HYPERNUMBERS: REAL OR IMAGINARY?

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On 22 August 1975, Dr. Charles Muses gave a talk at IIASA on hypernumbers, in which he has worked extensively. The talk was fascinating to me and he expressed many viewpoints with which I very much agree and which are not often voiced. He left reprints of two papers [1] [2] which I was eager to read. In general, his discussion and some of the claims he put forth stimulated me to re-examine a subject to which I had not given serious thought for perhaps fifteen years or more.

On reading the papers, however, I found myself confused by the notation, especially by apparent inconsistencies. Also, the proliferation of "species and subspecies" of numbers without any apparent motive, and the incomplete development of the theory were troublesome. Furthermore, the use of exponential forms and the introduction of "bimatrix arithmetic" before the set or sets of numbers and their arithmetic (on which Muses lays great stress, properly I think) are rigorously defined, gives an impression of sleight-of-hand. Finally, the statement in [2] that "specific details of method cannot be discussed explicitly at this time because of negotiations in progress," is a virtual invitation to examine the subject critically. This paper does so from one viewpoint, namely the use of matrix arithmetic as a convenient mechanism for calculation with quantities which are noncommutative and, in some ways, non-unique.

Rather than attempt to "straighten out" Muses'notation, I will simply start from the beginning with notation of my own, standard in so far as applicable. Accidental similarities with other parts of Muses' notation should not be assumed to imply equivalence.

The basic numbers are, of course, the positive integers. By use of the minus sign, we introduce both subtraction, which then requires zero, and negative numbers. As soon as ratios are needed we have the rationals and, to take roots, in general we need irrationals. Finally, trigonometric and exponential forms introduce transcendentals. At some time in the past, all of these "innovations" engendered lengthy and sometimes bitter controversy. People seemed unwilling at first to believe that such "unreal", "irrational" concoctions could have any meaning. However, the field of real numbers has been in use by some mathematicians since ancient times, whether explicitly called that or not.

Everything can be worked out fine with the reals until one has to take the square root of a negative number, in particular the final factor $\sqrt{-1}$. By simply defining $i = \sqrt{-1}$, a whole new world of mathematics opened up which was developed in the theory of functions of a complex variable during the nineteenth century -- one of the most brilliant and complete accomplishment of mathematical genius to date. It seems strange now that less than two centuries ago eminent mathematicians were almost afraid to mention the "imaginary number" i, a term which has unfortunately persisted.

Several notational forms have been used to represent i and complex numbers. Unfortunately, these have often been mixed, somewhat indiscriminately, which can easily lead to misconceptions. Also, the <u>rules</u> of notation have been violated at times leading to such concoctions as quaternions which fell into disuse some seventy-five years ago, although they remain an interesting side-line in texts on abstract algebra. Since the unnecessary terms one gets by "playing with symbols" form groups, there is always the temptation to try to impute some profound meaning to symbols which, under defined arithmetic rules, transform into a subset which is isomorphic to familiar quantities.

Nevertheless, the $\sqrt{-1}$ did introduce something new in kind. All the reals, from integers to transcendentals, have one unit element, 1, and one zero element, 0. But complex numbers require <u>two</u> units, which are usually designated 1 and i. <u>This is the</u>

mistake. Complex numbers are two dimensional and we use the real unit 1 to measure in both directions; hence we should call a unit along the real axis in the complex plane by another name, say r. Using the familiar 2 x 2 matrix notation, we have,

$$r = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} , \quad i = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Note that only real numbers go in the matrices. The matrices, themselves, represent units in the complex field.

Certain rules are involved in the use of these matrix representations. Only the diagonals are used for units. Hence they can be added to form complex numbers which then obey the rules of matrix arithmetic, a great convenience. For example,

$$r + i = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} , \quad r - i = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Both these matrices are nonsingular and, in fact, any matrix representation for

$$ar + bi \quad , \quad a, b \text{ real and not both zero}$$

is nonsingular. For, we have

$$\det \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = a^2 + b^2 > 0.$$

It is also readily verified that the product of any two such matrices is commutative and gives a product of the same form. It is obvious that the same is true for addition and subtraction. Matrix inversion represents reciprocation and also gives a result of the same form. Hence division is likewise defined. Finally, the zero element is unique, namely

$$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and has no nonzero factors. Hence complex arithmetic obeys all the laws of real arithmetic.

