cases} \mathbf{C}(\mathbf{h}) \cdot (\mathbf{f} - \mathbf{h}) & \geq 0 & \forall \mathbf{f} \\ \sum_k h_{ijk} & = g_{ij} & \forall i,j \\ \hat{v}_a & = \sum_{ijk} \delta^a_{ijk} \cdot h_{ijk} & \forall a \epsilon \hat{A} \\ g_{ij},\ h_{ijk} & \geq 0 & \end{cases}$$

where $\mathbf{C}(\mathbf{h})$ is the cost of travel on paths given path flows $\mathbf{h}$ and $\mathbf{f}$ is any feasible path flow solution. If the observed flow pattern is a user-equilibrium flow pattern, the extended entropy model of Fisk will have the same solution as a combined trip distribution and assignment model. This is shown in Fisk (1989). For combined models efficient solution algorithms exist. Earlier work includes the combined trip distribution/assignment model by Erlander et al.(1979) and the combined model by Fisk and Boyce (1983). These combined models have the number of trips originating in/attracted to each zone given ($O_i$ and $D_j$ respectively) and this may be expressed as constraints. One important difference between these models and the one by Fisk (1988) above is seen in the assignment constraints. The observed traffic counts do not appear in the combined model but are used to determine the value of the parameter $\mu$ in an estimation phase. One of the so called "doubly constrained" models considered by Fisk and Boyce has the optimization formulation:

$$\min_{g_{ij},h_{ijk}} \mu \sum_a [\int_0^{v_a} s_a(v) \cdot dv] + \sum_{ij} g_{ij} \cdot \ln(g_{ij}) \qquad (P_{FiBo})$$

$$s.t. \quad \begin{cases} \sum_{j \neq i} g_{ij} & = O_i & \forall i & (\alpha_i) \\ \sum_{i \neq j} g_{ij} & = D_j & \forall j & (\beta_j) \\ \sum_k h_{ijk} & = g_{ij} & \forall i,j & (\chi_{ij}) \\ v_a & = \sum_{ijk} \delta^a_{ijk} \cdot h_{ijk} & \forall a & \\ g_{ij},\ h_{ijk} & \geq 0 & & \end{cases}$$

where the Lagrange multipliers are indicated in the parenthesis in the right-hand column. The solution to the minimization problem ($P_{FiBo}$) can be obtained by forming the Lagrangian of the problem and setting the first order derivatives of the Lagrangian with respect to $h_{ijk}$ and $g_{ij}$ equal to zero:

$$\sum_a s_a(v_a) \cdot \delta^a_{ijk} - \chi_{ij} = 0 \ \ \forall\ k;\ \ h_{ijk} > 0 \tag{3.3}$$

$$\sum_a s_a(v_a) \cdot \delta^a_{ijk} - \chi_{ij} \geq 0 \ \forall\ k;\ \ \ h_{ijk} = 0 \tag{3.4}$$

$$g_{ij} = e^{-\alpha_i - \beta_j}\ e^{-\mu\chi_{ij}} = A_i O_i B_j D_j e^{-\mu\bar{c}_{ij}} \tag{3.5}$$

Equations (3.3) and (3.4) stipulate that the routes that are actually used, i.e. with $h_{ijk} > 0$, should have a minimal cost for travel between the relevant zone pair $ij$. The Lagrange multipliers $\chi_{ij}$ are thus equal to the minimum costs for travel. From now we set $\chi_{ij}$ to $\bar{c}_{ij}$, the minimal costs in the road network. Equation (3.5) include the Lagrange multipliers $\alpha_i$ and $\beta_j$ that are determined so that the trip production and attraction constraints are fulfilled. The balancing factors $A_i$ and $B_j$ are transformed Lagrange multipliers, $A_i\, O_i = e^{-\alpha_i}$ and $B_j\ D_j = e^{-\beta_j}$. The value of the parameter $\mu$ is undetermined in the problem formulation above and is calculated in an estimation phase.
The importance of this model stems from the relative ease with which a combined distribution and assignment model can be solved. Erlander et al. prove that the model may

be rigorously estimated and that the corresponding optimization problem has a unique solution if the sum of the integrals of all link cost functions is known:

$$\hat{C} = \sum_{a \epsilon A} \int_0^{\hat{f}_a} c_a(x)dx \tag{3.6}$$

The observed traffic counts are reproduced by a combined model if observed counts for all links are available and if they are consistent with user-equilibrium. Other model refinements have been performed by for example Erlander, Jörnsten and Lundgren for example in a technical report written in 1984. The problem of missing data in the base year, used in the estimation phase, is studied. The authors solve this problem by the prediction of values to be used in a prediction year.

If observed traffic counts only are available for a subset of the links in the network, Fisk and Boyce (1983) suggest how an estimate of $\hat{C}$ may be obtained, "weighting" average link costs by the importance of various link types in the network representation. The model of Fisk and Boyce will in general not reproduce the observed traffic counts. This may to some extent relate to the fact that the observed traffic counts might not constitute an equilibrium flow pattern. Different suggestions on procedures that remove internal inconsistencies have been proposed but no effective procedures for assuring the user-equilibrium property and internal consistency of the observed flow pattern seems to exist. See Yang (1994) for a recent treatment and review of these problems.

In Kawakami et al. (1992) the combined model of Fisk and Boyce is extended further to include two modes of travel, large size trucks and cars. An application to a medium sized network in the city of Nagoya, Japan, is reported. A target matrix appears to have been available but is not included in the model formulation.

In Tamin and Willumsen (1989) not only the gravity model but also the intervening opportunity model is considered, both models are of the doubly constrained type. Applications to a small test problem without congestion using a gravity model, an intervening opportunity model and a gravity-opportunity model are presented.

Nguyen (1977) presented one of the first formulations of the equilibrium based OD matrix estimation problem. The network was congested and Nguyen analyzed properties of the solution obtained thoroughly. The solution will reproduce the observed traffic counts. Though the solution is unique in link flow variables there normally exists many different OD matrices that correspond to these estimated link flows. It remains to choose a criterion for determining a unique OD matrix from all the different OD matrices that reproduce the observed counts.

