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Dynamic Model of Innovation: Optimal Investment, Optimal Timing, Market Competition

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Abstract

A dynamic model of investment process for a technology innovator in a market environment is designed. The "light" dynamics of the active innovator is described by the system of exponential trajectories in which one can quickly change growth parameters. It is assumed that the innovator operates in the inert market environment which can be presented by "heavy" exponential trajectories. The growth parameters of the market trajectories vary slowly and can be identified to some accuracy in the dynamic process of econometric observation basing on information of the current technology stock (the average market technology stock) and its rate (the average market technology rate). The model consists of three decision making levels for dynamical identification, optimization of the commercialization time and optimal control design. On the first level the innovator makes assessment for the market commercialization time using econometric characteristics of the current level of the market technology stock and the market technology rate. Since the market environment is inert and its acceleration (the second derivative) is small then information about the market technology stock (current position) and the market technology rate (current first derivative) gives an opportunity to estimate exponential parameters of the market growth trajectories, to forecast the market commercialization time and indicate its sensitivity. On the second level the innovator optimizes its commercialization time basing on its own current technology stock and taking into account the forecast of the market commercialization time. Two scenarios are possible for the innovator: the "slow" scenario with "large" time of innovation is oriented on the local extremum with usual level of sales of invented products, the "fast" scenario with "small" time of innovation maximizes the level of early innovation with bonus sales due to the market overtaking. On the third level the innovator solves an optimization problem for the investment policy basing on information about the chosen innovation scenario, the commercialization time, and the difference between the achieved technology stock and the demanded technology stock for starting commercialization. Dynamical optimality principles for optimizing discounted innovation costs on investment trajectories are used for finding the optimal investment plan and designing optimal feedback for its realization. Properties of sensitivity and robustness are investigated for the optimal profit result and innovation feedbacks.

Contents

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Introduction

In this paper we deal with a dynamic model of innovation for a technology innovator which operates in the competitive market environment. In this model three main interacting objectives of the innovator are in the focus. These three tasks can be formulated shortly as: (i) observation and assessment of the market potential innovation, (ii) selection of the possible innovation scenario and optimization of the commercialization time, (iii) optimal design for the investment level. The main feature of the model is in its dynamic setting: all three problems are considered as the time evolved processes. At each moment of time the innovator can make a decision on the new innovation scenario, optimal time of innovation and optimal investment level in the feedback interaction basing on information about the current econometric characteristics of its own technology stock, the market technology stock and the market technology rate. The problem is to find a policy strategy for assessing the potential market innovation, choosing a scenario, optimizing the commercialization time and the investment level.

In the problem of designing optimal investment level we use the basic constructions of the models of optimal growth with irreversible investment and allocation of resources for invention (see [Intriligator, 1971], [Arrow, 1985]). The construction of the benefit and expenditure functionals is based on integral payoffs for the problem of the optimal control with discount coefficients (see, for example, [Dolcetta, 1983]), and, in particular, on payoff patterns for the problem of allocation of drug control efforts (see [Dawid, Feichtinger, 1996]). We adapt the time-delay dynamics of the model of a firm's R&D investment (see [Griliches, 1984], [Watanabe, 1992, 1997]) for description of the controlled investment process. For dynamic selection of scenarios and optimization of the commercialization time we apply and develop the static model of optimal timing of innovations (see [Barzel, 1968]). This paper deals with the problem of determining the date for which an innovation is optimal. It discusses also the effect of premature timing of innovation and competitive trends which could enlarge or reduce the amount of resources devoted to innovating activity due to indeterminacy in assessment of benefits. In our research we use patterns of the differential games theory (see, [Krasovskii, Subbotin, 1988], [Ivanov, Tarasyev, Ushakov, Khripunov, 1993]) for modeling the identification process of the market innovation trajectories which can be interpreted as dynamics of a "heavy" object with the weak controllability. For composing dynamics of the market technology growth and the investment process of the

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innovator we adjust econometric measurement formulas for R&D activities with time lag in interaction between technology and economy (see [Watanabe, 1995]).

The dynamic model of optimal investment policy consists of three interacting levels of decision making: the econometric identification of the market technology trajectories, the selection of an innovation scenario and optimization of the commercialization time, optimal control design of investment level. In the identification part of the model we assume that the identified object - the market environment, has the "heavy" dynamics with weakly variable (controllable) parameters of exponential trajectories. This assumption gives the opportunity to describe the trajectories ensemble, to assess the attainability set of the market potential innovation, and to analyze sensitivity of the predicted commercialization times. Basing on the evaluated time of the market innovation the innovator can make decision on selection of the innovation scenario. There are two possible strategies. The first strategy is oriented on the local maximum of the profit function with the usual level of benefit sales. The second strategy tends to overtake the market potential innovation and to capture the local maximum of the profit function with the bonus sales of early innovation. In the second scenario the time of innovation is smaller then in the second one, but the amount of the technology investment resources of the fast innovation is too large in comparison with the slow trajectory. In parallel with identification and selection problems the innovator can dynamically optimize the investment level. The natural problem of minimizing innovation expenditures is posed on the trajectories of the investment dynamics. In solving this optimal control problem the innovator should reach the level of the technology stock which is necessary for starting commercialization at the time prescribed by the selected scenario.