However, in defining r and i, we used an arbitrary selection. The definition of r is more or less determined since it is the matrix identity. For i, we could just as well use

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} .$$

However, as i was defined above, this is -i. Hence, we have

$$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad r = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad -r = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad i = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad -i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

But note that, apart from negatives, we have ignored two other possibilities:

$$m = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad w = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and

$$m^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = r ,$$

$$w^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = r .$$

This observation seems to be the basis for speculation about hypernumbers. Can we attach a meaning to these additional square roots of unity and what laws do they obey? Note carefully that m and w contain <u>only real units</u>. Writing such mixtures as

$$\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

is really nonsense, although the above is formally the square root of m.

If one uses m and w with r and i, two things occur: multiplication is not always commutative and there are divisors (non zero factors) of zero. The following relationships are evident or readily verified:

1. For $x = r, i, m$ or w

   $x - x = 0$

   $rx = xr = x$

   $(-r)x = x(-r) = -x$

   $r^2 = (-r)^2 = m^2 = (-m)^2 = w^2 = (-w)^2 = r$

2. The multiplication of units is noncommutative:

r	m	i	w
m	r	-w	-i
i	w	-r	-m
w	i	m	r

3. $[\pm(m \pm i)]^2 = [\pm (i \pm w)]^2 = 0$ (8 square roots of 0)

4. Numerous forms have no reciprocals, for example:

$$(m - i)(m + i) = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = 2r - 2w \quad ,$$

$$(r - m)^2 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} = 2(r - m) \quad .$$

5. r has ten square roots:

   $\pm r, \pm m, \pm w, \pm \frac{1}{\sqrt{2}}(m - w), \pm \frac{1}{\sqrt{2}}(m + w) \quad ,$

6. $-r$, i and $-i$ each have two square roots:

   $\sqrt{-r} = \pm i, \quad \sqrt{i} = \pm \frac{1}{\sqrt{2}}(r + i) \quad , \quad \sqrt{-i} = \pm \frac{1}{\sqrt{2}}(r - i) \quad ,$

just as in complex arithmetic.

Notably missing are $\sqrt{\pm m}$ and $\sqrt{\pm w}$ in proper forms. Going to 3×3 matrices introduces inconsistences in forms since one sometimes must use one real unit and sometimes two for the same quantity. However, if we go to 4×4 matrices, the problem can be resolved. This is the analogue of going from 1×1 (i.e., real) to 2×2 (complex) to find $\sqrt{-1}$.

Starting with r, the only real unit in complex space, define four units as follows:

$$u_o = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}, \quad u_1 = \begin{bmatrix} r & 0 \\ 0 & -r \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 & -r \\ r & 0 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 0 & r \\ r & 0 \end{bmatrix} \quad .$$

Then clearly, since u_o is the 4×4 identity,

$u_o\ u_n = u_n\ u_o = u_n$ for $n = 0, 1, 2, 3$

$(-u_o)\ u_n = u_n(-u_o) = -u_n$ for $n = 0, 1, 2, 3$.

One can readily verify the following:

$u_n\ u_n = u_o \quad , \quad n \neq 2$

$u_2\ u_2 = -u_o$

u_o	u_1	u_2	u_3
u_1	u_0	$-u_3$	$-u_2$
u_2	u_3	$-u_o$	$-u_1$
u_3	u_2	u_1	u_o

Note that in going down rows of the multiplication table, even n progress to right, odd n to left. They also change sign at every move except when crossing the vertical line. (Arrows pointing outside are supposed to wrap around to the other side.) Note also how u_o and u_1 pair up, and u_2 and u_3.

We can now give a complete set of square roots.

$$\sqrt{0} : \pm(u_1 \pm u_2) \ , \quad \pm(u_2 \pm u_3) \ , \tag{8}$$

$$\sqrt{u_o} : \pm u_o \ , \quad \pm u_1 \ , \quad \pm u_3 \ , \quad \pm \frac{1}{\sqrt{2}} (u_1 \pm u_3) \tag{10}$$

$$\sqrt{-u_o} : \pm u_2 \ , \tag{2}$$

$$\sqrt{u_2} : \pm \frac{1}{\sqrt{2}} (u_o + u_2) \ , \tag{2}$$

$$\sqrt{-u_2} : \pm \frac{1}{\sqrt{2}} (u_o - u_2) \ , \tag{2}$$

$$\sqrt{u_1} : \pm \begin{bmatrix} r & 0 \\ 0 & i \end{bmatrix} \ , \tag{2}$$

$$\sqrt{-u_1} : \pm \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \ , \tag{2}$$

To find the square roots of $\pm u_3$, let

$$d_1 = i - r = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}, \quad d_2 = -i - r = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix},$$