Jörnsten and Nguyen (1979) and later LeBlanc and Farhangian (1982) started from the assumption of equilibrium assignment and formulated models of the entropy maximizing and the "minimum least squares" type, respectively. The first approach does not require a target OD matrix while the model of LeBlanc and Farhangian also relies on a known target OD matrix. The model development starts from work suggested by LeBlanc in Gur et al. (1980). The motivation for least squares formulation is to obtain the OD matrix having a user-optimal behaviour of travelers and being "closest" (in a GLS meaning) to the target OD matrix:

$$\min_{g_{ij}} \sum_{ij} (g_{ij} - \hat{g}_{ij})^2 \tag{$P_{LF82}$}$$

$$s.t. \quad \begin{cases} \sum\limits_a \int_0^{v_a} s_a(v)\, dv \; - \; \sum_{ij} \bar{c}_{ij}\, g_{ij} & = F \\ \sum\limits_k h_{ijk} & = g_{ij} \qquad \forall i,j \quad (\bar{c}_{ij}) \\ v_a & = \sum\limits_{ijk} \delta^a_{ijk} \cdot h_{ijk} \quad \forall a \\ g_{ij},\; h_{ijk} & \geq 0 \end{cases}$$

where $F$ is the objective function at optimum of Nguyen (1977)'s equilibrium based problem. This is a variable demand problem. The test problems presented in these two works are small.

Most formulations of the congested OD matrix estimation problem have a bilevel structure. The "upper" level problem estimates the OD matrix (assuming link flow volumes given) and the "lower" level problem is the equilibrium assignment problem that determines a link flow pattern of the equilibrium type. On the lower level the demand for travel, the OD matrix, is assumed given. Efficient solution algorithms for OD estimation problems having a bilevel structure are discussed in section 3.3.

## 3.2. Statistical Inference approaches

### 3.2.1. Maximum Likelihood

The Maximum Likelihood (ML) approach maximizes the likelihood of observing the target OD matrix and the observed traffic counts conditional on the true (estimated) OD matrix. It is assumed that the elements of the target OD matrix $\hat{g}$ are obtained as observations of a set of random variables. The observed traffic counts $\hat{v}$ constitute another source of information about $g$, the OD matrix to be estimated, and $\hat{v}$ and $\hat{g}$ are usually considered to be statistically independent. The likelihood of observing $\hat{g}$ and $\hat{v}$ can be expressed as:

$$\mathcal{L}(\hat{g}, \hat{v}|g) \;=\; \mathcal{L}(\hat{g}|g) \cdot \mathcal{L}(\hat{v}|g) \tag{3.7}$$

Due to the independence assumption on the observed traffic counts and the target OD matrix, the likelihood of observing both sets is equal to the product of the two likelihoods. Applying the ML principle for this problem amounts to seeking the OD matrix $g$ that maximizes this likelihood. With the convention that $0 \cdot \ln(0) = 0$ we can as well maximize the logarithm of the product.

If the target OD matrix is obtained by simple random sampling in a region with a stable travel pattern, the target OD matrix may be assumed to follow a multinomial distribution. This is dependent on small sampling fractions $\alpha_i$: if $\hat{g}_{i\cdot}$ trips are sampled out of a total of $g_{i\cdot}$ trips at origin $i$ then $\alpha_i = \hat{g}_{i\cdot}/g_{i\cdot}$. For the logarithm of the probability $\mathcal{L}(\hat{g}|g)$ we have:

$$\ln \mathcal{L}(\hat{g}|g) = \sum_{ij} (\hat{g}_{ij} \;\; \ln(\alpha_i\, g_{ij})) \;\; + constant \tag{3.8}$$

This corresponds to $F_1$ in equation (2.2) of section 2. If the sampling fractions are sufficiently large, a Poisson probability distribution may be assumed for the target OD matrix and for the logarithm of $\mathcal{L}(\hat{g}|g)$ we receive:

$$\ln \mathcal{L}(\hat{g}|g) = \sum_{ij}(-\alpha_i \, g_{ij} \; + \; \hat{g}_{ij} \; \ln(\alpha_i \, g_{ij})) \; + constant \tag{3.9}$$

If the observed traffic counts also are assumed to be generated by a Poisson probability distribution and independent of the target OD matrix, a similar expression for the probability $\mathcal{L}(\hat{v}|v(g))$ is obtained, that is $F_2$ in equation (2.2) of section 2:

$$\ln \mathcal{L}(\hat{v}|v(g)) \; = \sum_{a \epsilon \hat{A}} (\hat{v}_a \; \ln(v_a(g)) \; - \; v_a(g)) + constant \tag{3.10}$$

where $v_a(g)$ denotes the flow volume on link $a$ resulting from an assignment of $g$. If a multivariate normal, MVN, distribution is assumed for the error terms of the observed traffic counts with zero mean and a variance-covariance matrix $W$, the expression for $F_2$ is:

$$\ln \mathcal{L}(\hat{v}|v(g)) \; = -\frac{1}{2}(\hat{v} \; - \; v(g))' \, W^{-1} \, (\hat{v} \; - \; v(g)) + constant \tag{3.11}$$

If (3.9), (3.10) and proportional assignment are valid assumptions, the OD matrix estimation problem can be formulated as:

$$\begin{aligned} \max \quad & \textstyle\sum_{ij}(-\alpha_i \, g_{ij} \; + \; \hat{g}_{ij} \; \ln(\alpha_i g_{ij})) \; + \; \sum_{a \epsilon \hat{A}}(\hat{v}_a \ln(v_a(g)) \; - \; v_a(g)) \\ s.t. \quad & \begin{cases} \sum_{ij} p_{ij}^a \; g_{ij} = v_a, \quad \forall a \epsilon A \\ g_{ij} \geq 0 \end{cases} \end{aligned} \tag{3.12}$$

This is one of the optimization problems considered by Spiess (1987). To solve problem (3.12) Spiess proposes an algorithm of the cyclic coordinate ascent type. The test examples of Spiess are small.
In the case of equilibrium assignment, problem (3.12) will have a bilevel structure with the assignment problem on the lower level and the estimation of the OD matrix on the upper level. See section 3.3 for solution methods of bilevel programs.