For solution of identification, selection and optimal control problems in the dynamic model of investment policy we use dynamic programming principle (see [Bellman, 1961]), Pontryagin's maximum principle of optimal control theory (see [Pontryagin, Boltyanskii, Gamkrelidze, Mishchenko, 1962]) and methods of differential games (see [Krasovskii, Subbotin, 1988]).

Using the principle maximum of Pontryagin we find the optimal plan (programming control) for the investment strategy which depends on initial and final technology stocks. We extract the optimal feedback equivalent to the optimal plan. This optimal feedback is based on the current situation of the technology stock and generates the optimal trajectory of the technology growth in the interactive procedure. Substituting optimal investment plans to the integral of discounted expenditure we derive the collection of value functions parametrized by the commercialization times. Basing on the principle of dynamic programming one can prove that solution for the joint optimization problem of investment policy can be decomposed into two levels: on the first level the optimal investment feedback is designed and value functions of innovation expenditures are calculated for the parametrized commercialization times; on the second level the balance in the profit function between benefit from the parametrized amount of sales and innovation expenditures is optimized by the parameter of the commercialization time. Two possible scenarios of investment arise naturally due to the difference in amounts of sales: the first scenario is oriented on advancing the commercialization time beyond the market and on the bonus sales; the second scenario optimizes the commercialization time for the profit function with the usual amount of benefit sales. These scenarios correspond to two local maximum points of the profit function which does not have properties of convexity or concavity. The proper selection of the global maximum among two local maximum points depends on the correct assessment of the market commercialization time. To assess market trajectories we use the model of the inert market environment. In this model the dynamic observation of the market technology stock and its rate allows to identify parameters of the market exponential growth. Under assumption of small acceleration rate we evaluate the market commercialization time and analyze its sensitivity.

Combining all three levels of the model: identification of the market commercialization time, selection of the innovation scenario, optimal control design of the investment level, we obtain the optimal dynamic algorithm for the investment policy organized on the feedback principle. We show that the optimal feedback strategy of innovation constitutes the saddle type equilibrium in the game interaction of the innovator with the market environment and guarantees the value of the profit function at this equilibrium. The sensitivity and robustness properties of the optimal profit value and the optimal strategy for scenarios selection are analyzed and corresponding linear estimates with respect to values of acceleration are given.

1 Dynamical Model of Innovation Policy

We consider the dynamical model of innovation policy for an innovating firm which includes three interacting objectives of decision making: (i) econometric assessment of the market technology trajectories and prediction of the market commercialization time; (ii) selection of the innovation scenario with optimizing the innovator commercialization time; (iii) the feedback design for dynamical optimization of the investment level.

In the problem (iii) of the optimal investment we assume that the current technology stock $x(t)$ is subject to the growth dynamics with the time-delay and obsolescence effects (see, for example, [Griliches, 1984], [Watanabe, 1992, 1997])

$$\dot{x}(t) = -\sigma x(t) + r_a^{\gamma}(t) \tag{1.1}$$

Here parameter $\sigma > 0$ is coefficient of technology obsolescence, the control parameter $r_a(t)$ is the index of R&D investment, parameter γ, $0 < \gamma < 1$ is the time-delay exponential coefficient. Let us note that the "light" dynamics (1.1) describes the energetic behavior of the innovator since the controlled investment $r_a(\cdot)$ influences directly on the technology rate $\dot{x}$.

The innovator starting the innovation process at time t_0 from the initial level x_0 of the technology stock $x(t)$ should reach at the commercialization time t_a the technological level x_a, $x_a > x_0$ which is necessary for launching commercialization. In this investment process the innovator is minimizing its expenditures

$$J(t_0, x_0, t_a, x_a, r_a(\cdot), \gamma, \lambda, \sigma) = \int_{t_0}^{t_a} e^{-\lambda s} r_a(s) ds \tag{1.2}$$

$$r_a = r_a(s) = r_a(s, t_0, x_0, t_a, x_a, \gamma, \lambda, \sigma)$$

Here parameter $\lambda > 0$ is a constant rate of discount.

The dynamic optimization problem with dynamics (1.1) and the functional of expenditures (1.2) can be treated in the framework of optimal control theory (see [Pontryagin, Boltyanskii, Gamkrelidze, Mishchenko, 1962], [Arrow, 1985]).

Assume that the problem (iii) is solved. Denote by the symbol $r_a^0 = r_a^0(s)$ the optimal investment intensity, and by the symbol $x^0 = x^0(s)$ the corresponding scenario of the technology growth. Substituting the optimal intensity into the functional (1.2) one can calculate the optimal total investment

$$w(t_0, x_0, t_a, x_a, \gamma, \lambda, \sigma) = \int_{t_0}^{t_a} e^{-\lambda s} r_a^0(s) ds \tag{1.3}$$

Fixing in relation (1.3) parameters t_a, x_a, γ, λ, σ and varying initial positions $(t_0, x_0) = (t, x)$ one can consider the series of optimal result functions (value functions)

$$(t, x) \to w(t, x, t_a, x_a, \gamma, \lambda, \sigma) \tag{1.4}$$

parametrized by variables t_a, x_a, γ, λ, σ. In the problem (ii) of selecting the innovation scenario we will be interested in dependence of the series $w(\cdot)$ (1.4) with respect to the commercialization time t_a.