Then

$$d_1{}^2 = -2i, \quad d_2{}^2 = 2i$$

$$d_1 d_2 = d_2 d_1 = 2r$$

So,

$$\begin{bmatrix} d_1 & d_2 \\ d_2 & d_1 \end{bmatrix} \begin{bmatrix} d_1 & d_2 \\ d_2 & d_1 \end{bmatrix} = \begin{bmatrix} 0 & 4r \\ 4r & 0 \end{bmatrix} = 4u_3$$

Hence,

$$\sqrt{u_3} = \pm\frac{1}{2} \begin{bmatrix} d_1 & d_2 \\ d_2 & d_1 \end{bmatrix} = \pm \begin{bmatrix} \sqrt{-i} & \sqrt{i} \\ \sqrt{i} & \sqrt{-i} \end{bmatrix}$$

To get $\sqrt{-u_3}$, multiply each part by i. Now

$$id_1 = d_1 i = d_2, \quad id_2 = d_2 i = -d_1,$$

as easily verified from the definitions. So,

$$\begin{bmatrix} d_2 & -d_1 \\ -d_1 & d_2 \end{bmatrix} \begin{bmatrix} d_2 & -d_1 \\ -d_1 & d_2 \end{bmatrix} = \begin{bmatrix} 0 & -4r \\ -4r & 0 \end{bmatrix} = -4u_3,$$

Hence,

$$\sqrt{-u_3} = \pm\frac{1}{2} \begin{bmatrix} d_2 & -d_1 \\ -d_1 & d_2 \end{bmatrix} = \pm \begin{bmatrix} \sqrt{i} & -\sqrt{-i} \\ -\sqrt{-i} & \sqrt{i} \end{bmatrix}.$$

Note that $\sqrt{u_3}\, u_2 \neq u_2 \sqrt{u_3}$ and neither gives $\sqrt{-u_3}$. The parts must be multiplied individually by i.

Hence we have all square roots in terms of 4×4 matrices of real numbers or 2×2 of complex numbers. Hence, in this sense

$$u_o, \quad u_1, \quad u_2, \quad u_3,$$

is a complete system just as r, i. However, we have used diagonal combinations which are not in terms of real units r. Also we have introduced divisors of zero and a class of numbers with no reciprocals. Muses talks about divisors of infinity, apparently meaning reciprocals of divisors of zero. But these numbers have no reciprocals since, along with all numbers (except 0) without reciprocals, their matrices have rank less than 4.

One can define other sets by using i, m or w to define the 4×4 units. For example:

$$v_0 = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}, \quad v_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix},$$

Then,

$$v_0 v_n = v_n v_o = -u_n \quad \text{for } n = 0,\ 1,\ 2,\ 3.$$

Hence this algebra is isomorphic to the first. Using m, one gets

$$m_0 = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \quad m_1 = \begin{bmatrix} m & 0 \\ 0 & -m \end{bmatrix}, \quad m_2 = \begin{bmatrix} 0 & -m \\ m & 0 \end{bmatrix}, \quad m_3 = \begin{bmatrix} 0 & m \\ m & 0 \end{bmatrix}.$$

Then,

$$m_0 m_n = m_n m_0 = u_n \quad \text{for } n = 0,\ 1,\ 2,\ 3$$

again isomorphic. Clearly the same is true for w_0, w_1, w_2, w_3 defined with w, since ww = r.

If one goes to hybrid sets, then something new does occur. For example:

$$h_0 = \begin{bmatrix} r & 0 \\ 0 & i \end{bmatrix}, \quad h_1 = \begin{bmatrix} r & 0 \\ 0 & -i \end{bmatrix}, \quad h_2 = \begin{bmatrix} 0 & -i \\ r & 0 \end{bmatrix}, \quad h_3 = \begin{bmatrix} 0 & i \\ r & 0 \end{bmatrix}.$$

Then,

$$h_0 h_0 = \begin{bmatrix} r & 0 \\ 0 & -r \end{bmatrix} = u_1 ,$$

$$h_0 h_1 = h_1 h_0 = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} = u_0 .$$

But

$$h_0 h_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = v_2 ,$$

$$h_2 h_0 = \begin{bmatrix} 0 & r \\ r & 0 \end{bmatrix} = u_3 ,$$

$$h_0 h_3 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = v_3 ,$$

$$h_3h_0 = \begin{bmatrix} 0 & -r \\ r & 0 \end{bmatrix} = u_2 \quad .$$

It is not clear what is to be gained by such a weird contrivance where sets transform among each other. But let us try one more where each unit is defined in "its own style".