### 3.2.2. Generalized Least Squares

The target OD matrix $\hat{g}$ may be assumed to be obtained from the estimated, "true" OD matrix $g$ with a probabilistic error term. In the same way the traffic counts may be viewed as obtained from a stochastic equation:

$$\begin{aligned} \hat{g} \; &= \; g \; + \; \eta \\ \hat{v} \; &= v(g) \; + \; \epsilon \end{aligned} \tag{3.13}$$

where $\eta$ is the probabilistic error that relates $\hat{g}$ with $g$ and $\epsilon$ the error that relates the observed traffic counts $\hat{v}$ with $v(g)$. Often both $\eta$ and $\epsilon$ are assumed to have zero means, $E(\eta) = 0$ and $E(\epsilon) = 0$.
Note that in the derivation of the generalized least squares (GLS) estimator below no distributional assumptions need to be made for the random parts $\eta$ and $\epsilon$ in (3.13). There is only a requirement on the existence of dispersion matrices. In the absence of accurate dispersion matrices often unity matrices (with diagonal elements equal to 1) have been used. There also exist a couple of reports on the sensitivity of the GLS model to using approximate dispersion matrices. Cascetta (1984) concludes that estimations or even "heavy approximations" of the dispersion matrix produced better results than a maximum entropy estimator. This independence of distributional assumptions is one important advantage of the GLS approach. One experience seems to be that the models are much more sensitive to variations and inaccuracies in the traffic count data and the target OD matrix than to values of the parameters, see e.g. Cascetta (1984) or Bierlaire and Toint (1994). The parameters include the dispersion matrices $Z$ and $W$ used in the problem formulation (3.14) below. As in the ML approach the target OD matrix and the observed traffic counts are assumed to be mutually independent. If the target OD matrix $\hat{g}$ has an error with a variance-covariance matrix $Z$ and the dispersion matrix of the traffic counts is $W$, the GLS estimator may be obtained by solving:

$$\begin{aligned} \min \quad & \tfrac{1}{2}(\hat{g} - g)' Z^{-1} (\hat{g} - g) + \tfrac{1}{2}(\hat{v} - v(g))' W^{-1} (\hat{v} - v(g)) \\ s.t. \quad & g_{ij} \geq 0 \end{aligned} \tag{3.14}$$

The estimated OD matrix $g$ should of course be constrained to be non-negative. Cascetta (1984) develops expressions for the mean and variance of the GLS estimator when non-negativity constraints are not active. An important property of the GLS approach is that the two sources of information, the observed traffic counts and target OD matrix, are readily combined. For example, if either dispersion matrix is close to zero, reflecting a great belief in this part of the information, the matrix inverse is very large. This means that the weights on the corresponding deviations are large and hence this part of the observed information is reproduced by the model when the minimum is attained.
The dispersion matrix Z can be approximated in different ways. If origin-based simple random sampling is adopted, an approximation that becomes sparse may be developed (see e.g. Cascetta and Nguyen (1988)). The dispersion matrix $W$ is often considered to be diagonal and hence no covariances between the different traffic counts are assumed.
Bell (1991) presents an algorithm that solves the problem (3.14). This algorithm explicitly considers the non-negativity constraint on the estimated OD matrix $g$. Both Bell (1991) and Cascetta (1984) assume proportional assignment. Bell derives the solution to the estimation problem (3.14) to be:

$$g = \hat{g} + ZP^{'}(PZP^{'} + W)^{-1}(\hat{v} - P\hat{g}) + (Z - ZP^{'}(PZP^{'} + W)^{-1}PZ)\mu \tag{3.15}$$

where $P$ is the assignment matrix and $\mu$ are the Lagrange multipliers associated with non-negativity constraints on the OD matrix $g$. From (3.15) one can see that the target OD matrix is "adjusted", "changed" by two terms. The first term is related to the deviation of the observed traffic counts to those obtained by an assignment of the target OD matrix. The second term is related to active non-negativity constraints, i.e. constraints with non zero Lagrange multipliers corresponding to elements $g_{ij}$ equal to zero.

If the traffic counts and the target OD matrix (their error terms $\eta$ and $\epsilon$) are assumed to follow MVN distributions, the GLS estimator can be shown to coincide with a ML estimator. The same result is also produced using a Bayesian inference approach (Maher 1983) under the assumption of MVN distributions of the traffic volumes and the target OD matrix. In an earlier paper, Bell (1984) has shown that the minimum information approach of Van Zuylen and Willumsen (1980) is approximated by the GLS approach if the traffic counts are known to a high degree.
Yang et al. (1992) extend the GLS model by integrating equilibrium assignment of the OD matrix into the model. The OD matrix estimation problem is formulated as a bilevel program with the generalized least squares problem on the upper level and the equilibrium assignment problem on the lower level. The resulting problem becomes difficult to solve. A heuristic algorithm is suggested and applied to small problems (for other bilevel formulations and solution techniques, see section 3.3 below.)
If the observed traffic data is of the user-equilibrium type and available for all links an OD matrix that reproduces the observed link volumes can be obtained as a solution to an underspecified equation system. It is difficult to determine if the observed traffic data is of the user-equilibrium type. If the underspecified equation system is feasible, Yang et al. (1994) show that the traffic flow is of the user-equilibrium type. Adjusting observed traffic counts to be compatible with user-equilibrium is however difficult and no definite procedure has been suggested.

### 3.2.3. Bayesian Inference

The Bayesian inference approach considers the target OD matrix as a prior probability function $\Pr(g)$ of the estimated OD matrix $g$. If the observed traffic counts are considered as another source of information about $g$ with a probability $\mathcal{L}(\hat{v}|g)$, then Bayes theorem provides a method for combining the two sources of information. For the posterior probability $f(g|\hat{v})$ of observing $g$ conditional on the observed traffic counts we then have:

$$f(g|\hat{v}) \approx \mathcal{L}(\hat{v}|g) \cdot \Pr(g) \tag{3.16}$$

The posterior probability function allows, in principle, for a determination of a confidence region for $g$ but due to practical computational complications only point estimators may be obtained. This may take the form of the maximum value of the logarithm of the posterior distribution, the $g$ that maximizes $\ln f(g|\hat{v})$. For the first term in (3.16), the observed traffic counts, a Poisson probability or a MVN distribution is usually assumed. The expressions for the logarithm of $\mathcal{L}(\hat{v}|g)$ will then be (3.10) or (3.11). For the probability function $\Pr(g)$ a multinomial distribution may be assumed. Then, for the logarithm of $\Pr(g)$, using Stirling's approximation we have:

$$\ln \Pr(g) = -\sum_{ij} g_{ij} \ln(\frac{g_{ij}}{\hat{g}_{ij}}) + constant \tag{3.17}$$

This is the minimum information function. A similar function is also obtained with a Poisson approximation of the multinomial distribution. If a multivariate normal distribution is assumed to hold for $\Pr(g)$, with mean $q$ and dispersion matrix $Z_q$ Maher (1983) obtains:

$$\ln \Pr(g) \approx -\tfrac{1}{2}(g-q)' Z_q^{-1}(g-q) + constant \qquad (3.18)$$

Maher assumes that proportional assignment holds. For the traffic counts Maher makes the MVN assumption and shows that in this case the estimated OD matrix also becomes MVN distributed. The model of Maher is applied to a very small transportation network. The optimization problems of the Bayesian Inference approach contain, just as in the ML and GLS approaches, the sum of two parts. The first relates to the target OD matrix $\hat{g}$, (3.8) or (3.9), and the second concerns the observed traffic counts. The Bayesian Inference approach is a statistical inference technique with properties in common with the ML and the GLS approaches. But, as Cascetta and Nguyen note the roles assumed by $g$ in the classical inference approaches (ML and GLS) and the Bayesian Inference approach differ. In the first case, the true $g_{ij}$ are parameters of the likelihood function $\mathcal{L}(\hat{g}, \hat{v}|g)$ and in the second case the $g_{ij}$ are random variables with given prior distributions.

