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Interim Report **IR-00-045**

Timber Harvesting with Variable Prices and Costs

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2 August 2000

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Abstract

In its history, IIASA's Forestry (FOR) Project has produced a number of models casting future scenarios of the forest industry. At the same time, FOR has also been involved in ecosystem modeling and forest resource assessments. Linking the processes of forest industry development to its resource use will be of particular interest in the activity that FOR started this year – Information Technology and Structural Change of the Global Forest Sector. In this particular work, carried out in cooperation with the Finnish Forest Research Institute in Helsinki, Markku Penttinen establishes a decision tool that will help in endeavoring to answer the question of whether information technology will or will not improve the sustainability of the world's forest resources.

About the Author

Markku Penttinen received his Ph.D. in business economics in 1983 from the University of Vaasa, Finland. In 1997, he was appointed as the Deputy Coordinator of the Working Party "Economic Evaluation of Multifunctional Forestry" of the International Union of Forest Research Organizations (IUFRO). Markku Penttinen is currently responsible for the 'Management of Forestry and Timber Working Industries' at the Finnish Forest Research Institute (METLA) in Helsinki, Finland and is also Adjunct Professor at the Department of Economics, University of Helsinki.

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Background

This paper solves the optimal harvesting time problem of a non-industrial private forest (NIPF) owner, assuming that the owner has a forest management plan and merchantable forest stands. The optimal harvesting time is defined in a volatile market situation. The infinite period problem has also been formulated, allowing variable stumpage prices and reforestation costs in a two-period framework, the first of which covers the near future with *dynamic* price and cost functions and the second consists of the rest of the infinite future with *trend* price and cost functions.

The existence and uniqueness of an optimal policy is shown, based on the explicitly quasi-concavity of the objective functions. First, the solutions are constructed with prices and costs depending only on stand age. Cases with the same prices and costs for all periods and with dynamic ones for the first period and trend ones for subsequent periods are both tackled. Second, the age-dependent functions are multiplied, separately by the calendar time dependent exponential terms. Solutions are given both for the case with the same age-dependent functions, and for the case with the dynamic ones for the first period and trend ones for subsequent periods

The sensitivity analysis is performed with respect to the interest rate, price and cost change, both analytically and numerically. Optimal rotation solutions are given with alternative competing volume growth functions. Final results are provided by a gross income growth function. Competing optimization models are discussed, and alternative volume growth models and a value growth model are compared.

The key notion of the study is analysis of the sensitivity of the optimal rotation solutions with respect to roundwood prices, reforestation costs and interest rates. Different local market parameter and alternative growth data estimates are applied in testing the impact of price, cost and interest rate parameters. The purpose of the study is to provide tools for day-to-day decision-making in the changing world of forestry and also to compare the silvicultural recommendations with the solutions.

Many NIPF owners have a tendency of trying to sell only during the peak price periods. Their behavior is compared with policy results obtained using empirical data on the turbulent market place with fluctuating prices, and the optimal rotation models developed.

1 Introduction

1.1 Competing Views in Forest Economics

The history of optimal rotation age research includes solutions based on different approaches. A traditional misunderstanding, for example, is to define the optimal rotation age of the standing timber as in the case of wine aging, and therewith to forget the necessity of felling the forest before the land can be used again. Another example is biologically maximum sustained yield (MSY), which ignores economic evaluation, and sustained yield forestry (Waldreinertragswirtschaft), which assumes that the interest rate is zero (Löfgren, 1990). Moreover, a generally used criterion, the rate of growth of capital, known by terms such as internal rate of return (IRR), has been shown to be incorrect by Samuelson (1976). The IRR criterion may actually lead to unsatisfactory conclusions from either the infinite or zero wealth increment (Hirshleifer, 1970). Recall that, when using the IRR maximization, one hypothesis is that the area of land available for forestry is infinite and that access to all capital markets is closed (Newman, 1988).

In firms, the primary criterion of decision making is profitability and is based on the theory of interest (Fisher, 1931) and its consequent return on investment (ROI). Profitability has been seen as the best available measure of efficiency (see, e.g., Brozik, 1984). The traditional interest theory approach, as outlined by Fisher (1931; 1954, p. 159), requires that the rate of return over cost must exceed the rate of interest.

In forestry, the marginal rate of return approach has been supported by Duerr among others (see, e.g., Duerr *et al.*, 1979). The Finnish local tradition applied to forestry practice was to use according to Duerr a predefined marginal interest rate as suggested, for example, by Nyyssönen (1958; 1997). This notion has been inspired by the portfolio management approach and the theory of interest. When cutting the forest, for example, the "average" rate of growth is replaced by a *marginal* rate of growth (Fisher, 1954, p. 165). Moreover, timber and timber investments are capital goods and should be managed at a rate of return equal to the return of other capital investments in the economy, i.e., at the market interest rate according to Hirshleifer (1974). Some proposals, as in the case of "financial rotation periods" are, however, based on the maximization of the *average* relative profitability, in which the value of both stocks and land form the bounded capital (Speidel, 1984, p. 172).

In forest economics, however, the main line has been the net present value (NPV) as the objective variable of the optimization. The exclusive position of the König-Faustmann present value (PV) has, however, been criticized by Oderwald and Duerr (1990) among others. Finally, their criticism has been contradicted by Chang (1990). In all, the König-Faustmann PV maximization according to arguments already presented by the classic article of Samuelson (1976) still remains the basis of the standard approach (see also, Newman, 1988). [1]

[1] Note that the PV maximization solution of the rotation is well-defined provided that (i) the capital market is perfect, (ii) the future price of timber is known, (iii) forest land can be bought and sold in a perfect market, and (iv) the future technical lumber yields are known (Löfgren, 1990).

1.2 Previous Work in Deterministic Optimal Rotation Modeling

Faustmannian deterministic[2] net present value (NPV) results are provided in numerous books, e.g., by Johansson and Löfgren (1985). The "Faustmann" (NPV) solution, the "Fisher" (ROI) solution, internal rate of return (IRR) and maximum sustained yield (MSY) have been compared by Samuelson (1976). Rideout (1986) compared the Fisher and Faustmann solutions and also suggested the benefit-cost ratio maximization. The differences between the optimal rotation lengths when applying the Faustmann and maximum-sustained yield has been studied by Binkley (1987). Löfgren (1990) summarized the 'profitability war' concentrating on PV and land rent approaches, and summarized conditions under which the optimal rotation problem is well defined. However, the König-Faustmann tradition has been criticized by Oderwald and Duerr (1990), who suggest optimizing the firm's investment in timber stock in terms of marginal revenues and cost per unit of capital, an approach which provoked more criticism than support. Recall that the state of the art in the optimal forest rotation has been summarized by Newman (1988).

This study is based on the dynamic rotation modeling contributions of McConnell *et al.* (1983), Hardie *et al.* (1984), Newman *et al.* (1985), Yin and Newman (1995), and Chang (1998). McConnell *et al.* (1983) presented optimal rotation solutions for subsequent rotations with exponential prices and costs.[3] Hardie *et al.* (1984) used the dynamic programming approach and solved an optimal rotation problem with price, cost and yield forecasts assuming steady state rotations after some fixed numbers of rotations. Newman *et al.* (1985) analyzed the optimal rotation of subsequent rotations with evolving prices and solved the problem with exponential prices. Yin and Newman (1995) considered a case in which prices and costs increase exponentially, either of which can grow faster. Chang (1998) defined optimal solutions relating subsequent rotations and provided numerical solutions with variable prices.

The comparative static analysis is the main stream of optimal rotation research. Chang (1982) studied the impact of different forest taxation systems on optimal rotation age. He has also dealt with the influence of different factors such as price, interest rate, regeneration cost and taxation on the rotation age (Chang, 1983). Chang (1984) developed the sensitivity results in his comparative static analysis. The response of optimal rotation and management intensity to changes in prices, management costs, and discount rates has been analyzed by Nautiyal and Williams (1990).

The maximum principle and the control theory approach has been applied to the Faustmann formulation. A steady state solution to forestry management problems with, e.g., variable harvesting costs, has been advocated by Heaps (1984). Recently, *in situ* versions of optimal forest rotation results have been analyzed using the control theory approach by Kuuluvainen and Tahvonen (1999) and Tahvonen and Salo (1999).

[2] Stochastic optimal rotation models have been actively researched, many contributions applying stochastic differential equations (see, e.g., Brazee and Mendelson, 1988; Yin and Newman, 1997). However, stochastic optimal rotation results will be presented in a complementary publication.

[3] The cost of the regeneration delay has been provided by Brodie and Tedder (1982) and Lappi (1983) among others.

Local growth and yield functions as well as local forest management planning (FMP) tradition form the cornerstones for implementing the results in Finland. The optimal rotation problem has been scrutinized for practical purposes by Nyyssönen (1958; 1997), based on the marginal rate of return. The rotation dilemma has been studied as part of a forest stand planning problem by Kilkki (1968), and as a dynamic programming problem by Kilkki and Väisänen (1969). Local growth functions have been investigated by Nyyssönen and Mielikäinen (1978), as well as by Vuokila and Väliaho (1980), among others. Fridh and Nilsson (1980) suggested a simplified growth function. Foundations of the optimal rotation have been presented by Vuokila (1980) and others. The relationship between the value increment and the volume increment has been analyzed by Nyyssönen and Ojansuu (1982).

Recommendations concerning the optimal rotation for forestry practice have been published by the Forestry Centre Tapio (1994). A decision about the prerequisites of a regeneration cutting as an implementation of the Forest Act has been given by the Finnish Ministry of Agriculture and Forestry (1997)

1.3 Objectives of the Study

The objectives of the study are the following:

1. This study aims at supporting timber harvesting decisions for a forest owner. A forest management plan (FMP) could be extended to include the order in which the merchantable forest stands should be harvested and harvesting time recommendations as functions of interest rate, price change, etc.

2. The purpose is to produce an implementation decision support system (DSS) in the form of personal computer (PC) programs for day-to-day decision-making. By using the FMP data as input a forest owner could study the dynamic conditions of varying interest rates, prices, costs, etc. Such programs are able to describe the interactions between the various features of the planning situation.

3. The empirical part of the study applies different volume and income yield functions, and local data in order to review the restrictions of the forest law and the forest management recommendations of the Forestry Development Centre Tapio.

4. The roundwood sales behavior of forest owners is assessed. They primarily follow stumpage prices, but also the silvicultural costs and interest rates' developments carefully. In practice, many forest owners try to sell only at peak prices — this behavior is reviewed.

2 Material and Methods

2.1 The Optimal Rotation Problem Definition

The traditional Faustmann optimal rotation, also called the harvest time problem, can be formulated not only in a discrete but also in a continuous time framework as the point of time when the value of continued growth equals the opportunity cost for waiting (Brazee and Mendelson, 1988):

$$p\, q'(\tau) = r\, p\, q(\tau) + r\; b, \tag{1}$$

where p is the constant stumpage price, b is the value of the bare land, r is the interest rate, and $q(\tau)$ is the yield function describing the timber volume at *age* τ. The steady state problem can be defined as wealth maximization according to Binkley (1987)[4]

$$\max_{\tau} w(\tau) = -c + p\, q(\tau) \exp(-r\,\tau) + w(\tau) \exp(-r\tau), \tag{2}$$

where $w(\tau)$ is the wealth as a function of rotation age τ, and c is the constant regeneration cost.

