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Interim Report **IR-99-044**

A Game with Program Control and Variable Terminal Moment

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September 1999

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A Game with Program Control and Variable Terminal Moment

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In the theory of positional differential games, it is well known that continuous strategies, generally speaking, do not provide the best possible result. In the present paper, a simple example is given that shows that a similar property holds for conflict control systems in which one of the two players chooses a program control, and the other one chooses the moment of termination of the game, in which the payoff is calculated. The payoff functional depends on the phase state at this time moment. It is shown that in this control system no rule for choosing the terminal time moment described by a continuous mapping can guarantee the corresponding player a nonzero result. However, simple discontinuous mappings can ensure the desired nonzero result. Two examples of such discontinuous rules are given. One of them requires measuring the phase state at one time moment only.

Introduction

Positional differential games [1–5] have been intensively studied in the last decades. Properties of continuous strategies in comparison with discontinuous ones were studied in [6]. Multivalued upper semicontinuous strategies were considered in [7]. Alongside with other questions, in [8] properties of strategies described by Carathéodory functions were investigated. Systematic introduction into the theory of positional differential games requires discussing these questions in some form. Thus, in [2] on pp 17-24, the class of positional strategies and the corresponding motions of controlled system were discussed. In this connection, on pp 18-21 in [2] an example of a two-dimensional differential game was given, for which the guaranteed result was found with respect to the classes of discontinuous and continuous strategies. The corresponding proof in [2] employed the Schauder fixed point theorem.

In [9] the above mentioned differential game was modified so that one of the two players can choose the time moment when the payoff should be calculated. The payoff depends on the norm of the phase vector at this time moment. Thus, the quality index contains a variable point, which can specify the game's terminal moment assigned by feedback on the basis of measurements of current values of the

*The research was done with financial support from Russian Foundation for Basic Research, grant 97-01-00160.

phase vector. It was shown in [9] that in this problem the methods of feedback control described by continuous mappings are of limited capability even in case of practically complete current information on the process (including its history) at all the time moments. On the other hand, a simple discontinuous mapping was constructed that describes a control law that provides the desired result using but very limited current information and modest possibilities to influence the process. The proof in [9] used the fixed point theorem of Kakutani. (The corresponding mapping in [9] turned out to be multivalued due to technical reasons.) The efficiency of a control law in the mentioned problem is directly connected with the existence or nonexistence of solutions to some nonlinear boundary value problem in ordinary derivatives with a solution dependent point in one of the two boundary conditions. The corresponding boundary condition is described by a nonlinear functional of special form. Let us note that nonlinear boundary value problems for functional differential equations with solution dependent points in boundary conditions arising in connection with thermal conflict control systems were considered in [10, 11].

The aim of article [9] was to consider a modification of a well-known control system used in the theory of differential games as a sample example. However, similar properties connected with the choice of termination moment are possessed by some simpler systems. The present paper gives such an example. The control system considered below is described by a simple scalar differential equation, which allows to simplify significantly the proofs and to do without fixed point theorems. Instead we use the well-known fact that a scalar continuous function that takes values of different signs at the ends of an interval vanishes at some point.

1 A Scalar Game Problem

Let the current state of a controlled object be described by a scalar x. The evolution of the value x on a time interval $[0, 2]$ is governed by a differential equation

$$\dot{x} = t + u, \quad 0 \le t \le 2. \tag{1}$$

Equation (1) should hold for almost all time moments t. The function $x(\cdot)$ is assumed to be absolutely continuous. The initial state is zero

$$x(0) = 0. \tag{2}$$

The payoff functional has the form

$$\gamma(x(\cdot)) = |x(\vartheta_0)|.$$

Note that the payoff $\gamma(x(\cdot))$ depends not only on the function $x(\cdot)$, but also on the number ϑ_0.

There are two players. One of them chooses the parameter $-1 \le u \le 1$ as a Lebesgue measurable function of time $u : [0, 2] \to [-1, 1]$. The aim of this player is to minimize the value of quality index $\gamma(x(\cdot))$. The other player chooses the number $0 \le \vartheta_0 \le 2$ with an aim to maximize the index $\gamma(x(\cdot))$.

For a fixed program control $u(\cdot)$, the solution $x(\cdot)$ to the initial value problem (1),(2) has the form

$$x(t) = \frac{t^2}{2} + \int\limits_0^t u(\tau)d\tau. \tag{3}$$

If the choice of control parameter ϑ_0 is known beforehand to the player that chooses the control u, then this player can make $\gamma(x(\cdot)) = 0$ putting for example $u(t) \equiv -\vartheta_0/2$.

Thus, the corresponding player can not choose the constant ϑ_0 so that a nonzero result is guaranteed for any possible disturbance $u(\cdot)$. However the player can try to ensure a nonzero result by choosing the number $\vartheta_0 = \vartheta_0(x(\cdot))$ on the basis of feedback, that is on the basis of measurements of the current phase state $x(t)$ for some time moments t. Sometimes it is possible to measure the disturbance u also. In this case the player can assign the number ϑ_0 in the form $\vartheta_0 = \vartheta_0(x(\cdot), u(\cdot))$.