### 3.3. Gradient based solution techniques

The most general optimization problems (with congested networks) of the previous sections 3.1and 3.2 have a bilevel structure. The OD matrix estimation problem that determines $g$ appears on the upper level and the equilibrium assignment problem on the lower level. Such problems were considered by Spiess (1990), Drissi-Kaïtouni and Lundgren (1992), Yang et al. (1992), Florian and Chen (1993) and Chen (1994). The related solution algorithms have been applied to large city networks.

In gradient based solution techniques, the target OD matrix is taken as an initial solution to the OD matrix estimation problem. The target OD matrix is "adjusted" or "changed" to reproduce the traffic counts by iteratively calculating directions based on the gradient of the objective function. The link volumes are implicit functions of OD flows and obtained by the assignment procedure, $v(g) = assign(g)$, of the user-equilibrium type. This means that the OD matrix estimation problem (using traffic counts) can be formulated in terms of $g_{ij}$ variables only:

$$\min_{g \geq 0} \quad F(g) = \gamma_1 F_1(g, \hat{g}) + \gamma_2 F_2(v(g), \hat{v}) \qquad (3.19)$$

where the $F_i$ are appropriate distance measures. Except for the nonnegativity constraints on $g$ this problem is unconstrained. Drissi-Kaïtouni and Lundgren propose general descent algorithms for solving the problem (3.19). Florian and Chen study a Gauss-Seidel type and an Augmented Lagrangian type method. Of these the former is more suitable for applications to problems of larger networks. Also Yang et al. have suggested a Gauss-Seidel type method. The algorithm of Spiess is more approximate and does in the practical applications presented performe well.

In theory all contributions except for the one by Spiess consider both $F_1$ and $F_2$ in the upper level problem. In applications to problems on networks having considerable size supplied by both Spiess and Drissi-Kaïtouni and Lundgren only $F_2$ appears in the objective function. Chen (1994) and a comparative report by Denault (1994) contain results where also the travel matrix variables, $g_{ij}$, are explicitly considered in the upper level objective function. Higher goodness-of-fit seems to result from including the target OD matrix. As mentioned, an advantage of this approach is its computational tractability. Spiess presents applications to several large scale problems. The problems include an urban application of Bern, Switzerland, with about 2 700 links and one interregional application to the road networks of Finland with about 12 500 links. The method of Spiess is approximate since proportional assignment is assumed to hold locally and the method does not necessarily converge to a solution of the stated optimization problem. With this assumption the gradient of the objective function becomes easy to compute, attainable from the solution of two equilibrium assignment problems. The close relation to one of the methods suggested by Drissi-Kaïtouni and Lundgren is shown by the latter ones. The descent directions employed in the Gauss-Seidel method of Florian and Chen may be interpreted as more elaborate and hence providing a less approximate solver to the estimation problem. As noted by Florian and Chen this interpretation also applies to the Gauss-Seidel method of Yang et al..

The results obtained by Spiess are reasonable with a significant improvement of the goodness-of-fit. The method of Spiess is available within the "commercial" EMME/2 (1990) transportation planning system. The methods of Drissi-Kaïtouni and Lundgren have been applied to problems from the city of Hull, Canada, with about 800 links. Drissi-Kaïtouni and Lundgren investigate different gradient based descent directions possibly scaled by second order information. The results indicate that the computations involved are reasonable. The emphasis is on the quality of the search directions and not on attaining a computationally efficient OD matrix estimator. Chen applies a Gauss-Seidel type method to a medium size network of Winnipeg, Canada, with 2983 links, 154 zone centroids and with observed counts for about 2.3% of the links. The inclusion of OD matrix variables explicitly (i.e. $F_1$) estimates more precise results.

## 4. Annotated bibliography

### 4.1. Traffic Modelling Based approaches

#### 4.1.1. Erlander, Nguyen and Stewart (1979)

On the calibration of the combined distribution-assignment model.

The models considered are of the combined distribution-assignment type similar to model ($P_{FiBo}$) as stated in section 3.1. The contribution of the paper is an investigation into sufficient conditions for calibrating the models. The observed values on the entropy or the total assignment cost for travel, eq. (3.6), are both shown to be sufficient to determine a unique value of the parameter of the model. On the other hand, by giving a counter example the total cost for travel is shown insufficient.

Erlander et al. show that if traffic counts are available for all links in the network the corresponding optimization problem has a unique solution, resulting in a unique OD matrix. For combined models the reader is referred to e.g. Boyce et al. (1983) or Abrahamsson and Lundqvist (1996). This formulation of the OD matrix estimation problem can handle large-scale applications.

### 4.1.2. Fisk and Boyce (1983)

A note on trip matrix estimation from link traffic count data.

Through the calibration of a combined distribution and assignment model an estimated OD matrix is obtained. Equilibrium assignment is assumed to hold and the observed traffic counts can be used to estimate the sum of the integrals of the link cost functions $\hat{C}$.
In practice, only a sample of traffic counts is available normally. The sample is generally not random and Fisk and Boyce proposes an unbiased procedure for estimating $\hat{C}$. They stratify the sample into $K$ groups with corresponding mean values of the cost function $\hat{C}_k$:

$$\hat{C}_k = \frac{1}{n_k} \sum_{a \epsilon A_k} \int_0^{\hat{f}_a} c_a(x)\, dx$$

where $A_k$ is the set of links in group $k$ and $n_k$ the number of links in group $k$. The total cost may then be estimated as:

$$\hat{C} = \sum_{k=1}^{K} p_k \hat{C}_k \tag{4.1}$$

where $p_k$ is the proportion of all network links in group $k$. With the estimate (4.1) of the total cost a doubly constrained combined distribution-assignment model is formulated. This problem has a unique solution and efficient solution algorithms exist, see e.g. Boyce et al. (1983) or Abrahamsson and Lundqvist (1996). The observed traffic count data are used in a very aggregate way and there is no requirement of a reproduction of the observed traffic counts. The relation to a bilevel formulation of the estimation problem as investigated by Fisk (1988 and 1989) should be mentioned.
No target matrix is required and no applications are reported. However, an extension of the model to include two travel modes has been developed and applied to the city of Nagoya, Japan, by Kawakami et al. in 1992.