In equation (1) the value b of the bare land poses a two-period problem. With the available information, as the crucial limiting constraint, one has to admit that the information concerning the stumpage prices, silvicultural costs and interest rates in the near future are essentially better known than those of subsequent rotations. In all, the basic discrete deterministic problem of the optimal rotation with variable prices p(t) and costs c(t) is that of dynamic programming (DP) as applied by Hardie *et al.* (1984) [Amidon and Akin (1968) have already used DP for choosing optimal thinning rules; see also, Gong (1992), Filius and Dul (1992) and Salminen (1993)]:

$$w_n(T_n) = \max_{t} \{ [p(t)q(t) - c(t)]\exp(-rt) + w_{n+1}(t,T_{n+1})\exp(-r\,t) \} - c(0) \tag{3}$$

where regeneration happens in practice 2–5 years after harvesting, which is recognized in c(t) but not explicitly shown in the function form. The regeneration cost c(0) of the present generation cannot be affected and is ignored in equation (3). Note that the time t and the age τ_n of the tree generation n to be harvested are the same, $t = \tau_n$. The price and cost functions may change over calendar time t, but the growth is nevertheless a function of biological age $\tau_n, \tau_{n+1}, \tau_{n+2},\ldots$. The optimal harvesting age is $T_n, T_{n+1}, T_{n+2},\ldots$. The yield function q depends, as a matter of fact, both on the age τ and planting density m (see, Chang, 1983), but for the present generation some density has already been chosen, i.e., $g(\tau,m) = q(\tau,.)$ and is denoted by $q(\tau)$. Recall that according to empirical studies growth is quite deterministic compared with price (Lausti and Penttinen, 1998).

[4] The traditional soil expectation value (SEV), as the sum of all discounted future revenues, has been presented in forest economics books such as Johansson and Löfgren (1985).

It depends on age and on a number of forestry variables such as the stand basal area, the basal area median diameter, and the dominant height (local yield models are presented in Pukkala and Miina, 1997).

The DP requires *separability* and *monotonicity* (see, Nemhauser, 1966) the latter meaning that $w_{n+1}(t,T_{n+1})$ exp(-rt) depending on t is monotonic. Recall that the information available for prices and costs after the present rotation period is very limited in the sense of fluctuations, which means that only a non-increasing function $w_{n+1}(t,T_{n+1})$ for the subsequent period does not limit the applicability of the model. Then the monotonicity of $w_{n+1}(t,T_{n+1})$ exp(-rt) is fulfilled.

2.2 Optimal Rotation Solutions

A key dilemma of the general deterministic case is the discrepancy between calendar time t and age τ, i.e., whether the future NPV in equation (3), $w_{n+1}(t,T_{n+1})$, depends on t or not. If, for example in harvesting, the present generation n is delayed, the beginning of the biological age τ_{n+1} of the next generation is also delayed in calendar time t. Thus, the deterministic model with variable parameters consists of two parts:

(A) All the prices, profit ratios and costs depend on age τ $[w_{n+1}(t,T_{n+1}) \equiv w_{n+1}(T_{n+1})]$:

First, consider an extension of the traditional approach allowing the price $p(\tau)$ of timber to vary according to the biological age of the stand (Nautiyal and Williams (1990) and many others assume the price p to be constant). The selling price $p(\tau)q(\tau)$ may be set so that the net income is only a proportion $d(\tau)$ of the sales revenue. These charges, the share $1-d(\tau)$ of the sales, are items such as capital income tax, currently 29% in Finland, (for the impact of taxation on optimal rotation, see Chang, 1982), marketing and logging costs or something else depending on the sales revenue. Whenever the regeneration costs c(0) of the present generation are also included, the soil expectation value (SEV) is denoted by $V(\tau)$ and is

$$V(\tau) = [\, p(\tau)q(\tau)d(\tau)\exp(-r\,\tau) - c(0)]/[1-\exp(-r\tau)], \quad (4)$$

where the yield function $q(\tau)$, varying prices $p(\tau)$, constant reforestation costs[5] c(0) and net profit ratio $d(\tau)$ of the present generation have been used. The optimal rotation period T is then defined (see the Appendix) by

$$p'(T)/p(T) + q'(T)/q(T) + d'(T)/d(T) = r\,[1 - c_r(T)] \,/\, [1-\exp(-rT)]\,, \quad (5)$$

where notation $c_r(T)$ denotes the relative reforestation cost, i.e., the ratio $c_r(T) = c(0)/[p(T)q(T)d(T)]$. Supposing that the ratios $p'(\tau)/p(\tau)$ and $d'(\tau)/d(\tau)$ are simple functions or even constants, the above transcendental equation can be easily solved using numerical methods. Where $c_r(T)$ can be assumed to be constant, the solution can be even obtained manually using growth tables.

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[5] This traditional approach of using c(0) suffers from the necessity to estimate parameters which have, e.g., 80 years time difference as c(0) and p(T).

The maximum solution can be shown to be the global one already based on the explicitly quasi-concavity of $V(\tau)$ for some τ_0, $\tau_0 > 0$ (see the Appendix), which allows at most one sign change in the derivative $V'(\tau)$. Unfortunately, $V(\tau)$ is not even quasi-concave for the whole positive axis. Moreover, c(0) is not relevant for the present period. Additionally, c(0) is not necessarily a correct estimate for $c(T_n)$, $c(T_{n+1})$,... .[6] In all, the alternative favored in literature of using c(0) as the regeneration cost is rejected here. Therewith, $V(\tau)$ is not subsequently accepted as any relevant model for the analysis.

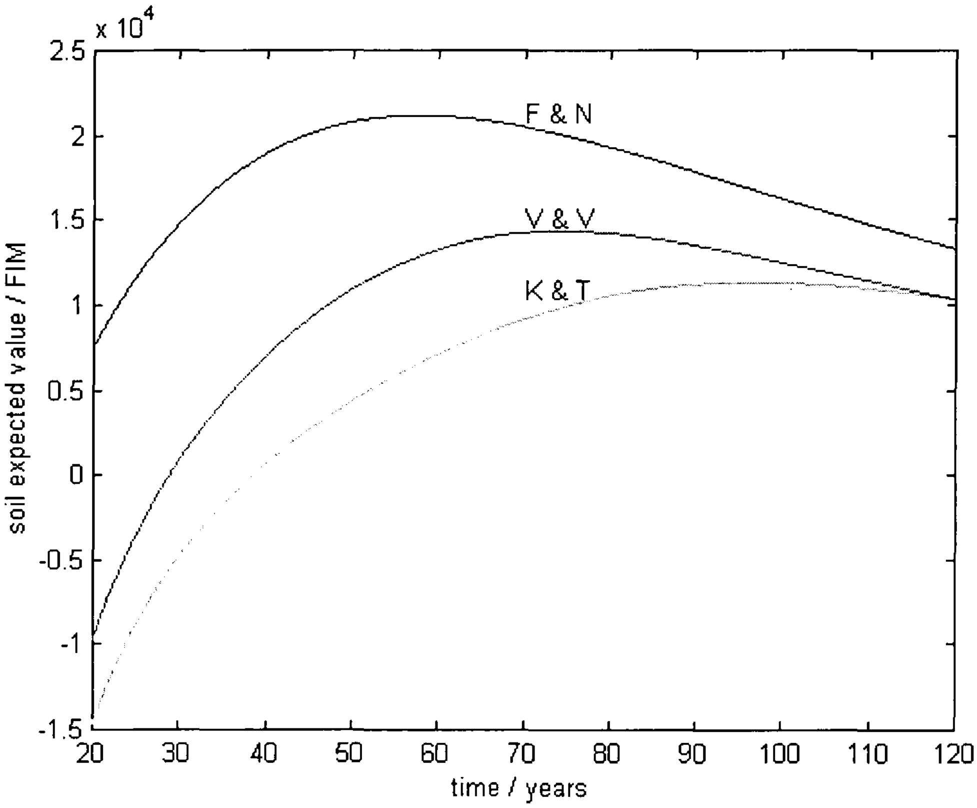

Figure 1: The soil expected value S FIM/ha, when r = 3%, p(T) = FIM 240/m^3, p/p = 1%, d(T) = 0.71, d' =0, c(T) = FIM 5000/ha, c'/c = 0.5% and the growth and yields are that in equation (16) (Fridh and Nilsson, 1980), in equation (17) (Kuuluvainen and Tahvonen, 1999), and in equation (18) (Vuokila and Väliaho, 1980).

[6] Note that Finnish forest legislation links final felling and reforestation of the next rotation to be performed within 5 years from the final felling.

Second, to include the varying reforestation costs in the optimal rotation model, the explicit reforestation costs c(t) of the next generation[7] can be used. The soil expectation value (SEV) denoted by $S(\tau)$ is then

$$S(\tau) = \{[p(\tau)q(\tau)d(\tau)-c(\tau)] \exp(-r\tau)\}/\{1-\exp(-r\tau)\} - c(0) . \qquad (6)$$

The corresponding optimal rotation period T is defined by

$$p'(T)/p(T) + q'(T)/q(T) + d'(T)/d(T) = c_r(T)\, c'(T) / c(T)$$
$$+ r [1 - c_r(T)] / [1 - \exp(-rT)], \qquad (7)$$

where $c_r(T)$ denotes the reforestation cost ratio $c_r(T) = c(T)/[p(T)q(T)d(T)]$. The extreme is now also shown to be the global maximum based on the explicitly quasi-concavity of $S(\tau)$, which allows at most a single sign change pattern, from + to -, of the derivative $S'(\tau)$ as shown in the Appendix.

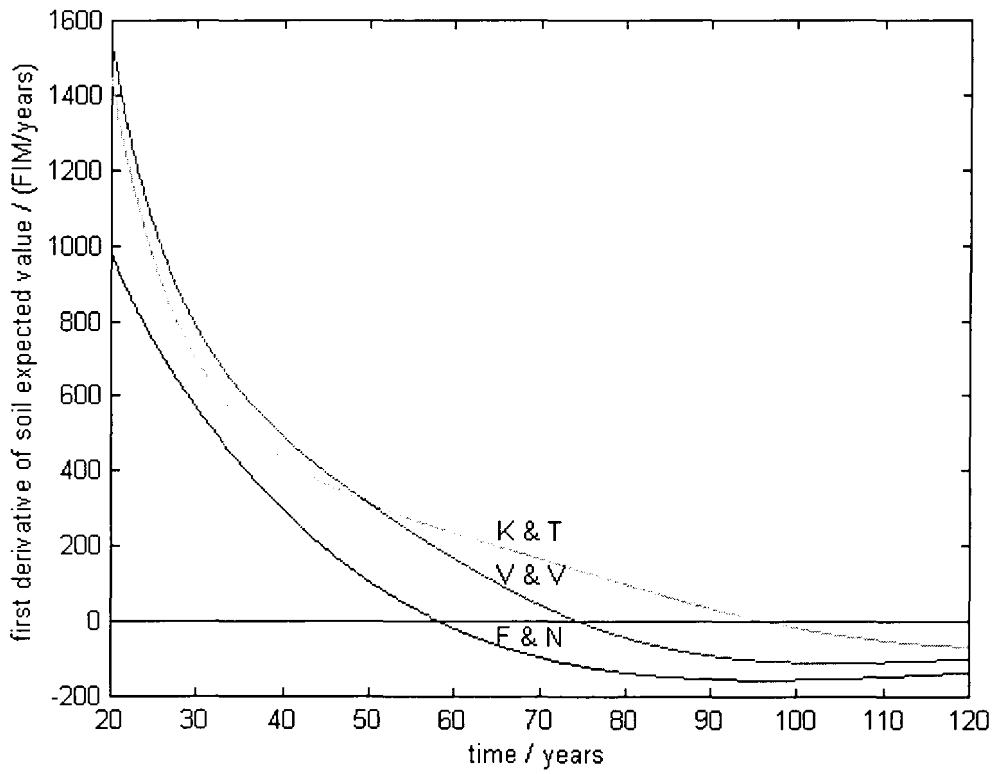

Figure 2: The derivative S' FIM/(ha year) of the soil expected value, when r= 3%, p(T) = FIM 240/m^3, p'/p = 1%, d(T) = 0.71, d' = 0, c(T) = FIM 5000/ha, c'/c = 0.5% and the growth and yield is that of Kuuluvainen and Tahvonen (1999), Vuokila and Väliaho (1980), and Fridh and Nilsson (1980).