2 Continuous Strategies

Assume now that the player that chooses the number ϑ_0 employs some continuous mapping

$$\vartheta_0 : C^0 \to [0, 2] \tag{4}$$

to appoint the value $\vartheta_0 = \vartheta_0(x(\cdot))$ on the basis of feedback. (Other possible restrictions on this mapping will be discussed below.) Here C^0 denotes the space of all continuous scalar functions endowed with the standard uniform norm. It turns out that whatever the continuous mapping (4) might be, it does not guarantee a nonzero result $\gamma(x(\cdot))$. For any continuous mapping $\vartheta_0 = \vartheta_0(x(\cdot))$ there exists a control u such that the payoff functional $\gamma(x(\cdot))$ vanishes. Here it suffices to employ constant controls $u \equiv \text{const}$ independent of the variable t. It is even possible to allow the number $\vartheta_0 = \vartheta_0(x(\cdot), u)$ to depend also on the parameter u.

One can formulate this fact as the following existence result for a boundary value problem with a solution-dependent point in boundary condition.

P r o p o s i t i o n. For an arbitrary continuous mapping $\vartheta_0 : C^0 \times [-1, 0] \to [0, 2]$ there exists a constant $u \in [-1, 0]$ and an absolutely continuous scalar function $x(t)$, $t \in [0, 2]$, that satisfy the boundary value problem (1),(2),(5) with a condition

$$x(\vartheta_0(x(\cdot), u)) = 0. \tag{5}$$

P r o o f o f P r o p o s i t i o n. To establish the solvability of boundary value problem (1),(2),(5), let us consider initial value problem (1),(2) and choose the parameter u so that the boundary condition (5) is valid. For a number $u \in [-1, 0]$ denote by $x_u(\cdot)$ the unique solution of the initial value problem (1),(2) that corresponds to the chosen value u. According to (3), one has the formula

$$x_u(t) = \frac{t^2}{2} + ut. \tag{6}$$

Thus, the inequalities hold

$$x_{-1}(t) \leq 0, \quad x_0(t) \geq 0 \tag{7}$$

for all $t \in [0, 2]$. Consider a function

$$\Psi(u) = x_u(\vartheta_0(x_u(\cdot), u)).$$

Note that $\Psi(u)$ is a continuous scalar function defined for all $u \in [-1, 0]$. It follows from inequalities (7) that $\Psi(-1) \leq 0$, $\Psi(0) \geq 0$. So, the function Ψ vanishes at some point in $[-1, 0]$. For this point u, the function $x_u(\cdot)$ satisfies boundary condition (5). Thus, boundary value problem (1),(2),(5) has a solution. Proposition is proved.

The continuous map (4) above is arbitrary. This does not lead to any contradiction if one formally considers the corresponding boundary value problem, or if one considers the control system associated with it assuming the variable t to be some coordinate. (The case of coordinate in somewhat similar problems was studied in [10, 11], where the feedback assigned points were not time moments but coordinates of point heat sources on a rod being heated and a class of conflict control problems was considered.) However, if the value t is treated as the time variable, it seems to make no sense to allow the player that chooses the time moment ϑ_0 to use information on the future evolution of the process. In this case the map (4) should be restricted by the following condition of non-anticipation.
(NA) For any $y(\cdot), z(\cdot) \in C^0$, if $y(s) \equiv z(s)$ for all $s \in [0, \vartheta_0(y(\cdot))]$, then $\vartheta_0(y(\cdot)) = \vartheta_0(z(\cdot))$.
This condition allows to consider ϑ_0 as the time of termination of the game. The non-anticipation condition for maps of the form $\vartheta_0 = \vartheta_0(x(\cdot), u)$ can be formulated in exactly the same way.

Condition (NA) or its analogues were not used in the proof of Proposition.

3 Discontinuous Strategies

Consider now the case when the map (4) is allowed to be discontinuous. To ensure a nonzero result, the corresponding player can choose the number ϑ_0 as follows

$$\vartheta_0(x(\cdot)) = \vartheta_{0(8)}(x(\cdot)) = \begin{cases} 1, & x(1) < -1/4, \\ 2, & x(1) \geq -1/4. \end{cases} \tag{8}$$

Thus the rule (8) is described by a scalar function of the scalar argument $x(1)$.

One has

$$x(2) \geq x(1) + 1/2 \tag{9}$$

for an arbitrary solution x to equation (1) with an admissible control function $u(\cdot)$. Indeed,

$$x(2) - x(1) = \int_1^2 (\tau + u(\tau))d\tau \geq \int_1^2 (\tau - 1)d\tau = 1/2.$$

Consequently, for either case in formula (8) one has an inequality

$$|x(\vartheta_{0(8)}(x(\cdot)))| \geq 1/4,$$

and the result $\gamma(x(\cdot)) \geq 1/4$ is guaranteed.