Let us consider now the problem (ii) of optimizing the commercialization time t_a. Assume that the benefit function $d(\cdot)$ of commercialization of the new technology depends on the difference of the commercialization times of the innovator t_a and the market t_b which gives the bonus sales S_b with respect to the usual amount of sales S_a

$$d = d(t_a, t_b, S_a, S_b, \lambda, \mu) = \int_{t_a}^{+\infty} S_a e^{-(\lambda-\mu)s} ds + \max\{0, \int_{t_a}^{t_b} S_b e^{-(\lambda-\mu)s} ds\} \tag{1.5}$$

Here parameter μ, $0 < \mu < \lambda$ is the rate of the discounted stream of the innovation benefits.

Let us note that the benefit function $d(\cdot)$ (1.5) is a generalization of the stream of benefits (see [Barzel, 1968]).

The sense of the benefit function is quite clear:

if the commercialization time t_a of the innovator is less than the market commercialization time t_b, $t_a < t_b$, then in the period $[t_a, t_b)$ the total amount of sales $S_0 = S_a + S_b$ for the innovator will be larger than the usual amount of sales S_a due to the bonus sales S_b of the early innovation compared to the market; in the period $[t_b, +\infty)$ after the market commercialization time t_b the benefit of the innovator will be measured by the usual amount of sales $S_0 = S_a$;

if the innovator can not overtake the market in its innovation $t_a \geq t_b$, then in the period $[t_a, +\infty)$ the benefit of the innovator is determined by the usual amount of sales $S_0 = S_a$.

Let us introduce the profit function $R(\cdot)$ of the innovation (the present value of the innovation) as the balance of the benefit function $d(\cdot)$ and the optimal investment expenditure $w(\cdot)$

$$R(t, x, t_a, x_a, t_b, S_a, S_b, \gamma, \lambda, \mu, \sigma) = d(t_a, t_b, S_a, S_b, \lambda, \mu) - w(t, x, t_a, x_a, \gamma, \lambda, \sigma) \tag{1.6}$$

The key problem of the innovator is to maximize its profit R in the dynamical investment process. This optimal solution essentially depends on the accurate assessment of the market commercialization time t_b. Identifying dynamically the market commercialization time t_b the innovator can select one of two possible scenarios of optimal innovation which correspond to the profit function $R(\cdot)$.

To model the market technology trajectories $y(\cdot)$ of the exponential growth we use the “heavy” dynamics which describes the inert behavior of the market environment with the large number of innovators

$$\begin{aligned} \dot{y}(t) &= -\sigma y(t) + r_b(t) = -\sigma y(t) + z(t)y(t) \\ \dot{z}(t) &= v(t), \quad |v(t)| \leq v_0 \end{aligned} \tag{1.7}$$

Here parameter $y(t)$ stands for the average market technology stock, parameter $r_b(t)$ denotes the average market investment, and variable $z(t) = r_b(t)/y(t)$ is the market R&D intensity.

The “heavy” market dynamics (1.7) with the small acceleration v_0 describes the exponential growth of the market technology stock $y(t)$ with the “nonintensive” variation $\dot{z}(t)$

(the second derivative $\ddot{y}(t)$) of the market R&D intensity $z(t)$ (the first derivative $\dot{y}(t)$). The market commercialization of the new technology starts at time t_b when the market technology stock $y(\cdot)$ achieves the commercialization level y_b, $0 \leq y(t) \leq y_b$, $0 \leq t \leq t_b$. In the dynamical identification process one can assess the market commercialization time t_b of the technology trajectory $y(\cdot)$ measuring econometric characteristics: the current market technology stock $y(t)$ and the current market technology rate $\dot{y}(t)/y(t)$. In this identification process one can assess sensitivity of the market commercialization time and estimate the reliability of its prediction.

Combining all three components of the model: identification of the market trajectories, scenarios selection and feedback optimization of the investment level, we obtain the dynamic design of the optimal innovation policy. The main feature of this model consists in the feedback interaction of three dynamic processes: identification, scenarios selection, optimization.

Objective I. Optimal Design of the Investment Level

2 Dynamic Optimality Principles and Investment Synthesis

Let us consider the first problem of optimal control design for the investment level. To reach this objective we are dealing with the investment dynamics (1.1) of the innovator and its expenditure functional (1.2). Introducing notations

$$u(t) = r_a^{\gamma}(t), \quad t_0 \leq t \leq t_a, \quad 0 < \gamma < 1 \tag{2.1}$$

we obtain the optimal control problem with the linear dynamics for the growth of the technology stock $x(t)$ depending on the scaled investment level $u(t)$

$$\dot{x}(t) = -\sigma x(t) + u(t) \tag{2.2}$$

and the exponential expenditure functional

$$J(t_0, x_0, t_a, x_a, u(\cdot), \alpha, \lambda, \sigma) = \int_{t_0}^{t_a} e^{-\lambda s} u^{\alpha}(s) ds \tag{2.3}$$

$$\alpha = \frac{1}{\gamma} > 1, \quad u = u(s) = u(s, t_0, x_0, t_a, x_a, \alpha, \lambda, \sigma)$$

The problem is to find the optimal investment level $u^0(\cdot)$ and the corresponding trajectory $x^0(\cdot)$ of the technology stock subject to dynamics (2.2) for minimizing the expenditure functional (2.3).