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[7] For example, the planting costs vary, typically slightly increasing, as a function of time (Oksanen-Peltola, 1989).

(B) The prices, costs and profit ratios depend on calendar time t $[w_{n+1}(t,T_{n+1})]$:

First, a two-period approach is considered. For periods n+1, n+2,..., the SEV is that of equation (6) $S(T_{n+1})$ in which c(0) is ignored. Recall that Hardie *et al.* (1984) proposed a solution in which only steady state rotations appear after k rotations. Note that the optimal rotation age T_{n+1}, T_{n+2},..., is the one solution obtained from equation (7), $T_{n+1} = T_{n+2}$,..., because the trend price p(τ), net profit ratio d(τ) and cost functions c(τ) of periods n+1, n+2,..., are the same for all subsequent periods. Now the SEV denoted by W(t) including all periods n, n+1, n+2,..., is

$$W(t) = c(0) + [\pi(t)q(t)\,\delta(t) - \gamma(t)]\exp(-rt) + S(T_{n+1})\exp(-rt) \tag{8}$$

where the "dynamic" price π(t), the net profit ratio δ(t) and the reforestation cost γ(t) functions associate only with the near future and with the first period n. Note that the separability, in fact additivity and monotonicity, have no increase in this case, which is required by dynamic programming (DP).

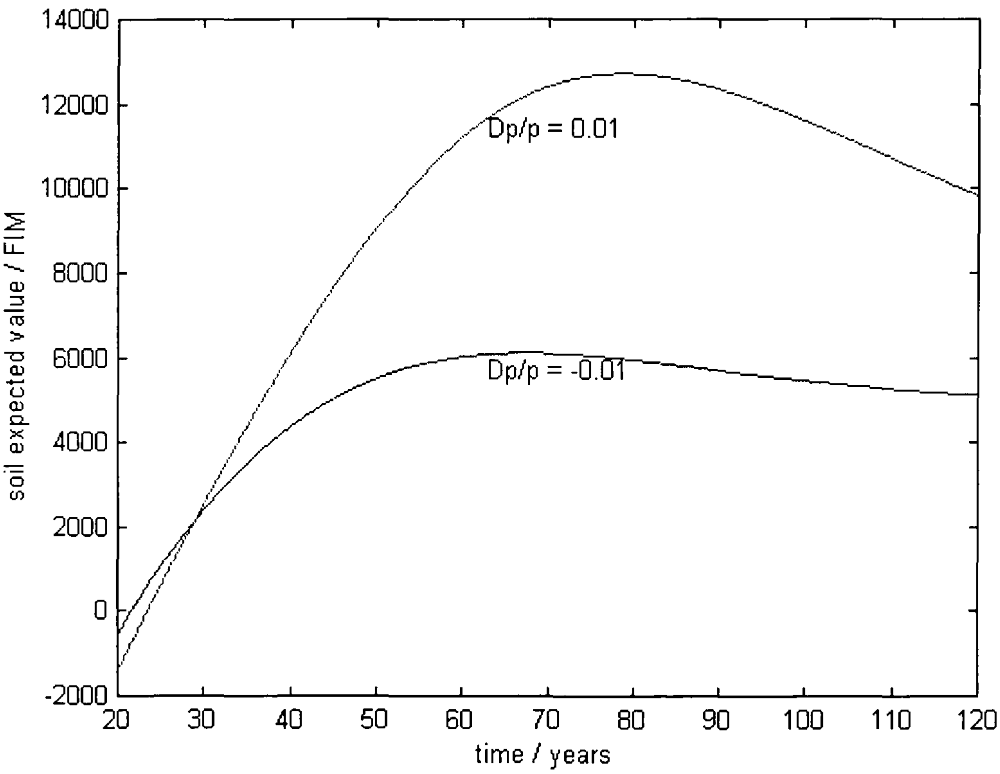

Figure 3: The soil expected value W FIM/ha, when r = 3%, π (T) = FIM 240/m^3, δ(T) = 0.71, δ'/δ = -0.05%, d'=0, γ(T) = FIM 5000/ha, γ'/γ = 0.5%, c'=0 and the yield is that in equation (18) of Vuokila and Väliaho (1980) with price changes π'/π = -1% and +1%, p'/p = 0.4%.

The optimal rotation age $T = T_n$ is defined by

$$\pi'(T)/\pi(T) + q'(T)/q(T) + \delta'(T)/\delta(T) - [\gamma'(T)/\gamma(T)]\,\gamma_r(T) =$$
$$r\{\,1 - \gamma_r(T) - S(T_{n+1})/[\pi(T)q(T)\delta(T)]\,\},\qquad (9)$$

where $\gamma_r(t)$ denotes the relative reforestation cost, $\gamma_r(t) = \gamma(t)/[\pi(t)q(t)\,\delta(t)]$.

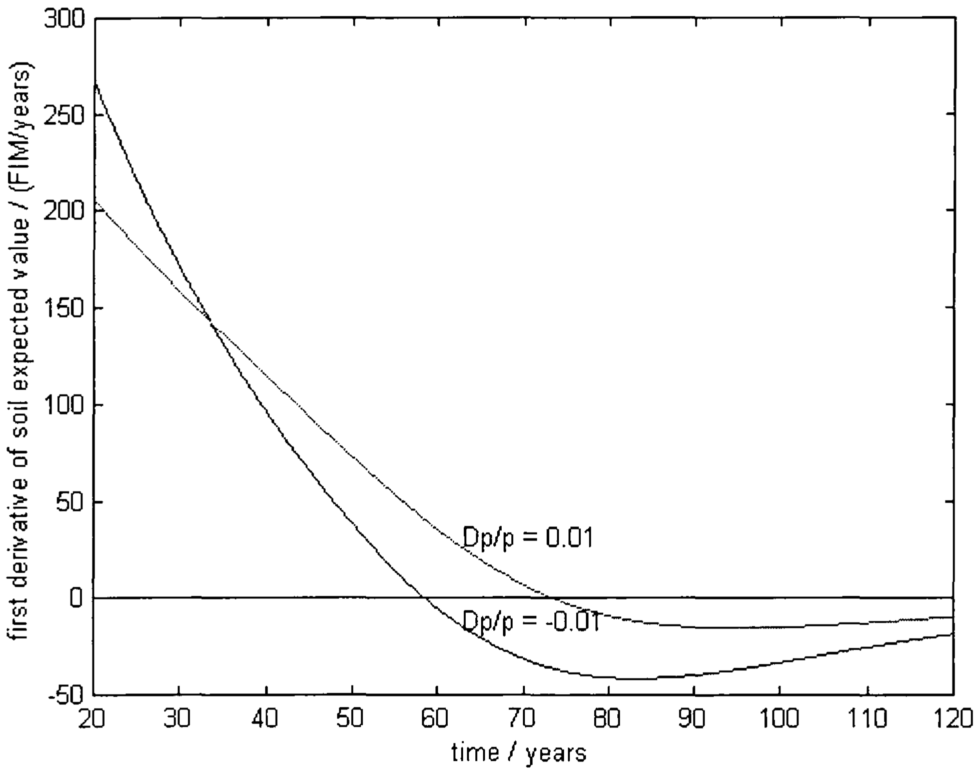

Figure 4: The derivative of the soil expected value W FIM/(ha year), when r = 3%, $\pi(T)$ = FIM 240/m^3, $\delta(T) = 0.71$, $\delta'/\delta = -0.05\%$, d'=0, c(T) = FIM 5000/ha, $\gamma'/\gamma = 0.5\%$,c' = 0 and the yield is that in equation (18) of Vuokila and Väliaho (1980) with price changes $\pi'/\pi = -1\%$ and +1%, p'/p = 0.4%.

The problem solution has two phases, the first of which consists of the definition of T_{n+1} as a solution of equation (7) by the trend price $p(\tau)$, net profit ratio $d(\tau)$ and cost $c(\tau)$ as well yield $q(\tau)$ functions and thereafter the calculation of $S(T_{n+1})$ is as defined in equation (6), without c(0) however.

In the second phase $S(T_{n+1})$ is considered constant and the optimal rotation period $T = T_n$ is calculated using equation (9). The optimum exists and is unique as shown in the Appendix.

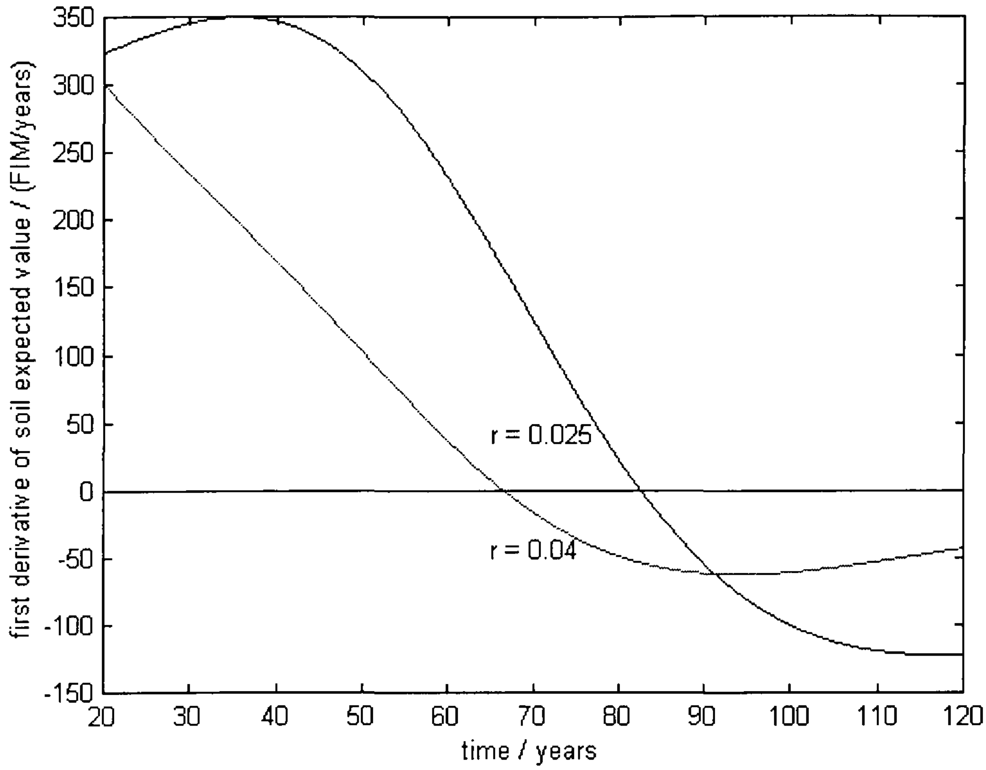

Figure 5: The derivative of the soil expected value W, when $\pi(T)$ = FIM 240/m^3, π'/π = p'/p = 1%, δ (T) = 0.71, δ'/δ = -0.05%, d' = 0, $\gamma(T)$ = FIM 5000/ha, γ'/γ = 0.5%, c' = 0 and the yield is that in equation (18) of Vuokila and Väliaho (1980) with interest rates r = 2.5%, and 4%.

Second, let all the prices, net profit ratios and cost functions of all periods vary in calendar time t. Limitations in empirical estimates for future prices and costs suggest that, for periods n+1, n+2,..., simple functions such as the linear or exponential ones are sufficient for the applications. Note also that exponential price and cost functions have already been suggested and applied by McConnell *et al.* (1983), and developed by Yin and Newman (1995).