### 4.1.3. Fisk (1988)

On combining maximum entropy trip matrix estimation with user optimal assignment.

The paper combines the entropy maximizing estimator of Van Zuylen and Willumsen with a user-equilibrium assignment procedure such as SATURN, Van Vliet (1982). In the latter work by Van Vliet, the user-equilibrium conditions are formulated as variational inequalities. By combining these two works, a mathematical programming problem with a bilevel structure is stated. No applications are reported. Prior information (target matrix) may be included.
See Fisk (1989) for linkages of this formulation to combined trip distribution and assignment models, e.g. Erlander et al.(1979), and outlines of solution methods.

### 4.1.4. Fisk (1989)

Trip matrix estimation from link traffic counts: the congested network case.

Three formulations for OD matrix estimation on congested networks are examined. When the observed traffic counts constitute a user-equilibrium pattern, the different formulations are shown to have equal solutions.

The first formulation is based on e.g. Jörnsten and Nguyen (1979) that determines an OD matrix which reproduces the observed traffic counts and has maximal entropy. The entropy maximizing model of Van Zuylen and Willumsen (1980), extended to the congested case by Fisk (1988), is the second formulation and finally formulations based on the combined distribution and assignment model, e.g Erlander et al. (1979), are studied. The latter model does not have the bilevel structure present in the other two formulations. No applications and solution techniques of the models are explicitly discussed. For the combined model efficient solution algorithms exist. All models are based on equilibrium assignment.
The travel patterns that solve the models are shown to be equal if the observed flow pattern is of the user-equilibrium type. From a computational perspective Fisk favours the combined trip distribution and assignment model approach. Very few methods for assessing the consistency of observed traffic counts with user-equilibrium and for removing potential inconsistencies have been presented, see e.g. Yang et al. (1994).

#### 4.1.5. Jörnsten and Nguyen (1979)

On the estimation of a trip matrix from network data.

The paper extends the model of Nguyen (1977) relying on equilibrium assignment by seeking the OD matrix with maximal entropy. Also exogenous information on the OD matrix, such as trip production and attraction and the total number of trips or an old (target) trip matrix, may be considered. The model by Nguyen (1977) fails, as Nguyen has observed, to find a unique trip matrix due to the underspecification problem. This problem is solved by requiring that the estimated matrix has maximal entropy. The resulting problem has a bilevel structure with equilibrium assignment on the lower level and a maximum entropy problem on the upper level. The observed traffic count data are internally consistent and are assumed to be of the user-equilibrium type and also available for all links.
An algorithm based on generalized Benders decomposition is developed. Three small numerical examples with up to 12 transportation links and a Sioux Falls, South Dakota, network with 76 links are reported.

#### 4.1.6. Jörnsten and Nguyen (1983)

Estimation of an OD trip matrix from network data: dual approaches.

Following Jörnsten and Nguyen (1979), methods based on user-equilibrium route choice and the minimum information principle are presented. A sequence of combined distribution and assignment problems are handled as part of solving the OD matrix estimation problem with a bilevel structure. Different formulations of the problem and the corresponding dual are derived and investigated. The computational requirements of the algorithm of Jörnsten and Nguyen (1979) are extensive and the dual approaches presented here are claimed to be computationally superior in large scale applications.
Only the first combined problem need a complete cold start optimization. Later problems can use the previous solution as an initial solution to the present combined distribution and assignment problem. The observed traffic count data are constrained to be reproduced by the model. Applications to small problems including a network with 7 links and a Sioux-Falls, South Dakota, network are referred to. The solution for the latter problem is better, with the same computational effort, as compared to the results of the approach in Jörnsten and Nguyen (1979).

### 4.1.7. Kawakami, Lu and Hirobata (1992)

Estimation of origin-destination matrices from link traffic counts considering the interaction of the traffic modes.

A combined trip distribution, modal split and assignment model is proposed. The model by Fisk and Boyce (1983) is extended to include also an entropy constraint with respect to mode choice. The trip distribution and modal split model is of the singly constrained nested combined type. If $p_{ijmr}$ is the proportion of traffic from zone $i$ to zone $j$ by mode $m$ and route $r$ the entropy constraint with respect to mode choice reads:

$$-\sum_{ijm}(\sum_{r} p_{ijmr}) \ln(\sum_{r} p_{ijmr}) \leq E_M$$

where $E_M$ is the observed entropy value with respect to mode choice. An algorithm for solving the combined model is proposed and an application to a simplified road network in Nagoya, Japan, is presented. The transportation network has about 500 transportation links with two modes for travel (large sized trucks and car). No target OD matrices are assumed in the model but observed OD matrices are available and reasonably well reproduced. The model is based on equilibrium assignment but the degree of congestion in the application is not obvious from the results presented. The OD matrix estimation application is concerned with all day (24 hour) traffic, which suggests that a detailed analysis of congestion is not aimed for.

### 4.1.8. LeBlanc and Farhangian (1982)

Selection of a trip table which reproduces observed link flows.

The paper extends the model of Nguyen (1977) relying on equilibrium assignment by seeking the OD matrix closest to a target OD matrix. In the bilevel formulation, a GLS problem is stated on the upper level and the problem on the lower level is of the user-equilibrium type. The relation to the method of Jörnsten and Nguyen (1979) is noted where the upper level problem is an entropy maximization problem. The lower level user-equilibrium problems are solved by the Frank-Wolfe method developed to use an evolving bounds technique.
Computational results from an application with 76 links – a Sioux Falls, South Dakota, network – are presented. The reported results are termed ‘attractive’ and the observed traffic count data are reproduced.

### 4.1.9. Sherali, Sivanandan and Hobeika (1994)

A linear programming approach for synthesizing origin-destination trip tables from link traffic volumes.

A linear programming model consistent with user-optimal behaviour of travellers is developed. The model recognizes that the observed traffic counts often do not comply with user-equilibrium and is capable of handling a prior target matrix.
Most notable is the small computational demands compared to a maximum entropy model and a network equilibrium approach (relying on an algorithm proposed by Gur et al. (1980)) on small networks. The model is also applied to a portion of the real network of Northern Virginia being of a modest size. The algorithmic tuning in terms of convergence criteria and other ’model refinements for such real-world applications’ are on the research

agenda of the authors. The model may have computational limitations when applied to problems of larger networks.