Another rule for choosing the number ϑ_0 can be given by the formula

$$\vartheta_0(x(\cdot)) = \vartheta_{0(10)}(x(\cdot)) = \min\{\tau : |x(\tau)| = 1/4\}. \qquad (10)$$

From relations (2),(9) and the continuity of function x it follows that the set of numbers τ in formula (10) is nonempty. As this set is closed, the minimum is attained. So, the number (10) is defined correctly, belongs to the interval $[0, 2]$, and the described mapping provides the result $\gamma(x(\cdot)) = 1/4$.

On the other hand, whatever the map (4) is, a better result can not be ensured for the player that chooses ϑ_0 because for the program control

$$u(t) = \begin{cases} -t, & 0 \le t < \dfrac{2-\sqrt{2}}{2}, \\ -1, & \dfrac{2-\sqrt{2}}{2} < t \le 2, \end{cases} \qquad (11)$$

the solution to initial value problem (1),(2) has the form

$$x(t) = \begin{cases} 0, & 0 \le t \le \dfrac{2-\sqrt{2}}{2}, \\ \dfrac{1}{4}(2t^2 - 4t + 1), & \dfrac{2-\sqrt{2}}{2} \le t \le 2, \end{cases} \qquad (12)$$

and satisfies the inequality $|x(t)| \le 1/4$ for all $t \in [0, 2]$.

Note that both (8) and (10) are discontinuous mappings in the sense (4). Both of them satisfy condition (NA).

Let us also note that in the considered control problem the number (10) satisfies the inequality $\vartheta_{0(10)}(x(\cdot)) < 2$. Indeed, if $\vartheta_{0(10)}(x(\cdot)) = 2$, then $x(2) = \pm 1/4$, inequality (9) implies $x(1) \le -1/4$, and consequently $\vartheta_{0(10)}(x(\cdot)) \le 1$, which contradicts the assumption $\vartheta_{0(10)}(x(\cdot)) = 2$.

However, solutions x of the control problem can be chosen for which the numbers $\vartheta_{0(10)}(x(\cdot))$ are as close to 2, as one wants. This can be shown by modification of functions (11),(12). Take

$$u(t, \varepsilon) = \begin{cases} -t, & 0 \le t < \dfrac{2-\sqrt{2}}{2} + \varepsilon, \\ -1, & \dfrac{2-\sqrt{2}}{2} + \varepsilon < t \le 2, \end{cases}$$

for $0 < \varepsilon \le \sqrt{2}/2$. The chosen control function satisfies the restriction $-1 \le u(t, \varepsilon) \le 1$. The corresponding solution to initial value problem (1),(2) has the form

$$x(t, \varepsilon) = \begin{cases} 0, & 0 \le t \le \dfrac{2-\sqrt{2}}{2} + \varepsilon, \\ \dfrac{1}{4}(2t^2 - 4t + 1 + 2\varepsilon\sqrt{2} - 2\varepsilon^2), & \dfrac{2-\sqrt{2}}{2} + \varepsilon \le t \le 2, \end{cases}$$

and satisfies the relation $x(t,\varepsilon) \geq x(1,\varepsilon) = -\dfrac{1}{4} + \dfrac{\varepsilon\sqrt{2}-\varepsilon^2}{2} > -\dfrac{1}{4}$ for all $t \in [0,2]$, $\varepsilon \in (0,\sqrt{2}/2]$. One has $\vartheta_{0(10)}(x(\cdot,\varepsilon)) = 1+\sqrt{1-\varepsilon\sqrt{2}+\varepsilon^2} < 2$ and $\lim\limits_{\varepsilon\to+0} \vartheta_{0(10)}(x(\cdot,\varepsilon)) = 2$. (However the function $x(\cdot,0)$ coincides with the solution (12), and $\vartheta_{0(10)}(x(\cdot,0)) = 1$, which confirms the discontinuity of the mapping (10).)

It is interesting to compare the rules (8) and (10) as two different ways of behaviour of the corresponding player. In the case (8), at the time moment $t = 1$ the player becomes aware of the value $x(1)$ and, basing on this information, a decision is taken either to stop the game immediately, or to wait until its end at $t = 2$. (Obviously, any nondegenerate rule of this type is described by a discontinuous mapping.) The rule (8) requires measuring the phase state x only at one point $t = 1$, whereas the mapping (10) requires to measure x continuously on the interval $[0,2]$. The time moment ϑ_0 calculated according to formula (10) is smaller than or equal to the one given by formula (8). By definition, for a fixed trajectory, the time moment given by (10) is the smallest possible one that achieves the result $1/4$. As was shown above, in the considered problem this time moment is always strictly smaller than 2, though it can be arbitrarily close to 2. The rule (10) always gives the guaranteed result $1/4$, whereas the rule (8) might provide a better result for the player that chooses ϑ_0 in case the opponent makes mistakes. One can also indicate other reasonable ways to appoint the time moment ϑ_0 that provide the ensured result.

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