Inspired by both the limitations in the availability of the estimates and previous studies, the model in which the calendar time emerges only in the exponential part of the functions, i.e., $\pi(t) = p(\tau)\exp(g_p\, t)$, $\delta(t) = d(\tau)\exp(g_d\, t)$, $\gamma(t) = c(\tau)\exp(g_c\, t)$ is considered. In case the growth rates g_p, g_d, g_c satisfy the relation $g_p + g_d = g_c$ the solution is trivial and obtained by replacing r by $r - (g_p + g_d)$. Denote for a while $r - g_p - g_d$ by α and $r - g_c$ by β, the net income or benefit is $p(\tau)q(\tau)d(\tau)$ by $b(\tau)$. In the first phase of the calendar variable functions model the functions of period n are of the same form as those of periods n+1, n+2,... . Then the SEV is in the form of

$$Z(t) = c(0) + b(t)\exp(-\alpha t) - c(t)\exp(-\beta t) + b(T_{n+1})\exp(-\alpha(t+T_{n+1})) - c(T_{n+1})\exp(-\beta\,(t+T_{n+1}) + b(T_{n+2})\exp(-\alpha(t+T_{n+1}+T_{n+2})) - c(T_{n+2})\exp(-\beta(t+T_{n+1}+T_{n+2}) + \dots \quad (10)$$

All the optimal rotation periods are the same as shown by applying dynamic programming (DB) and the induction axiom is (see the Appendix), i.e., $T = T_n = T_{n+1} = \ldots$, and a kind of simplified form emerges

$$Z_s(t) = c(0) + b(t)/[\exp(\alpha t)-1] - c(t)/[\exp(\beta t)-1] . \tag{10'}$$

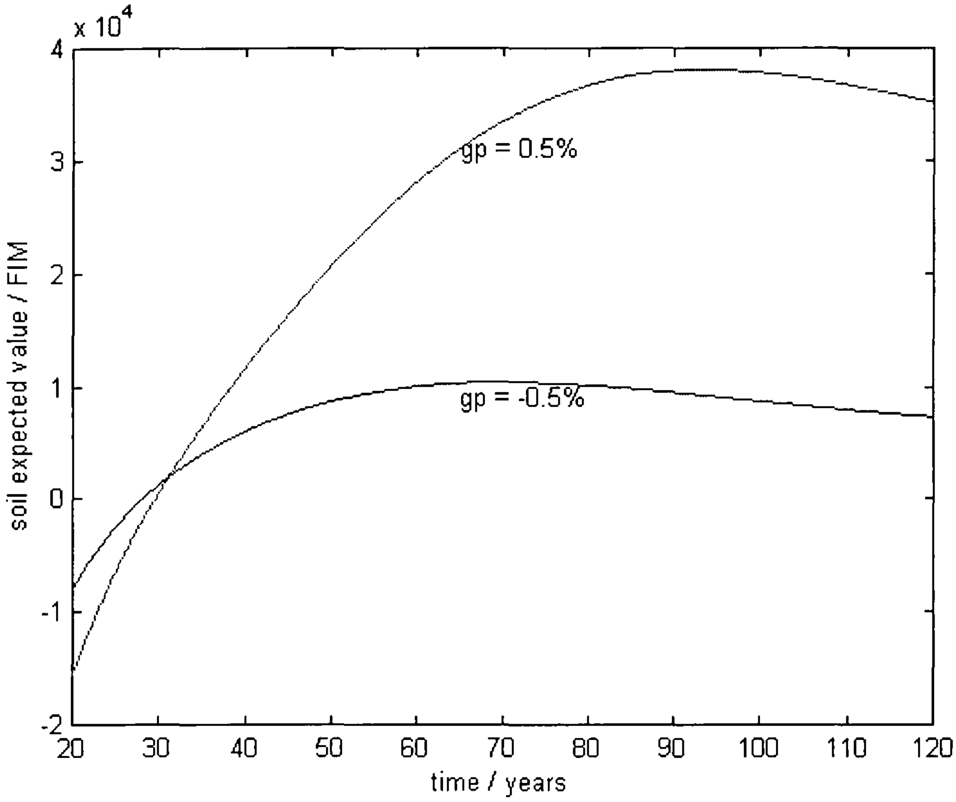

Figure 6: The soil expected value Z_s, when r = 3%, p(T) = FIM 240/m^3, p'/p = 1%, d(T) = 0.71, d' = 0, c(T) = FIM 5000/ha, c'/c = 0.5%, and the yield is that in equation (18) of Vuokila and Väliaho (1980).

Recall that Hardie *et al.* (1984) produced variable rotation lengths. Whenever the periods are equal the solution of $T = T_n = \ldots$ is defined by

$$[b'(T) - \alpha \exp(\alpha T)\, b(T)] / [\exp(\alpha T)-1] = [c'(T) - \beta \exp(\beta T)\, c(T))/[\exp(\beta T)-1] \tag{11}$$

which can be solved by using numerical methods. Note that the convergence requires that both $\alpha = r - g_p - g_d$ and $\beta = r - g_c$ are positive. The existence and uniqueness of the solution are discussed in the Appendix.

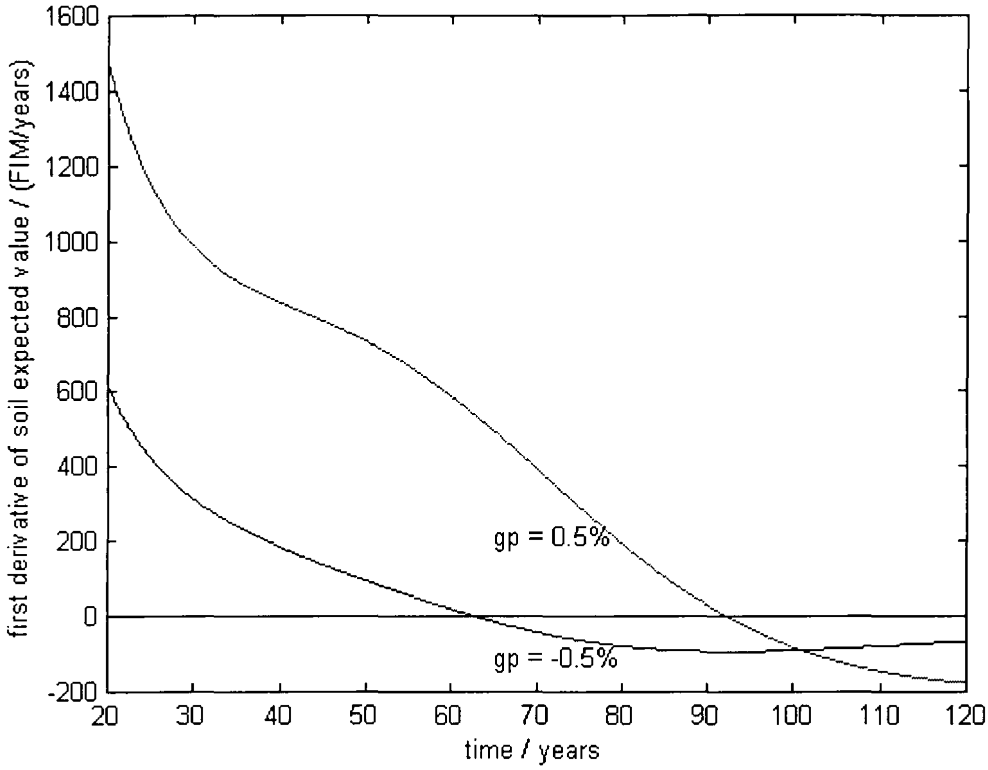

Figure 7: The derivative Z'_s of the soil expected value, when r = 3%, p(T) = FIM 240/m^3, p'/p = +0.5 and –0.5%, d(T) = 0.71, d' = 0, C(T) = FIM 5000/ha, c'/c = 0.5%, and the yield is that in equation (18) of Vuokila and Väliaho (1980).

Furthermore, the above solution can be used as that of T_{n+1}, T_{n+2} = ... and the first period might have "dynamic" functions $\pi(t)$, $\delta(t)$ and $\gamma(t)$ different from those of the subsequent periods. Denote $\pi(t)q(t)\delta(t)$ by φ (t). The first period optimal rotation period T = T_n would then be defined by

$$[\varphi'(T) - \alpha\,\varphi(T)]\exp(-\alpha T) - [\gamma'(T) - \beta\,\gamma(T)]\exp(-\beta T) =$$
$$\alpha\exp(-\alpha T)\,B(T_{n+1}) - \beta\exp(-\beta T)\,C(T_{n+1}), \qquad (12)$$

in which B and C represent the income and cost components of $Z_s(T_{n+1})$ for n+1, n+2,..., in equation (10'), and are defined by

$$B(T_{n+1}) = b(T_{n+1})/[\exp(\alpha T_{n+1})-1]$$

and

$$C(T_{n+1}) = c(T_{n+1})/[\exp(\beta T_{n+1}) - 1],$$

whenever the equal length optimal rotation periods, $T_{n+1} = T_{n+2}$, ..., are applied.

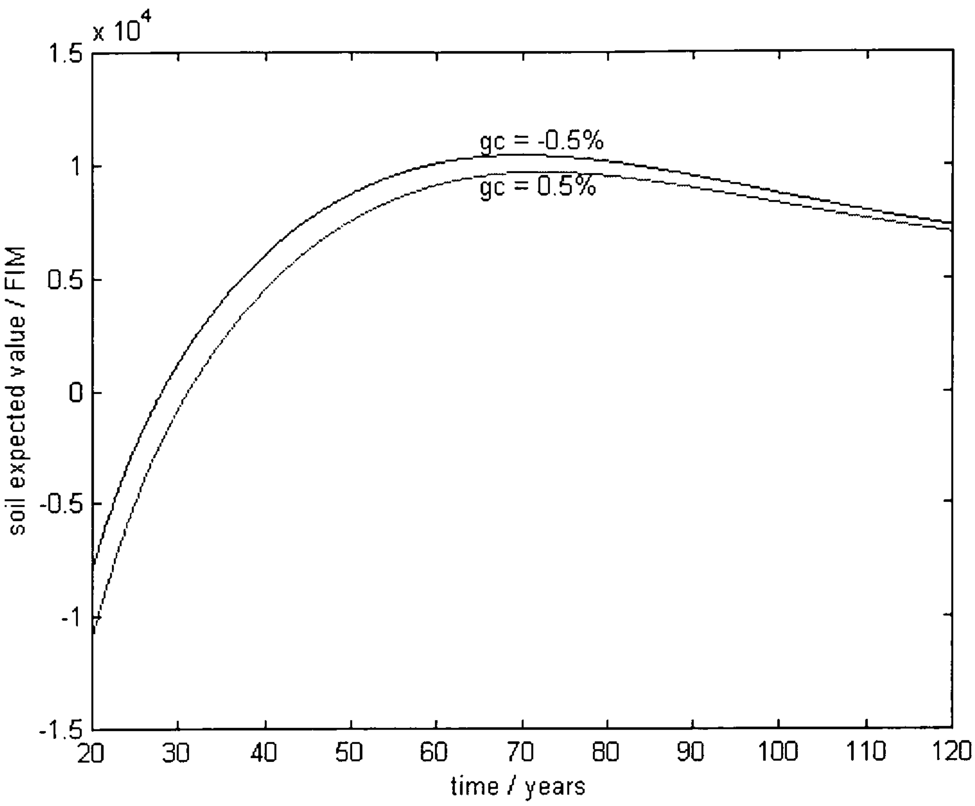

Figure 8: The soil expected value Z_S, when r = 3%, p(T) = FIM 240/m^3, p'/p = 1%, d(T) = 0.71, d' = 0, C(T) = FIM 5000/ha, c'/c = 0.5%, and the yield is that in equation (18) of Vuokila and Väliaho (1980).