#### 4.1.10. Tamin and Willumsen (1989)

Transport demand model estimation from traffic counts.

The three transport demand models considered by Tamin and Willumsen are the gravity, the intervening opportunity and a combined gravity/intervening opportunity model. All models are of the doubly constrained type. Three different estimation methods are investigated corresponding to each one of the models. Proportional assignment of traffic is assumed and both an all-or-nothing and a stochastic assignment technique are used. No target OD matrix is assumed in the models though the results are compared to a base year, target matrix.
A small application to real data from the town of Ripon, Great Britain, is presented. The transportation network has 188 transportation links. The gravity-opportunity model is found to produce the best fit in Ripon, i.e. the best matching of observed and estimated link volumes. If the estimated and observed OD matrices are compared the gravity model is however found to produce the best fit. The results indicate a dependence on the assignment technique. The amount of congestion in the Ripon application is negligible.

#### 4.1.11. Van Zuylen and Willumsen (1980)

The most likely trip matrix estimated from traffic counts.

The proposed models estimate OD matrices that reproduce the traffic counts. Proportional assignment is assumed. The algorithms iteratively adjust an initially given target matrix. The estimated OD matrix is constrained to reproduce the observed traffic counts. The models are based on minimum information and entropy maximizing principles. The problem of inconsistent observed traffic counts is discussed and a maximum likelihood method to remove inconsistencies and produce a better estimation of the observed link flows is described.
In applications to a small artificial network having 72 links the models are found to perform well for all but one case. In all test cases no target OD matrix was assumed known and the models were of the maximum entropy type. In the case with unsatisfactory results the authors note that these could certainly be improved with better prior information (a target OD matrix).

#### 4.1.12. Willumsen (1984)

Estimating time-dependent trip matrices from traffic counts.

The models proposed extend the entropy maximizing model by incorporating a target OD matrix. The objective function considered is:

$$- \sum_{ij} g_{ij}(log(\frac{g_{ij}}{\hat{g}_{ij}}) \ - \ 1)$$

Congestion effects in urban areas are here treated by letting the assignment matrix $P$ vary over time. The number of variables in the related problem is noted to be large. A heuristic solution approach is developed and some results are reported. The main focus of the paper is on the estimation of matrices that vary over time. This reference is included because of the minimum-information type of objective function.

## 4.2. Statistical Inference approaches

### 4.2.1. Maximum Likelihood, Spiess (1987)

A maximum-likelihood model for estimating origin-destination matrices.

In Spiess (1987) it is assumed that the elements of the target trip matrix $g$ are obtained by sampling, for all OD pairs, Poisson variables. In a first model, Spiess estimates the means of the random variables by the maximum-likelihood technique:

$$\min \quad \sum_{i,j}(\alpha_i g_{ij} - \hat{g}_{ij} \ln(\alpha_i g_{ij}))$$

$$s.t. \quad \begin{cases} \hat{v}_a = \sum_{ij} p_{ij}^a g_{ij}, \quad a \epsilon \hat{A} \\ g_{ij} \geq 0 \end{cases} \tag{4.2}$$

This model reproduces the observed traffic count data. If the feasible set defined by the constraints is non empty, there exists an optimal solution (this requires the constraints to be internally consistent). Proportional assignment is assumed. A convergent algorithm of the 'cyclic coordinate descent' type is developed. In a section on statistical inference, the validity of the model is displayed using tests derived from the asymptotic behaviour of the distribution.
Also a doubly constrained model and a model where the observed traffic counts are not required to equal the traffic volumes obtained by an assignment of the estimated OD matrix are studied. Solution algorithms to both these extensions are supplied. One important practical problem is mentioned: the derivation of the extended model (when the observed traffic counts are not required to equal the estimated traffic volumes) relies on the assumption of mutually independent traffic counts. This means that the observation of the traffic counts must take place at different times because, otherwise, a certain traveler might be counted on more than one link and the model may only be an approximate ML model.
An advantage of the model is the feasibility with respect to values of $\hat{g}$. The maximum likelihood method is always feasible while an entropy maximizing method might become infeasible because of zero valued elements of $g$. The test problems are small, i.e. networks with no more than 20 transportation links.

### 4.2.2. Generalized Least Squares, Bell (1991)

The estimation of origin-destination matrices by constrained generalized least squares.

The approach of Bell explicitly considers the non-negativity constraints on the OD matrix $g$. It is shown that this can improve the accuracy of the estimated OD matrix.
The route choice is assumed to follow a proportional assignment procedure. The optimization program of Bell is that of section 3.2.2 above with the solution stated in equation (3.15). It can be seen that active non-negativity constraints affect the estimated OD matrix $g$ through the last term in (3.15) (also strictly positive matrix elements $g_{ij}$ are affected). Bell further supplies a solution algorithm to the optimization problem with non-negativity constraints. The convergence is proved and expressions for variances and covariances of the estimated OD matrix are derived.
A small numerical example with 5 transportation links is reported.

### 4.2.3. Generalized Least squares, Bierlaire and Toint (1995)

MEUSE: An origin-destination matrix estimator that exploits structure.

The model is a developed GLS estimator, (Bell (1991) or Cascetta (1984)), that also contains terms derived from parking survey data. A type of augmented Lagranian algorithm is used to solve the problem. The determination of weights corresponding to the different term should 'ideally reflect the relative confidence one has in the associated terms' and the values of different parameters are determined beforehand. The model assumes proportional assignment.
In applications to both an artificial example and a real case study of Namur, Belgium, the performance is displayed. Comparisons with a GLS and an equilibrium based estimator (relying on the SATURN, Van Vliet (1982), software package) are supplied and the 'MEUSE-estimator' is found to estimate a reasonable OD matrix. The examination of the model sensitivity is noted to be important, in particular with respect to parameter variation. The results are found more sensitive to variations in the observed traffic counts than to changes in the parameters of the model.

### 4.2.4. Generalized Least Squares, Cascetta (1984)

Estimation of trip matrices from traffic counts and survey data: A generalized least squares estimator.