For convenience let us consider the new variable

$$w(t) = \int_{t_0}^{t} e^{-\lambda s} u^{\alpha}(s) ds \tag{2.4}$$

for the accumulated effective R&D investment and substitute the problem with the integral functional (2.2), (2.3) by the terminal optimal control problem

$$\begin{aligned} \dot{x}(t) &= -\sigma x(t) + u(t) \\ \dot{w}(t) &= e^{-\lambda t} u^{\alpha}(t) \end{aligned} \tag{2.5}$$

with the following boundary conditions

$$\begin{aligned} &x(t_0) = x_0, \quad x(t_a) = x_a, \quad w(t_0) = w_0 \\ &t_a > t_0 \geq 0, \quad x_a > x_0 \geq 0, \quad w_0 \geq 0 \end{aligned} \tag{2.6}$$

For dynamics (2.5) it is necessary to minimize the terminal boundary value of coordinate $w(t)$ at time t_a

$$w(t_a) \longrightarrow \min_{(u(\cdot),x(\cdot),w(\cdot))} \tag{2.7}$$

or equivalently to maximize the terminal boundary value of negative coordinate $-w(t)$ at time t_a

$$-w(t_a) \longrightarrow \max_{(u(\cdot),x(\cdot),w(\cdot))} \tag{2.8}$$

We solve the problem of optimal investment (2.5), (2.8) using the Pontryagin's maximum principle (see [Pontryagin, Boltyanskii, Gamkrelidze, Mischenko, 1962]). We find the optimal investment process $t \to (u^0(t), x^0(t), w^0(t))$ as the planned scenario, starting from the initial position (t_0, x_0, w_0). Then we synthesize the equivalent optimal feedback procedure $u = u(t, x)$ which react in the interactive regime on the current position (t, x) of the technology stock and generate the same optimal trajectory $t \to x^0(t)$. Finally we calculate the optimal accumulated R&D investment $w(\cdot)$ as the function of the problem's parameters t_0, x_0, t_a, x_a, α, λ, σ – the value function.

Introducing prices $\psi_1 = \psi_1(t)$ for the technology stock $x = x(t)$ and $\psi_2 = \psi_2(t)$ for the accumulated effective R&D investment $w = w(t)$ we compile the Hamiltonian of the problem (2.5), (2.8)

$$H(s, x, w, \psi_1, \psi_2) = -\psi_1 \sigma x + \psi_1 u - \psi_2 e^{-\lambda s} u^{\alpha} \tag{2.9}$$

which measures the current flow of utility from all sources.

The maximum value of the utility flow is achieved when the optimal condition takes place

$$\frac{\partial H}{\partial u} = \psi_1 - \alpha \psi_2 e^{-\lambda s} u^{(\alpha-1)} = 0 \tag{2.10}$$

at the optimal investment level

$$u^0 = \left(\frac{e^{\lambda s} \psi_1}{\alpha \psi_2} \right)^{\frac{1}{(\alpha-1)}} \tag{2.11}$$

For prices ψ_1, ψ_2 one can compose the equilibrium dynamics of adjoint equations

$$\begin{aligned} \dot{\psi_1}(s) &= -\frac{\partial H}{\partial x} = \sigma \psi_1(s) \\ \dot{\psi_2}(s) &= -\frac{\partial H}{\partial u} = 0 \end{aligned} \tag{2.12}$$

which balances the increment in flow and the change in price.

The general solution of the adjoint equations (2.12) is given by relations

$$\begin{aligned} \psi_1(s) &= A_1 e^{\sigma s} \\ \psi_2(s) &= A_2 \end{aligned} \tag{2.13}$$

with positive constants $A_1 > 0$, $A_2 > 0$.

Substituting solutions (2.13) for prices ψ_1, ψ_2 into relation (2.11) for the optimal level u^0 we obtain the structure of the optimal plan

$$u^0(s) = K e^{\frac{(\lambda+\sigma)}{(\alpha-1)} s}, \quad K = \left(\frac{A_1}{\alpha A_2} \right)^{\frac{1}{(\alpha-1)}} \tag{2.14}$$

Remark 2.1 *The optimal investment plan $u^0(s)$ (2.14) is the exponential growing function of time s on the time interval $[t_0, t_a]$ with the growth rate $(\lambda+\sigma)/(\alpha-1)$.*