Recall that in the calculation of T_n now c(0) of $Z_s(T_{n+1})$ in equation (10') is ignored. Conditions under which the solution exists and is unique are discussed in the Appendix.

2.3 The Sensitivity Analysis

Sensitivity is a key question in analyzing optimal policies. Risk and return studies (Lausti and Penttinen, 1998) demonstrate that the volatility of the return arises from the price component. The forest owner is a price taker, but he/she can speculate with the optimal roundwood selling time.

The interest rate may change. As a matter of fact, the interest rate r has traditionally been the most important sensitivity parameter of the optimal rotation problem.

The sensitivity of the proposed solutions with respect to the economic parameters, especially to prices and interest rates, are relevant to forest owners. In this study, it is investigated both analytically at the optimum point T and numerically in the area of the optimum.

Here the SEVs S(t) in equation (6) are used to establish analytically the sensitivity measures of the optimal rotation point T by differentiating the implicit function S'(T) = 0. Recall the partial derivative $\partial T/\partial r = -(\partial S'(T)/\partial r)/(\partial S'(T)/\partial T)$, etc. Numerical sensitivity studies are performed using both S(t) in equation (6) and W(t) in equation (8).

The interest rate r is considered first. Derivative $\partial T/\partial r$ is at point T (see the Appendix):

$$\partial T/\partial r = - f'(T)[1-\exp(-rT)]\{2\,T + T\exp(-rT) - 1/[r(1-\exp(-rT))]\} / \{f''(T)-rf(T)[\exp(rT)-1]\} , \qquad (13)$$

in which f(T)=p(T)q(T)d(T)-c(T).

Consider the case of Figure 1 with yield function in equation (18) of Vuokila and Väliaho (1980). At the optimum point T the rotation decreases by 0.8 years for every 0.1% percent increase and roughly 8 years for every percent increase in the interest rate r from 3% (Table 1).

Table 1: Sensitivity of S- and W-function with respect to interest rate r.

Interest rate r / %	**Optimal rotation time of s-function / years**	**Optimal rotation time of w-function / years**
1.9	79.5	83.2
2.0	78.7	82.3
2.1	78.0	81.4
2.9	71.9	74.5
3.0	71.1	73.7
3.1	70.3	72.9
3.9	64.1	66.3
4.0	63.3	66.5
4.1	62.4	64.6

The impact is graphically quite remarkable.

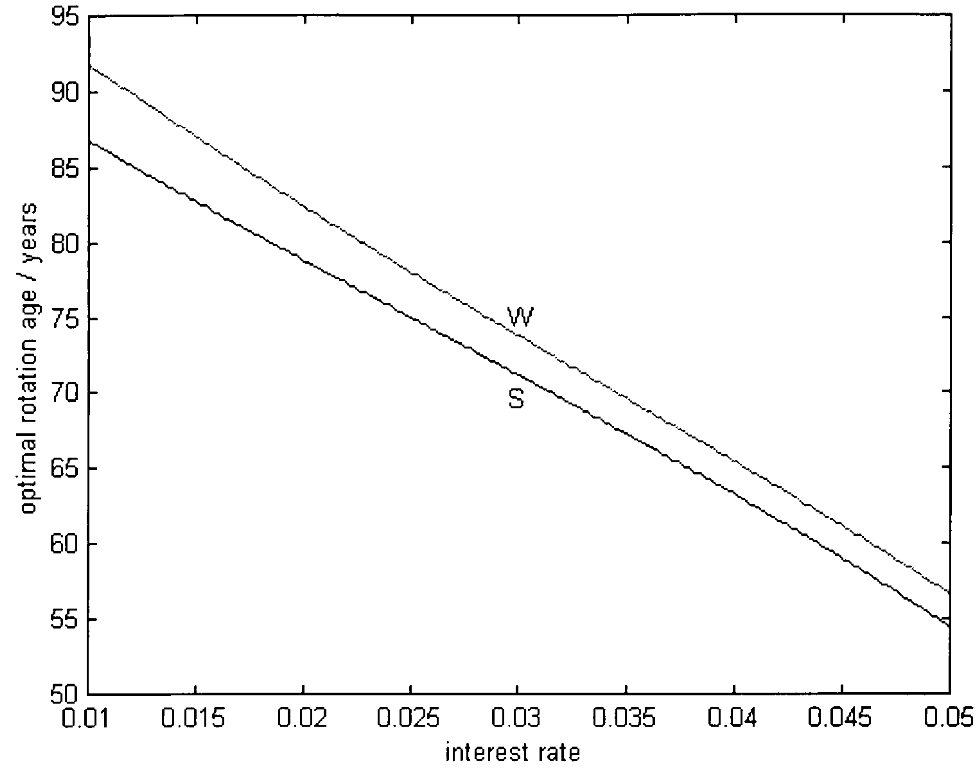

Figure 9: The optimal rotation with changing interest rate r in the SEV W and S when p(T) = FIM 240/m^3, p'/p = 1%, d(T) = 0.71, d' = 0, C(T) = FIM 5000/ha, c'/c = 0.5%, and the yield is that in equation (18) of Vuokila and Väliaho (1980).

Second, the impact of the stumpage price rate of change $p'(\tau)/p(\tau)$ on the optimal rotation period T is analyzed and demonstrated graphically. Let $p(\tau)$ be exponential, $p(\tau) = p0 \exp(p1\, \tau)$. The derivative $\partial T/\partial p1$, at point T is then (see the Appendix):

$$\partial T/\partial p1 = \{b(T)[\, p1\, T - 1]\}/\{f''(T) - r\, f'(T)\, \{[\exp(rT)-1]\} \,, \qquad (14)$$

in which $b(T) = p(T)q(T)d(T)$ and $f(T) = p(T)q(T)d(T)-c(T)$.

Consider the case of Figure 1 and yield function in equation (18). The graph shows that the impact of the price change on the optimal rotation is essential (Figure 10). More precisely, if the price change p1 increases by 0.1 per cent from zero, the rotation increases by roughly 1.2 years and by one percent by even some 12 years (Table 2).

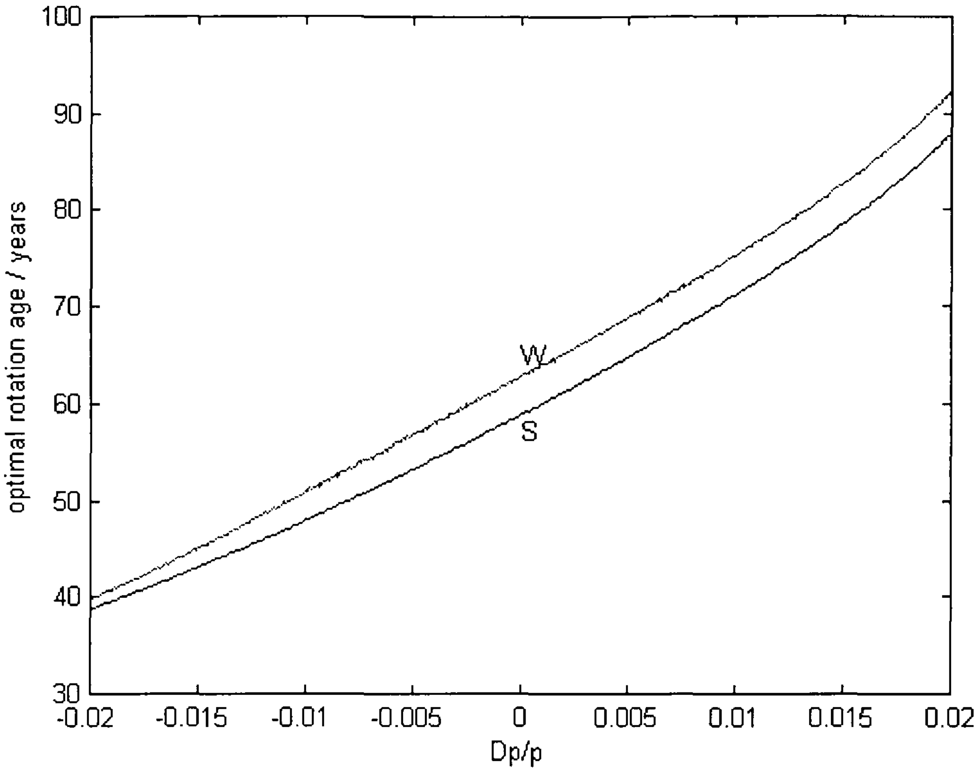

Figure 10: The optimal rotation with changing price increase p'/p in the SEV S and W, when r = 3%, p(T) = FIM 240/m^3, d(T) = 0.71, d' = 0, C(T) = FIM 5000/ha, c'/c = 0.5%, and the yield is that in equation (18) of Vuokila and Väliaho (1980).

Table 2: Sensitivity of S- and W-function with respect to stumpage price change p1.

Stumpage price change p1 / %	Optimal rotation time of s-function / years	Optimal rotation time of w-function / years
-1.1	46.9	49.7
-1.0	47.9	50.9
-0.9	49.0	52.0
-0.1	57.7	61.4
0.0	58.8	62.8
0.1	60.0	63.9
0.9	69.8	73.9
1.0	71.2	75.3
1.1	72.6	76.6

Furthermore, consider the change in reforestation cost $c'(\tau)/c(\tau)$. The owner can influence the reforestation costs, but only react to prices. The impact of these costs has been analyzed by Chang (1983) among others. He showed that both higher site preparation costs and higher planting costs mean a longer optimal rotation period.

Assume that $c(\tau) = c0 \exp(c1\ \tau)$. The derivative of the optimal rotation period at the optimum point T with respect to the change in silvicultural cost $\partial T/\partial c1$ is then (see the Appendix):

$$\partial T/\partial c1 = c(T)\ \{c1 - rT/[1-\exp(rT)]\}/\{\ [f''(T) - r\ f'(T)]\ [\exp(rT)-\exp(-rT)-1]\ \}\ . \quad (15)$$

The increase in the cost change c1 by 0.1 per cent affects only a rotation increase of 0.2 years or by one percent affects only an increase of less than 2 years (Table 3).

Table 3: Sensitivity of S- and W-function with respect to reforestation cost change.

Reforestation cost change c1 / %	Optimal rotation time of s-function / years	Optimal rotation time of w-function / years
-2.5	65.6	69.8
-2.0	66.3	70.2
-1.5	67.1	70.8
-0.5	69.0	72.1
0.0	70.0	72.9
0.5	71.1	73.7
1.5	73.2	75.1
2.0	73.6	75.0
2.5	72.9	73.4

Figure 11 shows also that the impact of cost change is quite modest. In the area where the cost increase exceeds the price increase, a non-natural situation can been seen as an anomaly in the optimal rotation period.