The route choice is assumed to follow a proportional assignment procedure. First Cascetta derives the estimator when the observed traffic counts are not necessarily reproduced by the estimated traffic volumes. If the observed traffic counts are known with a high degree of certainty the corresponding dispersion matrix of the traffic counts ($W$ in equation 3.14) is very small and the related estimator is also studied. Statistical characteristics of the two GLS estimators are derived that include the expected value and the variance-covariance matrix of the estimators. The derivations rely on the assumption of a strictly positive OD matrix $g$, i.e. the nonnegativity constraints on the travel matrix do not become active and no OD flows equal to zero are allowed.
In an application to a small artificial network with 5 zones, the effects of using approximate dispersion matrices and a comparison with an entropy maximizing estimator reproducing the traffic counts are studied. Approximate dispersion matrices only affect the characteristics of the estimator in a minor way and the GLS estimator has a mean square error superior to that of the entropy maximizing estimator.
Cascetta discusses the possibility of using an equilibrium assignment model and among others refers to models following e.g. Jörnsten and Nguyen (1979). In the equilibrium assignment case it is claimed to be difficult, if ever possible, to obtain expressions for the statistical characteristics.

### 4.2.5. Generalized Least Squares, Yang et al. (1992)

Estimation of origin-destination matrices from link traffic counts on congested networks.

The OD matrix estimation problem is formulated as a bilevel program. Generalized least squares terms constitute the objective function of the upper level. On the lower level equilibrium assignment is assumed. Similar bilevel formulations have also been suggested by LeBlanc and Farhangian (1982) and Jörnsten and Nguyen (1979) with a generalized least squares and entropy maximum objective function on the upper level respectively. The latter entropy maximizing model is discussed as an alternative formulation. The models of

Yang et al. do not require the observed traffic counts to be internally consistent nor to be of the equilibrium type. This is further discussed in comparisons with other equilibrium based models such as the one by Nguyen (1977) and Fisk (1988) that are interpreted as special cases obtained for certain parameter values. Fisk has discussed a version of the proposed solution algorithm. The important difference is that the observed traffic counts are viewed as observations of random variables and appear as a part of the upper level objective function and not as constraints to be fulfilled.
The difficulty with solving the bilevel program is noted and a heuristic solution procedure is proposed. The upper level and the lower level (user-equilibrium) problems are solved iteratively using proportional assignment from the lower level solution in the next upper level problem. The algorithm was shown to converge satisfactory in an application to a small test problem with a transportation network that has 24 transportation links. The method can be interpretated as a gradient based approach. Some statistical measures of error for the estimated OD matrix are calculated and show improvement. High quality traffic counts, with small internal variation, give more significant improvement.

### 4.2.6. Generalized Least Squares, Yang et al. (1994)

The equilibrium-based Origin-Destination matrix estimation problem

The observed traffic data are assumed to be of the user-equilibrium type and available for all links in the network. Then, it is shown that an OD matrix reproducing the traffic counts can be obtained by solving an underspecified system of linear equations. The problem of identifying if the observed traffic counts are of the user-equilibrium type can be solved by finding out if these equations have a feasible solution. Among the OD matrices that are compatible with the equation system, the OD matrix solving a generalized least squares or an entropy maximizing problem is taken to be the estimated OD matrix. The resulting model does not have the bilevel structure present in other models solving the congested OD matrix estimation problem.
Before the actual estimation problem is solved the observed traffic counts must be expanded to all links. An approach that determines the volumes on all links from an equilibrium assignment of the target matrix resulting in volumes of the missing links is suggested. These link volumes are scaled so that they become compatible with the total magnitude of the observed volumes. A small OD matrix estimation problem is supplied. Further work to develop estimation methods that also consider uncertainties in and non user-optimality of observed traffic volumes remains, compare Yang et al. (1992).

### 4.2.7. Bayesian Inference, Maher (1983)

Inferences on trip matrices from observations on link volumes: A Bayesian statistical approach.

Maher assumes that the target (prior) OD matrix and the observed traffic counts follows multivariate normal (MVN) distributions and that proportional assignment of route choice holds. Traffic count data for all links are assumed available. The estimated OD matrix becomes MVN distributed.
An advantage, as to traffic modelling approaches, is the allowance of different beliefs in the target OD matrix and the observed traffic counts. The relative beliefs are expressed in the variance-covariance matrices related to the target OD matrix and observed traffic counts.

An iterative solution method that updates the target OD matrix and the related dispersion matrix is suggested. At a general level, with dense dispersion matrices the computational demands are considerable and Maher studies some special cases with related updating equations. These cases include independent observations, observations without error and least-informative prior.
A small test example, taken from Bell (1983), with 6 OD-pairs and 4 links is presented. The minimum information (entropy maximizing) approach is seen to be one extreme case of a whole range of possibilities as to beliefs in the target OD matrix and the observed traffic counts. The minimum information approach represents a mimimal belief in the target matrix.

## 4.3. Gradient based solution techniques

### 4.3.1. Chen (1994)

Bilevel programming problems: Analysis, algorithms and applications.

The PhD thesis studies the bilevel programming problem thoroughly. Both historical and theoretical developments are reviewed. This includes results on the existence of solutions and the complexity of algorithms solving the problem. Our interest focuses on real world applications of the bilevel OD matrix estimation problem.
Two methods are suggested, an Augmented Lagranian method and a Gauss-Seidel type method. They were implemented by using macros of the "commercial" EMME/2 (1990) software. The first method shows satisfactory results on a small network taken from Nguyen and Dupuis (1984). Though only local convergence is guaranteed the estimated OD matrixes differ only marginally for different initial (target) OD matrices.
Chen argues that for large-scale problems algorithms that work directly in the link flow space (as opposed to the route flow space) and that do not require repeated objective function evaluations are needed. A heuristic Gauss-Seidel method is developed and applied to problems from Winnipeg, Canada, with a network containing 2982 road links, 154 zone centroids and about 2.5% of the links with observed traffic counts. It is concluded that the estimated results show higher quality if a target OD matrix is included in the objective function.

### 4.3.2. Denault (1994)

Étude de deux méthods d'adjustement de matrices origine-destination à partir des flots des véhicules observés (in French).

Two OD matrix adjustment methods applied to problems of congested networks are compared. The first is an Augmented Lagranian approach suggested by Chen (1994) and secondly a gradient based method proposed by the author is studied. The latter relies on sensitivity analysis developed by Tobin and Friesz (1988), compare the gradient based method of Drissi-Kaïtouni and Lundgren (1992). The performance of the methods were studied on a small network and a medium sized network of Winnipeg, Canada. The first method performed well on the small network but problems related to the large number of parameters were experienced in the Winnipeg application. The gradient method resulted in excellent results and was prefered because of its relative ease of use.

### 4.3.3. Drissi-Kaïtouni and Lundgren (1992)

Bilevel origin-destination matrix estimation using a descent approach.