Using the structure of optimal control (2.14) in the Cauchy formula we obtain the general solution of the optimal technology dynamics (2.2)

$$x(t) = x_0 e^{-\sigma(t-t_0)} + \int_{t_0}^{t} e^{-\sigma(t-s)} u(s) ds =$$
$$x_0 e^{-\sigma(t-t_0)} + K \frac{(\alpha-1)}{(\alpha\sigma+\lambda)} (e^{\frac{(\alpha\sigma+\lambda)}{(\alpha-1)}t} - e^{\frac{(\alpha\sigma+\lambda)}{(\alpha-1)}t_0}) e^{-\sigma t} \tag{2.15}$$

The constant K in the Cauchy solution (2.15) can be identified from the boundary conditions $x(t_a) = x_a$ (2.6)

$$K = \frac{(\alpha\sigma+\lambda)}{(\alpha-1)} \frac{(e^{\sigma(t_a-t_0)} x_a - x_0)}{(e^{\frac{(\alpha\sigma+\lambda)}{(\alpha-1)}(t_a-t_0)} - 1)} e^{-\frac{(\lambda+\sigma)}{(\alpha-1)}t_0} \tag{2.16}$$

Combining relations (2.14), (2.16) we obtain the final expression for the optimal plan

$$u^0 = u^0(s, t_0, x_0, t_a, x_a, \alpha, \lambda, \sigma) = \frac{(x_a e^{(t_a-s)\sigma} - x_0 e^{-(s-t_0)\sigma})\rho}{(e^{(t_a-s)\rho} - e^{-(s-t_0)\rho})} \tag{2.17}$$

Here function $\rho = \rho(\alpha, \lambda, \sigma)$ is given by relation

$$\rho = \rho(\alpha, \lambda, \sigma) = \frac{(\alpha\sigma+\lambda)}{(\alpha-1)} \tag{2.18}$$

3 Sensitivity Analysis of Optimal Investment Plan

Let us examine the sensitivity of the optimal plan $u^0(\cdot)$ (2.17) with respect to parameters α, λ, σ.

Proposition 3.1 *For the range of time s*

$$s \in [t_0, (t_0 + t_a)/2] \tag{3.1}$$

the level of the optimal plan $u^0(s)$ (2.17) is decreasing to zero, while the discount parameter λ is growing to infinity, or parameter α is declining to unit.

If time s is located in the second half of the time interval $[t_0, t_a]$

$$s \in ((t_0 + t_a)/2, t_a) \tag{3.2}$$

then the level of the optimal plan $u^0(s)$ (2.17) is first growing and then declining to zero, while the discount parameter λ is growing to infinity, or parameter α is declining to unit.

For time $s \in [(t_0 + t_a)/2, t_a)$ the level of the optimal plan $u^0(s)$ (2.17) is first growing and then decreasing to zero, while the obsolescence parameter σ is growing to infinity. For time $s \in [t_0, (t_0 + t_a)/2)$ there are two alternatives for the level of the optimal plan $u^0(s)$ (2.17) depending on the values of parameters $t_0 < t_a$, $x_0 < x_a$, and $\alpha > 1$, $\lambda > 0$: it can strictly decline to zero, or it can first grow and then decline to zero, while the obsolescence parameter σ grows to infinity.

At the final moment of time

$$s = t_a \tag{3.3}$$

the level of the optimal plan $u^0(t_a)$ (2.17) is growing to infinity, while the discount parameter λ is growing to infinity, or the obsolescence parameter σ is growing to infinity, or parameter α is declining to unit.

Proof. Let us present the optimal plan u^0 (2.17) in the following way

$$
\begin{aligned}
&u^0(s,t_0,x_0,t_a,x_a,\alpha,\lambda,\sigma) = u_1^0(s,t_0,x_0,t_a,x_a,\sigma)u_2^0(s,t_0,t_a,\rho)\\
&u_1^0(s,t_0,x_0,t_a,x_a,\sigma) = (x_a e^{(t_a-s)\sigma} - x_0 e^{-(s-t_0)\sigma})\\
&u_2^0(s,t_0,t_a,\rho) = \frac{\rho}{(e^{(t_a-s)\rho} - e^{-(s-t_0)\rho})}
\end{aligned} \tag{3.4}
$$

Calculating derivatives of optimal plan u^0 with respect to parameters α, λ we derive the following relations

$$
\frac{\partial u^0}{\partial \alpha} = u_1^0 \frac{\partial u_2^0}{\partial \rho}\frac{\partial \rho}{\partial \alpha} \tag{3.5}
$$

$$
\frac{\partial u^0}{\partial \lambda} = u_1^0 \frac{\partial u_2^0}{\partial \rho}\frac{\partial \rho}{\partial \lambda} \tag{3.6}
$$

Let us estimate signs of derivatives (3.5), (3.6). Note first that function u_1^0 is positive

$$
u_1^0 = (x_a e^{(t_a-s)\sigma} - x_0 e^{-(s-t_0)\sigma}) > 0 \tag{3.7}
$$

We indicate now the signs of derivatives of the function ρ (2.18)

$$
\frac{\partial \rho}{\partial \alpha} = -\frac{\sigma}{(\alpha-1)^2} < 0, \quad \frac{\partial \rho}{\partial \lambda} = \frac{1}{(\alpha-1)} > 0, \quad \frac{\partial \rho}{\partial \sigma} = \frac{\alpha}{(\alpha-1)} > 0 \tag{3.8}
$$