$$t_0 = u_0, \quad t_1 = m_1, \quad t_2 = v_2, \quad t_3 = w_3 \quad .$$

Then,

$$t_0t_n = t_nt_0 = t_n, \quad n = 0,\ 1,\ 2,\ 3 \quad ,$$

$$(-t_0)t_n = t_n\ (-t_0) = -t_n, \quad n = 0,\ 1,\ 3 \quad ,$$

$$t_nt_n = t_0 \quad \text{for all } n \quad .$$

And,

$$t_1t_2 = t_2t_1 = t_3 \quad ,$$

$$t_1t_3 = t_3t_1 = t_2 \quad ,$$

$$t_2t_3 = t_3t_2 = t_1 \quad .$$

We can summarize in a table:

t_0	t_1	t_2	t_3
t_1	t_0	t_3	t_2
t_2	t_3	t_0	t_1
t_3	t_2	t_1	t_0

This is symmetric and there are no noncommutativities, but we seem to have lost $\sqrt{-t_0}$. However, we know a square root of $-t_0$, for example,

$$\sqrt{-t_0} = \sqrt{-u_0} = u_2 = -v_0v_2 = -v_0t_2 \quad .$$

Similarly, we can find a square root of t_1 and $-t_1$:

$$\begin{bmatrix} & w \\ i & \end{bmatrix} \begin{bmatrix} & w \\ i & \end{bmatrix} = \begin{bmatrix} m & \\ & -m \end{bmatrix} = m_1 = t_1 \quad ,$$

$$\begin{bmatrix} & i \\ w & \end{bmatrix} \begin{bmatrix} & i \\ w & \end{bmatrix} = \begin{bmatrix} -m & \\ & m \end{bmatrix} = -m_1 = -t_1 \quad .$$

How were these guessed? By looking at the basic multiplication table for r, m, i, w and noting order of signs. In the case of $\sqrt{-t_0}$, a direct calculation was possible. Note that we would like $\sqrt{-t_0}$ to be some factor times t_2 just as $\sqrt{-u_0} = u_2$. The term $-v_0$ is like a change of coordinates. It is not so easy to find one for the m_n set since $\sqrt{\pm u_1}$ are not simple expressions to begin with.

However, we cannot find valid square roots for t_2 and t_3 or their negatives. The reason is that the determinants of these matrices are -1. Hence any square root must have a determinant of i which means that i must appear in place of a real number, not as a 2 × 2 submatrix. Thus we have lost all the advantages we gained.

Consequently, we conclude that the u_n constitute the only set of possible interest or value. However, there is nothing exotic or weird about them, except for divisors of zero and an oversupply of square roots of unity.

Muses seems to identify "species" of numbers with units, which is extravagant of terminology. I would say we have thus far displayed three species: real, complex, and the set based on u_n. (We might call the latter the "quadriforms".) Quaternions could be regarded as another species, somewhat intermediate between complex and quadriform. However, proliferating other units by writing i as if it were a real number is, in my opinion, nonsense.

Note that, although it is well established that complex numbers contain the reals as a subset, no such fundamental theorem has been established for hypernumbers. True, it seems obvious that quadriform numbers of the form

$$au_0 + 0u_1 + bu_2 + 0u_3, \quad a, b \text{ real}$$

are indeed the complex field, just as complex numbers of the form

$$ar + 0i, \quad a \text{ real},$$

are the reals. But we do not introduce divisors of zero and nonzero numbers without reciprocals by making the coefficient of i to a nonzero value. Furthermore, the complex numbers $ar + 0i$ are not actually the reals, but isomorphic to them, which is not exactly the same thing. Or is it?

Development of a rigorous theory of hypernumbers would seem desirable, particularly if some need for them can be shown. (Remember that complex numbers were developed to solve actual problems.) But it must be done carefully with sound premises and rules and not just by playing with symbols, except as a means of stimulating imagination and insight.

Post-Script

In a subsequent paper we show that the set u_n is, in fact, neither consistent nor useful. However, the 2×2 set - r, i, m, w - is consistent and useful if certain limitations are accepted.

References

[1] Muses, C. Hypernumbers and their Spaces: a Summary of New Findings. Reprinted from The Journal of Consciousness, Vol. 5, No. 2 (1972/73).

[2] Caianiello, E.R., (ed.) New Concepts and Technologies in Parallel Information Processing. NATO Advanced Study Institutes Series.