Following the formulation in equation (3.19) a general descent algorithm consistent with equilibrium assignment is proposed. The gradient of the objective function $F$ is:

$$\nabla F(g) \quad = \gamma_1 \nabla F_1(g) \; + \; \gamma_2 \nabla F_2(v(g)) \tag{4.3}$$

The difficulty with computing this gradient is related to the Jacobian $J = \nabla_g v(g)$ where $v(g)$ are the link volumes obtained by the assignment procedure. Following Tobin and Friesz (1988), who proved the Jacobian, $J$, to be unique, the computation of these derivatives will in theory require two matrix inversions. By solving a set of quadratic problems the computations become reasonable and with some minor additional calculations $\nabla F(g)$ is attained. It is noted that using a 'projected gradient' determined with a minimal computational effort, is equivalent to the approach of Spiess (1990). The method of Spiess is in essence equivalent to making simplifying assumptions in the quadratic problems. The search directions based on the gradient can be improved by using second order information obtained from the *Hessian* of $F$, the matrix of second order derivates. Computing the diagonal of this *Hessian* can be performed through calculations similar to those for obtaining the Jacobian $J$. Scaling the gradient with the inverse of this diagonal matrix then result in a method using second order information.
In the algorithm, equilibrium assignment subproblems are repeatedly solved and since path-flow variables are used each assignment problem can be solved by reoptimizing the previous one. The traffic assignment procedure uses a code by Larsson and Patriksson (1992). However, the requirements for memory space are considerable. In theory the descent algorithm can be shown to converge but no guarantee can be given that the limit point is a global or even a local minimum. Since the algorithm is a descent method Drissi-Kaïtouni and Lundgren however notes that the objective function is always improved from the starting point (the starting matrix), which leads to some sufficiently improved OD matrix.
The approach is implemented and applied to test problems from the city of Hull, Canada with 798 links and 146 OD pairs with non zero demand for traffic. The upper level objective function only includes link flow variables and a least squares function is employed: $Z(g) \; = \; \frac{1}{2} \sum_{a \epsilon \hat{A}} (v_a(g) \; - \; \hat{v}_a)^2$. The main emphasis is not on the computational efficiency of the methods but rather on the quality of the search directions in the solution process. In the applications traffic counts were assumed to be available for 5 % of the total number of links. The results of methods using various directions are all reasonably accurate and no large differences were visible in the final OD matrix estimated. The more elaborate search directions (using second order information) do render a faster objective function improvement than the gradient based search directions.

### 4.3.4. Florian and Chen (1993)

A coordinate descent method for the bilevel OD matrix adjustment problem,

The OD matrix estimation problem is formulated as a bilevel program with user-equilibrium assignment on the lower level. A practical algorithm to solve the OD matrix estimation problem when the flows in the network are distributed according to the user-optimal principle is developed.
It is argued that using path-flow variables in large networks is impractical. A Gauss-Seidel type method is proposed that does not use path information explicitly and iterates between the upper and lower level problems. The path dependent information is used implicitly

and it is devised how the algorithm may be implemented in the EMME/2 (1990) software package. In each iteration at most three equilibrium assignment problems are solved.
The algorithm is applied to two test problems. The first is a small network taken from Nguyen and Dupuis (1983) with 20 links and 4 OD pairs. The results from an application to a larger Winnipeg network is termed 'very encouraging' but it is stressed that the method indeed is heuristic and very dependent on the correct structure of the initial (target) OD matrix relative to the observed traffic counts.

### 4.3.5. Spiess (1990)

A gradient approach for the OD Matrix adjustment problem,

The problem of Spiess has a bilevel structure and uses a steepest descent method for the solution of the upper level problem. On the lower level, equilibrium assignment is assumed to hold. The optimization problem is the following:

$$\begin{aligned} \min \quad Z(g) \ &= \ \tfrac{1}{2} \textstyle\sum_{a \epsilon \hat{A}} (v_a(g) \ - \ \hat{v}_a)^2, \\ s.t. \ \ v \ &= assign(g) \end{aligned} \tag{4.4}$$

This is the same problem as considered by Drissi-Kaïtouni and Lundgren (1992). The approach of Spiess uses a gradient based on the relative change in the demand, and can be written as:

$$g_i^{l+1} = \quad \begin{cases} \hat{g}_i, & l = 0 \\ g_i^l(1 - \lambda^l[\frac{\partial Z(g)}{\partial g_i}]_{g_i^l}), & l = 1, 2, 3, ... \end{cases} \tag{4.5}$$

An expression for the gradient $\frac{\partial Z(g)}{\partial g_i}$ is easily determined if proportional assignment is assumed to hold locally within an iteration. An explicit expression for the optimal step length $\lambda^*$ can also be determined.
It is shown how this gradient method can be implemented using the standard version of the EMME/2 (1990) transportation planning software. Each iteration of the gradient method only corresponds to one or two equilibrium assignments plus some minor additional calculations. Even though this represents a substantial effort, it is claimed to be computationally reasonable for the largest networks that can be handled within the EMME/2 system. The method is available as a macro DEMADJ.
The method has successfully been applied to several large scale problems in Switzerland, Sweden and Finland. The networks of Sweden and Finland are both national networks while the two problems of Switzerland are urban applications of Bern and Basel. The largest network has 469 traffic zones and 12 476 transportation links. The method has been found to perform satisfactorily.

## 5. Conclusions for future work

In our applications to data from the greater Stockholm region, models suitable for problems of big cities with congested transportation networks should be chosen. Accounting for congestion effects implies that assignment procedures of the equilibrium type need to be used. In the Stockholm region a target OD matrix is available. The transportation network is normally large with thousands of transport links and hundreds of zones. Only a few approaches that are actually tested on problems of large congested networks are available,

including the approaches of Spiess (1990), Drissi-Kaïtouni and Lundgren (1992), Florian and Chen (1993), Chen (1994) and the combined trip distribution and assignment model (e.g. Fisk and Boyce (1983)). The combined model should be developed to also consider a target OD matrix. A difference is seen in the problem formulations of Drissi-Kaïtouni and Lundgren and the other gradient based models. The former formulate the assignment problems in route flow variables and develop a computationally fast solution procedure. This leads to high memory requirements whereas the other methods have substantially larger computational demands but less memory space needed.
The methods of Spiess (1990) and Drissi-Kaïtouni and Lundgren (1992) and a combined distribution and assignment model will be tested on Stockholm data, beginning with a medium sized network (46 zones, about 400 nodes and 1000 links).

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