Let us estimate derivative $\partial u_2^0/\partial \rho$

$$
\frac{\partial u_2^0}{\partial \rho} = \frac{((1-(t_a-s)\rho)e^{(t_a-s)\rho} - (1+(s-t_0)\rho)e^{-(s-t_0)\rho})}{(e^{(t_a-s)\rho} - e^{-(s-t_0)\rho})^2} \tag{3.9}
$$

The denominator in derivative $\partial u_2^0/\partial \rho$ (3.9) is positive. Let us consider function in the numerator for $\rho \geq 0$

$$
\begin{aligned}
&n(\rho) = (1-(t_a-s)\rho)e^{(t_a-s)\rho} - (1+(s-t_0)\rho)e^{-(s-t_0)\rho}) =\\
&(1+(s-t_0)\rho)e^{(t_a-s)\rho}\left(\frac{(1-(t_a-s)\rho)}{(1+(s-t_0)\rho)} - e^{-(t_a-t_0)\rho}\right)
\end{aligned} \tag{3.10}
$$

To estimate the sign of numerator $n(\rho)$ let us compare derivatives of the hyperbolic function

$$
g(\rho) = \frac{(1-(t_a-s)\rho)}{(1+(s-t_0)\rho)} -
$$

and the exponential function

$$
h(\rho) = e^{-(t_a-t_0)\rho}
$$

We have the following relations for derivatives at point $\rho = 0$

$$
\begin{aligned}
&g' = -\frac{(t_a-t_0)}{(1+(s-t_0)\rho)^2}|_{\rho=0} = -(t_a-t_0)\\
&g^{(2)} = \frac{2(t_a-t_0)(s-t_0)}{(1+(s-t_0)\rho)^3}|_{\rho=0} = 2(t_a-t_0)(s-t_0)\\
&g^{(3)} = -\frac{6(t_a-t_0)(s-t_0)^2}{(1+(s-t_0)\rho)^4}|_{\rho=0} = -6(t_a-t_0)(s-t_0)^2\\
&h' = -(t_a-t_0)e^{-(t_a-t_0)\rho}|_{\rho=0} = -(t_a-t_0)\\
&h^{(2)} = (t_a-t_0)^2 e^{-(t_a-t_0)\rho}|_{\rho=0} = (t_a-t_0)^2\\
&h^{(3)} = -(t_a-t_0)^3 e^{-(t_a-t_0)\rho}|_{\rho=0} = -(t_a-t_0)^3
\end{aligned}
$$

It is clear that the following relations take place

$$g(0) = h(0) = 1, \quad g'(0) = h'(0) = -(t_a - t_0)$$
$$g^{(2)}(0) = 2(t_a - t_0)(s - t_0) < (t_a - t_0)^2 = h^{(2)}(0), \quad t_0 \leq s < (t_0 + t_a)/2$$
$$g^{(3)}(0) = -\frac{3}{2}(t_0 + t_a)^3 < -(t_0 + t_a)^3 = h^{(3)}(0)$$

These relations imply that in a neighborhood $B(0, \varepsilon)$

$$B(0, \varepsilon) = \{\rho \,:\, 0 \leq \rho < \varepsilon\}$$

of the origin functions $g(\rho)$, $h(\rho)$ are connected by inequalities

$$g(\rho) < h(\rho), \quad \rho \in B(0, \varepsilon) \setminus \{0\}, \quad 0 \leq s \leq (t_0 + t_a)/2 \tag{3.11}$$

$$g(\rho) > h(\rho), \quad \rho \in B(0, \varepsilon) \setminus \{0\}, \quad (t_0 + t_a)/2 < s \leq (t_a + t_0) \tag{3.12}$$

Let us prove that the first inequality is valid for all $\rho > 0$

$$g(\rho) < h(\rho), \quad \rho > 0, \quad 0 \leq s \leq (t_0 + t_a)/2 \tag{3.13}$$

Really we have the chain of inequalities for $0 \leq s \leq (t_0 + t_a)/2$

$$g'(\rho) - h'(\rho) = -\frac{(t_a - t_0)}{(1 + (s - t_0)\rho)^2} + (t_a - t_0)e^{-(t_a - t_0)\rho} \leq$$
$$(t_a - t_0)(-\frac{1}{(1 + (s - t_0)\rho)^2} + e^{-2(s - t_0)\rho}) \leq 0 \tag{3.14}$$

Integrating inequality (3.14) on interval $[0, \rho]$, $\rho > 0$ and taking into account that $g(0) = h(0)$ we obtain the necessary inequality (3.13).