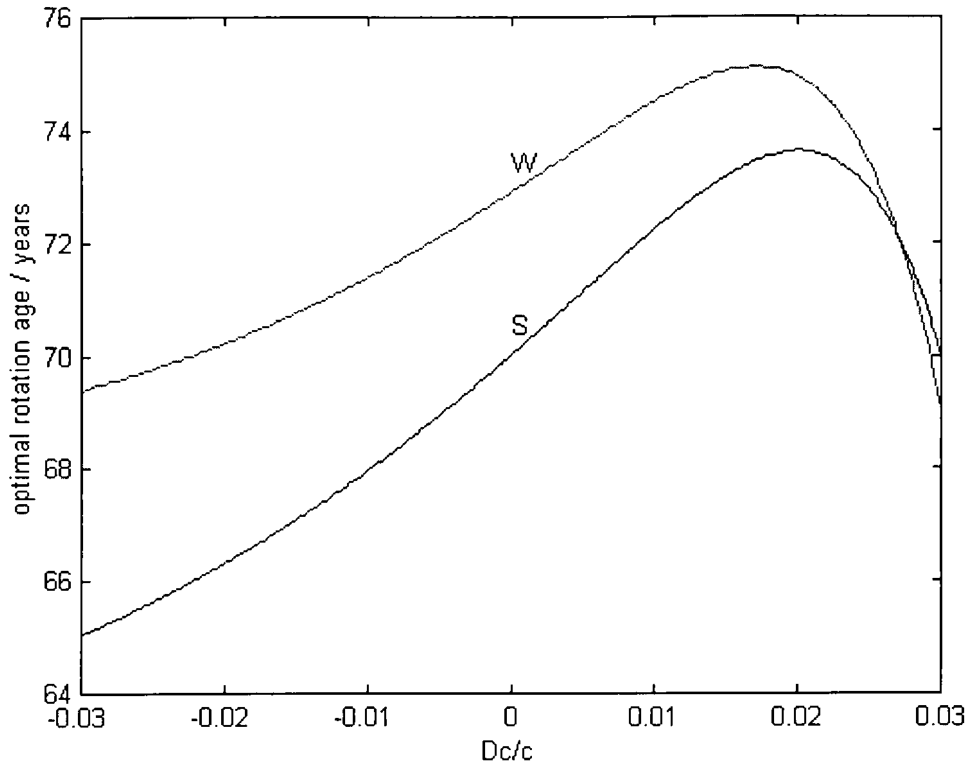

Figure 11: The optimal rotation with changing cost increase c'/c in the SEV S when r = 3%, p(T) = FIM 240/m^3, p'/p = 1%, d(T) = 0.71, d' = 0, C(T) = FIM 5000/ha, and the yield is that in equation (18) of Vuokila and Väliaho (1980).

One may note that in case of roughly 3% cost increase the profitability of forestry starts to deteriorate.

3 On the Numerical Solution of the Optimal Rotation

The optimal rotation solutions in equations (6)–(12) above can be based on growth tables or functions. The lower and upper bounds of the correction term 1/[1-exp(-rT)] above are defined when applying manual calculations.[8] The essential factors are p'(t)/p(t), in addition to the interest rate r, the relative cost change c'(t)/c(t) and the relative profit ratio change d'(t)/d(t). Whenever these are assumed to be constants, one can use the volume growth tables and the multipliers, which transform the volume growth to the value growth. When defining the first approximation manually, the reforestation cost ratio $c_r(t)$ can be assumed to be constant. Suppose that at least good empirical results are available, such as growth and yield tables (e.g., Vuokila and Väliaho, 1980). The solution can then be constructed, even manually.

[8] Suppose that the interest rate is, say, 3% per year. Assume initially that the optimal rotation period is between 70 and 90 years. The correction term is then between 1.13954 and 1.07205. This shows that the influence of future generations increases the interest rate r by only about 10%.

Recall that value growth is the correct measure related to the sales price, not the volume growth. The relationship between the value increment and the volume increment is roughly 1.5 according to Nyyssönen and Ojansuu (1982).

The growth during the next five years as a percentage of the present volume q_{v5}, is defined by Vuokila and Väliaho (1980). Yearly growth percentage functions are presented by Nyyssönen and Mielikäinen (1978).

The input variables needed to calculate q_{v5} for a particular stand are available in the forest management planning data. However, instead of the formulas of Vuokila and Väliaho (1980) with several input variables, three simple functions are applied in order to demonstrate the optimal rotation behavior. The first volume function used is simply (Fridh and Nilsson, 1980)

$$q_1(\tau) = [\mu \;\; a \;\; 1.6416]\,[\, 1. - 6.3582^{(-\tau/a)}\,]^{\,2.8967}, \qquad (16)$$

where μ is the maximum sustainable yield per hectare and a is the age of the stand when μ is obtained (see Lohmander, 1987; Gong, 1992).

Then standard techniques of numerical analysis are available. In this study, the calculations have been performed using MATLAB software. Here it is assumed that for a VT (Vaccinium Type) pine stand, which has dominant height H_{100} at the age of 100 years of 21 meters, one thinning with 35% removal is done. The maximum sustainable yield is then 4.1 m^3/ha/year and the maximum sustainable yield age is 100 years (Vuokila and Väliaho, 1980, p. 242). Here $\mu = 4$ m^3/ha/year and a = 90 years are used.[9]

An alternative yield function for testing purposes is that of Kuuluvainen and Tahvonen (1999), which is in the form of

$$q_2(\tau) = K / [1 - C \;\exp(-r\tau)], \qquad (17)$$

where $K = q_2(\infty) = 500$ m^3, $C = (q0\text{-}K)/q0$, $q0 = q_2(0) = 10$ m^3, and the growth rate $r = 0.048$ are the parameters proposed by the authors.

Consider the traditional and well known differential equation $q'(t)/q(t) = [q(t)\text{-}\, q_b][\, q_\infty - q(t)]\, g(t)$, where growth is related to the distance from both the bottom q_b and the ceiling q_∞. The solution q(t) of the differential equation above is affected by g(t), a special function of t (see, e.g., Hald, 1952, p. 659). However, the limitations in the amount of observations suggests that function g(t) is constant. Thus, the solution of the differential equation includes also a nonzero bottom q_b

$$q_3(t) = q_\infty / [\; 1 + C \exp(-t/a)] + q_b, \qquad (18)$$

where $C = [(q_\infty)/\, q_0] - 1$, $q_\infty = q_3(\infty) - q_b$, $q_0 = q_3(0)\text{-}q_b$ and age a is the shape parameter.

This is applied to curve fitting for the calculations after the last thinning (Vuokila and Väliaho, 1980, p. 242). The total production for the dominant height h 24 m is 450 m^3

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[9] The average sustained yield for a Vaccinium site type in Southern Finland is at most 4.7 m^3/ha/year and, according to the Central Forestry Board tables, 4.0 m^3/ha/year.

(Hynynen and Ojansuu, 1996, p. 73), with h 21 m the total production is roughly 420 m^3. Thinning means a removal of 65 m^3. Then the estimates for q_∞ and q_b are 355 m^3 and -65 m^3. The estimation based on the calculations of Vuokila and Väliaho (1980, p. 242), however, suggests 460 m^3 for the total production and -85 m^3 for the bottom q_b. The starting volume q_0 estimate is 22.8 m^3, the shape parameter is estimated to be 19.5 years or r as in Kuuluvainen and Tahvonen (1999), to 0.051 year. All the growth and yield table figures q respectively the estimates of q(t) are compared, and the maximum deviation turned out to be only 4.2%.

4 Results

Suppose that the annual interest rate of the national economy is r = 3% and the annual price increase is that of the local trend this century p'(T)/p(T) = +0.4% and the annual change in the gross profit ratio, say, d'(T)/d(T) = -0.1%. Assume, for the phase of manual calculations, that the reforestation ratio $c_r(T)$ is constant, $c_r(T) = c(0)/[p(T)q(T)d(T)] = 10\%$. Moreover, the correction term 1 / [1 - exp(-rT)] is for r = 3% / year between 1.07205 and 1.13954, whenever 70 < T < 90 years. The optimal rotation of the NPV approach in equations (6) and (7) is then defined by the value increase q'(T)/q(T) = 1.9–2.1% per year. The recommended optimal rotation of the Forestry Centre Tapio (1994) for a rather dry upland forest site (Vaccinium Type) under pine is 90–100 years. The volume growth tables (Vuokila and Väliaho, 1980) for pine on VT sites with three thinnings, and the value increment related to the growth increment of 1.5 (Nyyssönen and Ojansuu, 1982), suggest an approximately 90 year optimal rotation period.

Moreover, suppose that the increase in the yearly regeneration costs is 0.5–1% per year, i.e., c'(T)/c(T)= 0.5% per year. The cost of planting is c_a = 5000 FIM/ha and that of natural reforestation is c_n = 3000 FIM, i.e., c(T) = 5000 or 3000 FIM/ha. Additionally, assume that the standard present list prices for pine logs p(T) = FIM 240/m^3, and the gross profit ratio is d(T) = 71%, after a 29% sales tax. The volume increase is multiplied by 1.5 to obtain the value increase (Nyyssönen and Ojansuu, 1982), which is recognized by the linear price change between 40 and 80 years. When using growth in equation (16) with the parameters μ = 5.0 m^3/ha/year and the age a of the stand when μ is obtained a = 80 years, numerical methods give an optimal rotation of T = 73 years for planting and 71 years for natural regeneration. With yield function of equation (17) the corresponding optimal rotation periods are T = 78 years for natural regeneration and T = 81 years for planting.

The natural generation option means a shift at the beginning of the growth, say 5–10 years, which suggests that the actual rotation period recommendation is nearly 80 years.

When using the local growth and yield tables of Nyyssönen and Mielikäinen (1978) or Vuokila and Väliaho (1980), these volume growth percentages mean longer optimal rotation periods than those of growth function in equation (16).

The value growth ratio of 1.5 (Nyyssönen and Ojansuu, 1982) can be recognized by applying the log volumes of Vuokila and Väliaho (1980, p. 242). The most recent roundwood prices are FIM 271 / m^3 for pine log and FIM 87 / m^3 for pine pulpwood (Aarne and Linna, 1999). As before in equation (18) a logistic growth function (see, e.g., Hald, 1952, p. 659) is applied. The solution of gross income, price times quantity, based on Vuokila and Väliaho (1980, p. 242) including also a nonzero bottom n_b is

$$n(t) = n_\infty / [\ 1 + D \exp(-t/s)] + n_b, \tag{19}$$

where $n_\infty = n(\infty) - n_b =$ FIM 95,338, $D = [(n_\infty)/ n_0] - 1 = 106.0225$, $n_b = n_0 - n(0) =$ -FIM 7,473.4 and age s = 15.1507 is the shape parameter providing 1/s = 6.6%/year.

Furthermore, the-two period model W(t) of equation (8), which uses "dynamic" parameters for period n and the similar periods model S(T) of equation (6) for periods, n+1, n+2,..., with "trend" parameters, is analyzed using the gross income function of equation (19). The optimal rotation is then determined by using the case of Figure 1 with varying silvicultural costs and interest rates, applying gross income function of equation (19). Longer optimal rotation periods than in Figure 1 are obtained (Figure 12).

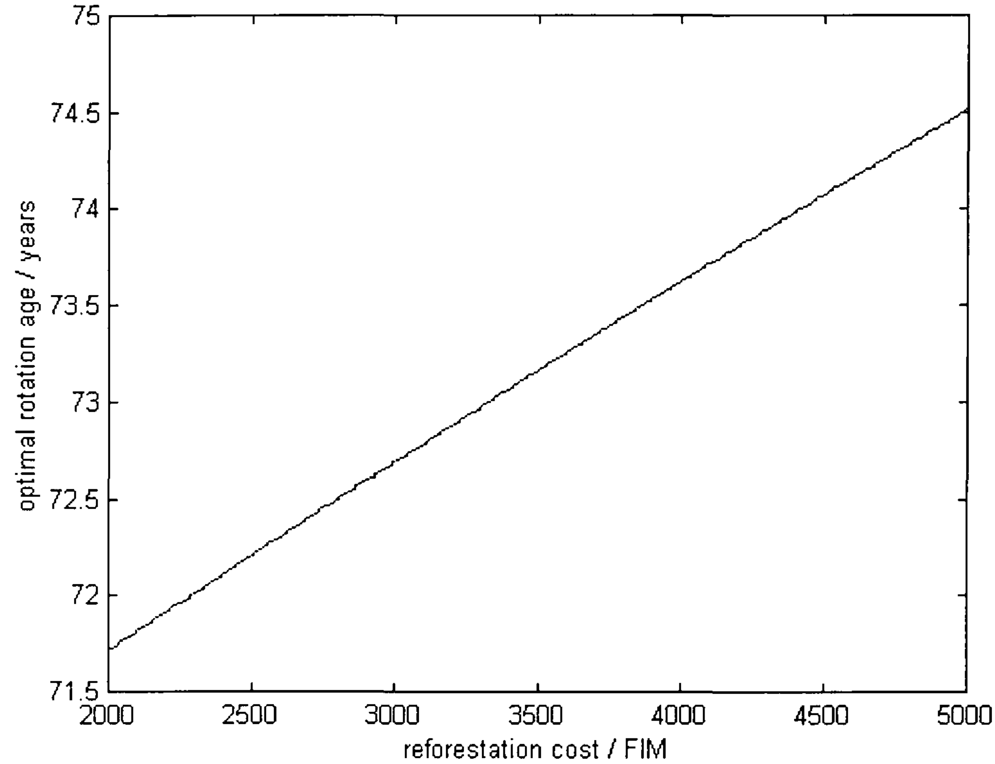

Figure 12: The optimal rotation period with soil expected value W in equation (8) varying present reforestation costs, when c'/c = 0.5%, p(pine pulpwood) = FIM 86, p(pine log) = FIM 271, r = 3%, d(T) = .71, d' =0, and the gross income is that in equation (19).