Let us prove that for parameters $(t_0 + t_a)/2 < s < t_a$ there exists threshold $\rho_s \geq \varepsilon > 0$ such that starting from it $\rho > \rho_s$ the opposite relation to the second inequality (3.12) takes place

$$g(\rho) < h(\rho), \quad \rho > \rho_s \geq \varepsilon > 0, \quad (t_0 + t_a)/2 < s < t_a \tag{3.15}$$

In the difference

$$\frac{(1 - (t_a - s)\rho)}{(1 + (s - t_0)\rho)} - e^{-(t_a - t_0)\rho}$$

the first hyperbolic term tends to the negative number $-(t_a - s)/(s - t_0) < 0$

$$g(\rho) = \frac{(1 - (t_a - s)\rho)}{(1 + (s - t_0)\rho)} \to -\frac{(t_a - s)}{(s - t_0)} < 0, \quad \rho \to +\infty$$

and the second exponential term tends to zero

$$h(\rho) = e^{-(t_a - t_0)\rho} \to 0, \quad \rho \to +\infty$$

It means that for a fixed parameter s, $(t_0 + t_a)/2 < s < t_a$ there exists a threshold $\rho_s \geq \varepsilon > 0$ starting from which $\rho > \rho_s$ the necessary inequality $g(\rho) < h(\rho)$ takes place.

Let us estimate now derivatives of the optimal plan u^0 (3.4) with respect to the obsolescence parameter σ. It is convenient to calculate them in the logarithmic form

$$\frac{\partial u^0}{\partial \sigma} = \frac{\partial (e^{\ln u^0})}{\partial \sigma} = \frac{\partial (\ln u^0)}{\partial \sigma} u^0 = u^0 (\frac{1}{\rho}\frac{\partial \rho}{\partial \sigma} - \frac{((t_a - s)e^{\rho(t_a - s)} + (s - t_0)e^{-\rho(s - t_0)})}{(e^{\rho(t_a - s)} - e^{-\rho(s - t_0)})}\frac{\partial \rho}{\partial \sigma} +$$
$$\frac{((t_a - s)x_a e^{\sigma(t_a - s)} + (s - t_0)x_0 e^{-\sigma(s - t_0)})}{(x_a e^{\sigma(t_a - s)} - x_0 e^{-\sigma(s - t_0)})}), \quad \frac{\partial \rho}{\partial \sigma} = \frac{\alpha}{(\alpha - 1)} > 0 \tag{3.16}$$

Let us indicate the sign of derivative (3.16) for large values of parameter σ. The first term is a positive, monotonically decreasing function with respect to parameter σ

$$H_1(s,\sigma) = \frac{\alpha}{(\alpha-1)\rho} = \frac{1}{(\sigma+\lambda/\alpha)} \quad \downarrow \quad 0, \quad \sigma \to +\infty \tag{3.17}$$

The second term in derivative (3.16) is a negative, monotonically increasing function

$$H_2(s,\sigma) = -\frac{\alpha}{(\alpha-1)} \frac{((t_a-s)+(s-t_0)e^{-\rho(t_a-t_0)})}{(1-e^{-\rho(t_a-t_0)})} \le -\frac{\alpha}{(\alpha-1)}(t_a-s) \tag{3.18}$$

The third term

$$H_3(s,\sigma) = \frac{((t_a-s)x_a+(s-t_0)x_0e^{-\sigma(t_a-t_0)})}{(x_a-x_0e^{-\sigma(t_a-t_0)})}$$

in relation (3.16) is the bounded function which monotonically decreases with respect to parameter σ

$$\frac{(t_a-t_0)x_a}{(x_a-x_0)} \ge \frac{((t_a-s)x_a+(s-t_0)x_0)}{(x_a-x_0)} \ge H_3(s,\sigma) \ge (t_a-s) \ge 0 \tag{3.19}$$

Combining the first and third terms (3.17), (3.19) together we obtain the following estimate: for arbitrary $\varepsilon > 0$ there exists threshold σ_ε starting from which $\sigma > \sigma_\varepsilon$ the chain of inequalities

$$(t_a-s) < H_1(s,\sigma)+H_3(s,\sigma) < (t_a-s)+\varepsilon \tag{3.20}$$

is valid.

Fixing parameter $\varepsilon = \varepsilon(s)$, $0 < \varepsilon < (t_a-s)/(\alpha-1)$ and combining inequalities (3.18), (3.20) we obtain the necessary estimate

$$H_1(s,\sigma)+H_2(s,\sigma)+H_3(s,\sigma) < -\frac{\alpha}{(\alpha-1)}(t_a-s)+(t_a-s)+\varepsilon < 0 \tag{3.21}$$

$$t_0 \le s < t_a, \quad \sigma > \sigma_{\varepsilon(s)}$$

which implies decrease of the optimal investment level $u^0(s)$ (2.17) to zero for parameters σ growing to infinity.

Let us examine the behavior of derivative (3.16) for small parameters σ, ρ. Combining the first and second terms together we obtain the indefinite ratio

$$H_1+H_2 = \frac{\alpha}{(\alpha-1)} \frac{((1-\rho(t_a-s))-(1+\rho(s-t_0))e^{-\rho(t_a-t_0)})}{\rho(1-e^{-\rho(t_a-t_0)})} \tag{3.22}$$

when parameter ρ tends to zero.