The optimal rotation period as a function of the "dynamic" price change for period n when applying the exponential price functions reveal the impact of the value increase factor.

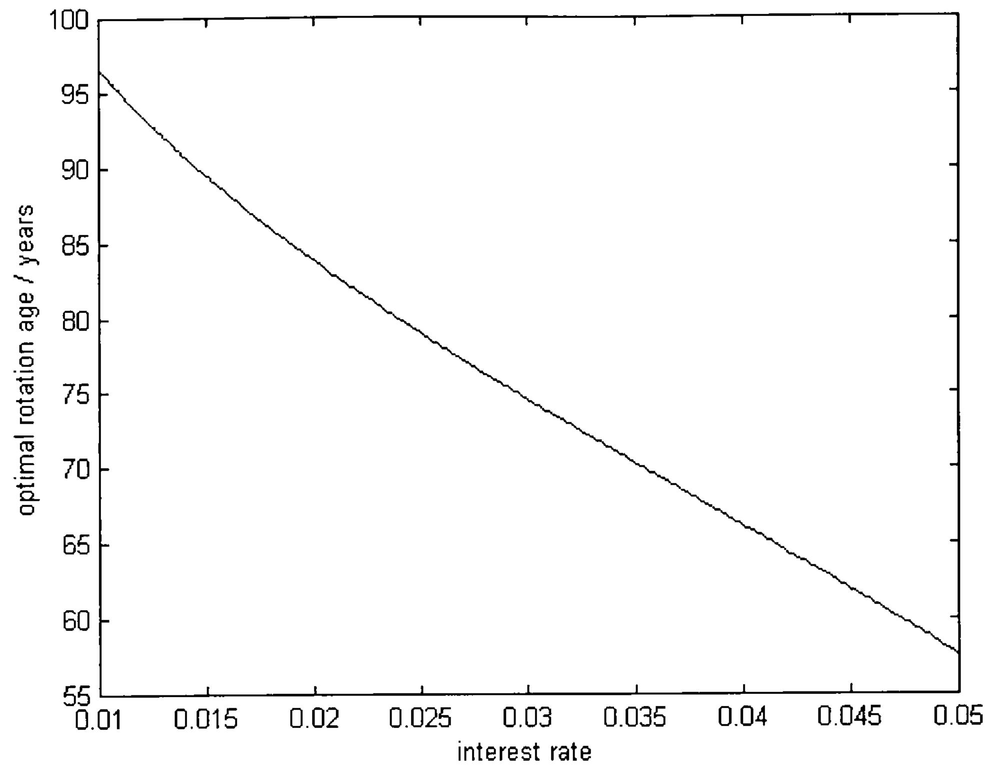

Figure 13: The optimal rotation period with soil expected value W in equation (8) varying interest rate r, when c(T) = FIM 5000, c'/c = 0.5%, p(pine pulpwood) = FIM 86, p(pine log) = FIM 271, d(T) = .71, d' =0, and the gross income is that in equation (19).

Finally, the optimal rotation age is analyzed in the case of changing log price p0 and constant pulp log prices of FIM 86/m^3.

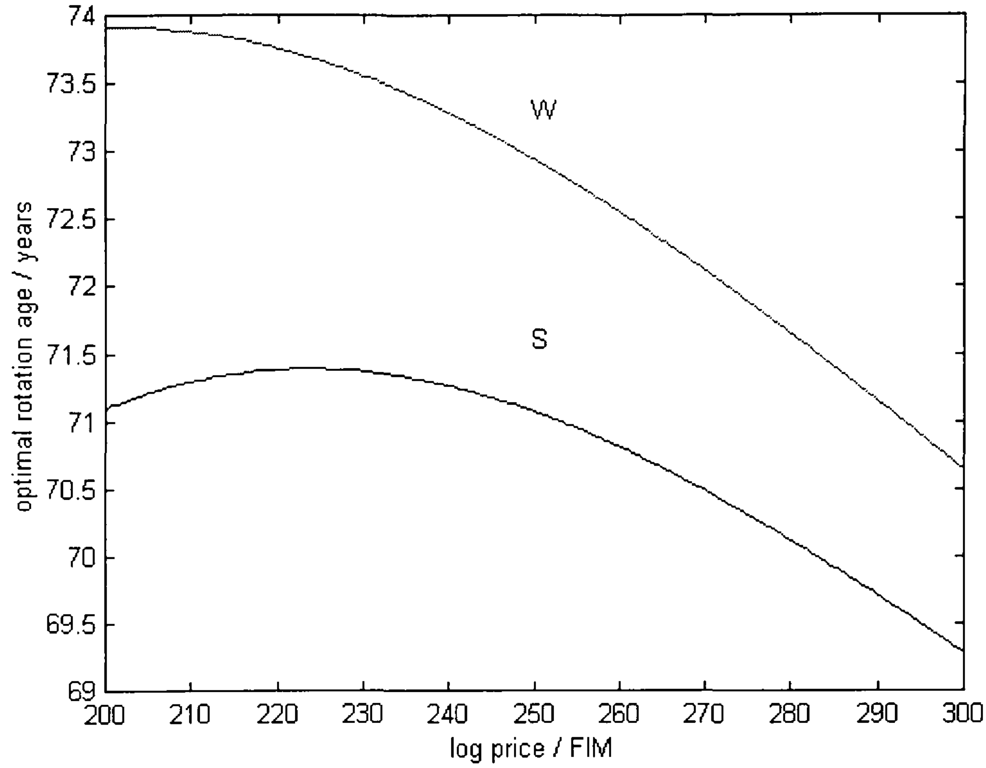

Figure 14: The optimal rotation period with soil expected value W in equation (8) varying log price p0, when c(T) = FIM 5000, c'/c = 0.5%, p(pine pulpwood) = FIM 86, r = 3%, d(T) = .71, d' =0, and the gross income is that in equation (19).

The level is clearly below 80 years. Stochastic prices might increase the optimum rotation slightly. The inclusion of an age decreasing relative harvesting costs may also increase the optimum. However, the 90 years recommendation of Forestry Centre Tapio (1994) is cautious. The prerequisite of regeneration felling is 80 years in Southern and Central Finland according to the Ministry of Agriculture and Forestry (1997), which roughly corresponds to the obtained results with 2.5% interest rate.

5 Discussion

The optimal rotation problem has been analyzed in this study for this and other dynamic situations, allowing economic items such as stumpage prices, reforestation costs, interest rates and gross profit ratios to vary in time according to both biological age and calendar time.

The methodological problem was tackled with the help of dynamic programming (DP). The existence and uniqueness of global optimal rotation periods was based on DP, the induction axiom and the explicitly quasiconcavity of the objective functions, which are first non-decreasing and then non-increasing. Forest economics has traditionally relied on concavity, which has been essentially relaxed to quasiconvexity.

DP produces two different cases: (i) the future prices and costs depend on age, and (ii) also depend on calendar time. All the combinations were investigated with the same and different functions for both the present and future periods. However, while the prices and costs of future rotations depend on age, they depend on calendar time only exponentially because of the limited availability of future estimate functions.

Different volume growth models and a value growth model for pine on the Vaccinium (VT) site type were applied in analyzing the optimal rotation and its sensitivity. However, models based on Vuokila and Väliaho (1980) were found the most practicable for the study. It turned out that planting regeneration results lead only to approximately three year longer optimal rotations, than those of natural regeneration, to which the delay at the inception of the growth should be added. Analogously, an increase in cost change velocity by one percent affects rotation only by roughly one year. However, a one percent increase both in the velocity of price change and interest rate produces a jump of some 10 years in rotation. All the models emphasize the sensitivity of the optimal rotation on the price change. The income growth obviously produces approximately a rotation period of five years longer than the corresponding volume growth model.

In all, price change affects the optimal rotation period length fundamentally, which suggests that forest owners should sell in peak seasons. Rotation lengths are also dominated by the interest rate, which cannot be affected. However, individual forest owners have different personal interest rates, caused by such things as loans. The impact of the reforestation cost is negligible. The results reveal strong dynamics produced by the market situation at present.

The key question is the availability of unbiased parameter estimates, as well as accurate value and volume growth functions for different tree species and forest stands. A striking outcome was the impact of the relative value growth on the optimum. The 90 years recommended by the Forest Development Centre Tapio turned out to be cautious. The rotation age limit of 80 years imposed by law, as interpreted by the Ministry of Agriculture and Forestry, was roughly similar to the final results at a 2.5% interest rate.

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Appendix

The traditional approach of optimal rotation studies, and even more generally forest economics, has relied on the *concavity*, i.e., proved that the second order derivative of the soil expected value (SEV) with respect to rotation age t is negative in order to guarantee a global unique maximum (see, e.g., Chang, 1983; 1984). Unfortunately, the concavity does not necessarily hold (Figure 15).

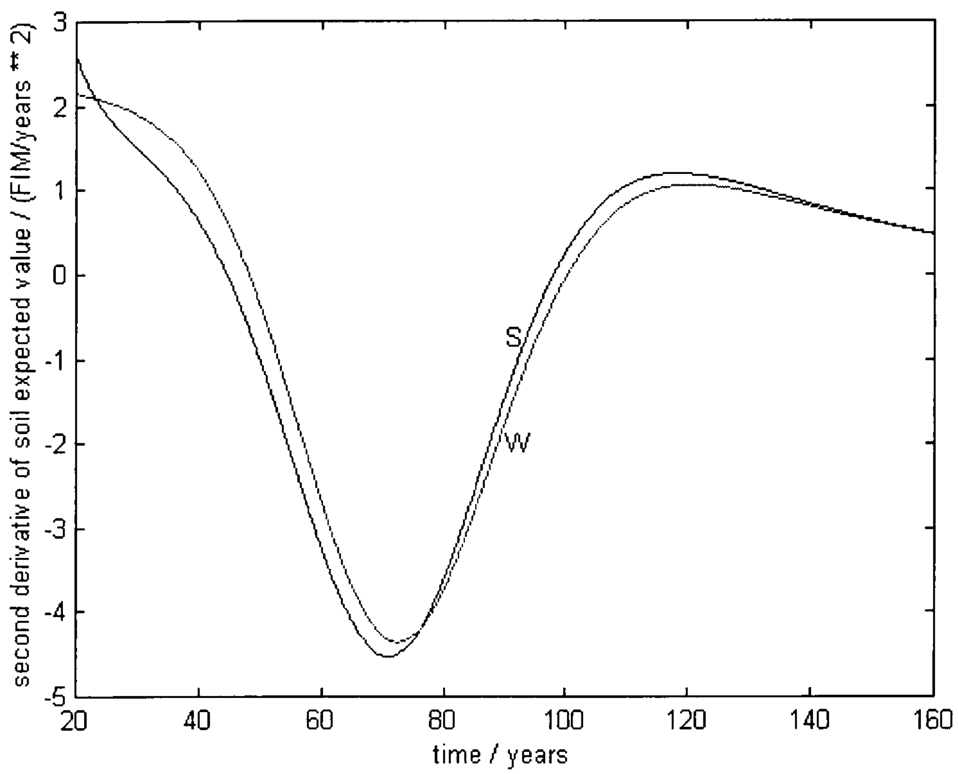

Figure 15: The second derivative soil expected values S and W, when r = 3%, $\pi(T)$ = FIM 24/m^3, π'/π = p'/p =1%, $\delta(T) = 0.71$, δ'/δ = d'/d = -0.05%, c(T) = FIM 5000/ha, γ'/γ =c'/c= 0.5%, and the yield is that in equation (18) of Vuokila and Väliaho (1980).

However, it has already been shown by Martos (1965) that a more general class of *explicitly quasiconcave*[10] functions is a *sufficient* condition, which guarantees that any local maximum is also a global maximum.

[10] A real-valued function f defined on a convex subset E of R^n is (Danao, 1992):

- quasiconcave on E if, and only if, x,y ε E, λ ε [0,1], and f(x) ≤ f(y) imply f(x) ≤ f[(1-λ)x+λ y];
- semistrictly quasiconcave on E if, and only if, x,y ε E, x ≠ y, λ ε (0,1), and f(x) < f(y) imply f(x) < f[(1-λ)x+λ y]; and
- explicitly quasiconcave on E if, and only if, it is quasiconcave and semistrictly quasiconcave on E.

Recall that a local maximum of a quasiconcave function f is a global one or f is constant in a neighborhood of the local maximum (Greenberg and Pierskalla, 1971). Note that the quasiconcavity of f is equivalent to the quasiconvexity of -f and vice versa (Martos, 1975). In the absence of constraints, given a continuous function f defined over a convex set, the explicitly quasiconcavity is also *necessary* for any local maximum to be also a global one (Netzer and Passy, 1975). Recently, the sufficient property to guarantee that a local maximum is also a global one is relaxed to the semistrictly quasiconcavity when semicontinuous functions are concerned (Daniilidis and Hadjisavvas, 1999). Moreover, if a *strictly quasiconcave*[11] function has a maximizer, then it is unique (Danao, 1992).

Recall the key notion that for an *explicitly quasiconcave* function from a convex set in R^n any local maximum is global (see, Martos, 1975, p. 89). Note also that a product of concave nonnegative functions is explicitly quasiconcave. Even a concave non-negative function divided by a convex positive function gives an explicitly quasiconcave function (Martos, 1975, p. 61–63).

Note that in order to avoid anomalies all functions here are assumed to be continuous, differentiable and possess derivatives of first and second order.

The derivation of the soil expectation value $V(\tau)$ in equation (4) is given by modifying a finding of Nautiyal and Williams (1990) as follows: instead of p $q(x,\tau)$, $p(\tau)q(\tau)d(\tau)$ is used. Denote $p(\tau)q(\tau)d(\tau) = b(\tau)$. The derivative $V'(\tau)$ is then simply

$V'(\tau) = \{[b'(\tau)-r\, b(\tau)]/[\exp(r\tau)-1]\ \} - r\,[b(\tau) - c(0)\exp(r\tau)]\ \}\ /\ \{[\exp(r\tau)-1]^2\}.$

Now $V'(T) = 0$ implies

$b'(T) - r\, b(T) = r[b(T)-c(0)]\exp(rT)/[\exp(rT)-1]$ and

$b'(T)[\exp(rTt)-1] - r\, b(Tt)\exp(rT) = -c(0)\exp(rTt)$, which gives equation (5).

Both the numerator $b(\tau)\exp(-r\tau) - c(0)$ and denominator $1-\exp(-r\tau)$ of $V(\tau)$ in equation (4) are differentiable, the denominator being even convex and > 0 for $\tau > 0$. $V(\tau)$ is explicitly quasiconcave, whenever the numerator is non-negative and concave (Martos, 1975, p. 63). The numerator is concave, whenever its derivative is non-increasing. The non-negativity holds after some $\tau_0 > 0$. Then $V(\tau)$ is explicitly quasiconcave after some $\tau_0 > 0$. Then any local maximum $V(\tau)$ after some $\tau_0 > 0$ is global.

The optimal rotation solution of equation (7) is based on $S(\tau)$ in equation (6), the derivative of which is analogous when using the notation $b(\tau)$

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[11] A real-valued function f defined on a convex subset E of R^n is (Danao, 1992):

- strictly quasiconcave on E if, and only if, $x,y \in E$, $x \neq y$, $\lambda \in (0,1)$, and $f(x) \leq f(y)$ imply $f(x) < f[(1-\lambda)x+\lambda\, y]$.

Note that a strictly quasiconcave function is explicitly quasiconcave according to Danao (1992). However, some earlier sources such as Greenberg and Pierskalla (1971) defined strictly quasiconvexity in such a way that it did not imply even quasiconvexity. In all, with lower semicontinuous functions the two definitions, explicitly quasiconvexity and strictly quasiconvexity coincide (Greenberg and Pierskalla, 1971). The same holds with quasiconcave and upper semicontinuous functions, because any convexity property of f is equivalent to respectively the concavity property of -f.

$S'(\tau) = \{[b'(\tau)-c'(\tau)] / [\exp(r\tau)-1] - \{r [b(\tau) - c(\tau)]\exp(r\tau)\}/ \{\exp(r\tau) - 1\}^2$.

Letting $S'(T) = 0$ and multiplying it by $[(\exp(rT)-1]$ and dividing by $b(T)$ and denoting $c(T)/[p(T)q(T)d(T)]$ by $c_r(T)$, one obtains equation (7).

Ignore c(0) for a while. The denominator $[\exp(r\tau)-1]$ of $S(\tau)$ is convex and > 0 for $\tau > 0$. Whenever the numerator $b(\tau)-c(\tau)$ is concave and non-negative, $S(\tau)$ is explicitly quasiconcave (Martos, 1975, p. 63). The concavity of the numerator is implied whenever the derivative of $b(\tau)-c(\tau)$ is non-increasing.

Consider W(t) in equation (8) ignoring the constant c(0). All terms are divided by exp(rt), which is both positive and convex. Recall that the sum of concave functions is concave. The constant $S(T_{n+1})$ is trivially concave. Provided that term $[\pi (t)q(t)\delta (t)-\gamma (t)]$ is concave, which happens when its derivative is non-increasing or the second derivative is negative, then W(t) is explicitly quasiconcave.

Consider Z(t) of equation (10) ignoring c(0) in the context of DP. The contributions of each period are separable, and each period is connected to the future with decreasing exponential multipliers. The DP type of problem definition is,

$$Z_n(t) = \max_t \{ b(t)\exp(-\alpha t) - c(t)\exp(-\beta t) + \exp(-\alpha t) Z_{+,n+1}(T_{n+1}) - \exp(-\beta t) Z_{-,n+1}(T_{n+1}) \},$$

in $Z_{+,n+1}(T_{n+1})$ stands for the benefit components $Z_{-,n+1}(T_{n+1})$ the cost components of the future. The formulation and the functions are exactly the same for each period n, n+1, n+2,... . Since the formulation holds for n, is assumed to hold for n+k, for any k > 0, and then holds for n+k+1 as shown above, the induction axiom implies that $T = T_n = T_{n+1} = T_{n+2} = T_{n+3} = \dots$. When the periods are equal equation (10) reduces to a kind of steady state form $Z_s(t) = c(0) + b(t)/[\exp(\alpha t)-1] - c(t)/[\exp(\beta t)-1]$, which gives equation (10').

Recall that $\alpha, \beta > 0$ is required in order to avoid the explosion of the model. Then both denominators $\exp(\alpha t)$ and $\exp(\beta t)$ above are non-negative and convex. Whenever b(t) is concave and c(t) is convex both items are explicitly quasiconvex. In case their sum b(t)-c(t) is concave, explicitly quasiconvex hull functions of $Z_s(t)$ can be constructed by replacing first α by β and then β by α in equation (10'). In practice, simple functions of $p(\tau)$, $d(\tau)$ and $c(\tau)$ such as exponential and/or linear ones are sufficient for the calculations. Then the explicitly quasiconcavity or even concavity of $Z_n(t)$ itself can be based on the concavity of its components.

Consider the existence and uniqueness of equation (12) of

$$W_z(t) = c(0) + \varphi(t) \exp(-\alpha t) - \gamma (t) \exp(-\beta t) + \exp(-\alpha t)\, b(T_{n+1})/[\exp(\alpha T_{n+1})-1] - \exp(-\beta t)c(T_{n+1})/[\exp(\beta T_{n+1})-1].$$

The terms multiplied by $\exp(-\alpha t)$, or divided by convex and positive $\exp(\alpha t)$, are explicitly quasiconcave whenever $\varphi(t)$ is concave. In the same way, the terms multiplied by $\exp(-\beta t)$ are explicitly quasiconcaver whenever $\gamma (t)$ is convex. Then explicitly quasiconcave hull functions can be constructed as above by replacing first α by β and then β by α in $W_z(t)$. Recall that in practice, simple functions of $p(\tau)$, $d(\tau)$ and $c(\tau)$ such as exponential and/or linear ones are typically used in the calculations. Then the explicitly quasiconcavity of the function $W_z(t)$ itself can be based on the concavity of its components.

The sensitivity of the solutions in the optimal rotation point T in case of S(T) is considered. The differentiation of implicit function $S'(T)= 0$ gives $\partial T/\partial r = - [\partial S'(t)/ \partial r] / [\partial S'(t)/ \partial T]$. Recall that $\partial S'(T)/ \partial T = S''(T)$ is, when using the optimum points condition $S'(T) = 0$, simply

$$S''(T) = \{ f''(T) + r f'(T)[\exp(rT))-1]]\}/[\exp(rT)-1],$$

where $f(T) = p(T)q(T)d(T)-c(T)$.

The derivative $\partial S'(T)/ \partial r$ is

$$\partial S'(T)/ \partial r = f'(T) \{2T + T \exp(-rT) -1/[r (1-\exp-rT)] \} / \{[1-\exp(-rT)][\exp(rT)-1]\}.$$

In all, the implicit derivative $\partial T/\partial r$ in the optimum point T is then

$$\partial T/\partial r = - f'(T)[1-\exp(-rT)]\{2 T + T \exp(-rT) - 1/[r (1-\exp-rT)] \}/ \{f''(T)-rf(T)[\exp(rT)-1]\},$$

which gives equation (13).

The price function is now assumed to possess form $p(t) = p0 \exp(p1\ t)$.

Then $\partial S'(T)/\partial p1$ is in the optimum point T simply

$$\partial S'(T)/\partial p1 = b(T)[1- p1\ T] /[\exp(rT)-1].$$

In all, the sensitivity of T with respect to p1 is therewith

$$\partial T/\partial p1 = \{b(T)[p1\ T - 1]\}/\{f''(T) - r f'(T) \{[\exp(rT)-1]\},$$

which gives equation (14).

If the cost function has the form $c(t) = c0 \exp(c1\ t)$, the derivative $\partial S'(T)/ \partial c1$ is

$$\partial S'(T)/ \partial c1 = c(T) \{-c1 + rT/[1-\exp(-rT)]\}/\{\exp(rT)-1]\}.$$

Finally, the sensitivity of T with respect to c1 is thus

$$\partial T / \partial c1 = c(T) \{c1 - rT/[1-\exp(rT)]\}/\{ [f''(T) - r f'(T)] [\exp(rT)-\exp(-rT)-1] \},$$

which yields equation (15).