Calculating the first and second derivatives of the numerator

$$N(\rho) = (1-\rho(t_a-s))-(1+\rho(s-t_0))e^{-\rho(t_a-t_0)}, \quad N(0)=0$$

and the denominator

$$D(\rho) = \rho(1-e^{-\rho(t_a-t_0)}), \quad D(0)=0$$

at point $\rho = 0$ we obtaining the following relations

$$N'(\rho) = -(t_a-s)-(s-t_0)e^{-\rho(t_a-t_0)}+(t_a-t_0)(1+\rho(s-t_0))e^{-\rho(t_a-t_0)}$$
$$N'(0)=0$$
$$N''(\rho) = (t_a-t_0)(s-t_0)+(t_a-t_0)e^{-\rho(t_a-t_0)}((s-t_0)-(t_a-t_0)(1+\rho(s-t_0)))$$
$$N''(0) = (t_a-t_0)((s-t_0)-(t_a-s))$$
$$D'(\rho) = (1-e^{-\rho(t_a-t_0)})+\rho(t_a-t_0)e^{-\rho(t_a-t_0)}, \quad D'(0)=0$$
$$D''(\rho) = 2(t_a-t_0)e^{-\rho(t_a-t_0)}-(t_a-t_0)^2\rho e^{-\rho(t_a-t_0)}, \quad D''(0)=2(t_a-t_0)$$

According to the L'Hospital rule the indefinite ratio (3.22) has the finite value

$$(H_1 + H_2)|_{\rho=0} = \frac{N''(0)}{D''(0)} = \frac{\alpha}{(\alpha-1)}(s - \frac{(t_0+t_a)}{2}) \tag{3.23}$$

The third term H_3 in derivative (3.16) for $\sigma = 0$ is strictly positive

$$H_3|_{\sigma=0} = \frac{((t_a - s)x_a + (s-t_0)x_0)}{(x_a - x_0)} > 0 \tag{3.24}$$

It is clear that for small enough parameters σ, ρ derivative (3.16) conserves the sign of the sum

$$(H_1 + H_2)|_{\rho=0} + H_3|_{\sigma=0} = \frac{\alpha}{(\alpha-1)}(s - \frac{(t_0+t_a)}{2}) + \frac{((t_a - s)x_a + (s-t_0)x_0)}{(x_a - x_0)} \tag{3.25}$$

This sign is definitely positive if $s \geq (t_0+t_a)/2$ and hence for such times s the optimal investment level $u^0(s)$ (2.17) is first growing with respect to the obsolescence parameter σ. For times $s \in [t_0, (t_0+t_a)/2)$ depending on parameters $t_a > t_0$, $x_a > x_0$, and $\alpha > 1$, $\lambda > 0$ the sign of relation (3.25) can be positive or negative and imply the initial growth or decrease of the optimal investment level $u^0(s)$ (2.17).

Finally we consider the case $s = t_a$. The derivatives of the optimal investment level u^0 (2.17) with respect to parameters α, λ, σ can be presented in the following form

$$\frac{\partial u^0}{\partial \alpha} = u^0(D_1 + D_2)\frac{\partial \rho}{\partial \alpha}, \quad \frac{\partial \rho}{\partial \alpha} = -\frac{(\lambda+\sigma)}{(\alpha-1)^2}$$

$$\frac{\partial u^0}{\partial \lambda} = u^0(D_1 + D_2)\frac{\partial \rho}{\partial \lambda}, \quad \frac{\partial \rho}{\partial \lambda} = \frac{1}{(\alpha-1)}$$

$$\frac{\partial u^0}{\partial \sigma} = u^0((D_1 + D_2)\frac{\partial \rho}{\partial \sigma} + D_3), \quad \frac{\partial \rho}{\partial \sigma} = \frac{\alpha}{(\alpha-1)}$$

$$D_1 = \frac{1}{\rho}$$

$$D_2 = \frac{((t_a - s) + (s-t_0)e^{-\rho(t_a-t_0)})}{(1 - e^{-\rho(t_a-t_0)})}|_{s=t_a} = \frac{(s-t_0)e^{-\rho(t_a-t_0)}}{(1 - e^{-\rho(t_a-t_0)})}$$

$$D_3 = \frac{((t_a - s)x_a + (s-t_0)x_0e^{-\sigma(t_a-t_0)})}{(x_a - x_0e^{-\sigma(t_a-t_0)})}|_{s=t_a} = \frac{(s-t_0)x_0e^{-\sigma(t_a-t_0)}}{(x_a - x_0e^{-\sigma(t_a-t_0)})}$$

For the sum $D_1 + D_2$ we have the following relation

$$(D_1 + D_2)|_{s=t_a} = \frac{(1 - (1 + \rho(t_a - t_0))e^{-\rho(t_a-t_0)})}{\rho(1 - e^{-\rho(t_a-t_0)})}$$

It is clear that for the numerator the following inequality takes place

$$g(\rho) = \frac{1}{(1 + (t_a - t_0)\rho)} > e^{-(t_a-t_0)\rho} = h(\rho), \quad \rho > 0$$

Hence both terms $D_1 + D_2$ and D_3 are strictly positive at the final moment of time $s = t_a$ and the level of the optimal plan $u^0(t_a)$ (2.17) is growing to infinity, while the discount parameter λ is growing to infinity, or the obsolescence parameter σ is growing to infinity, or parameter α is declining to unit. □

Remark 3.1 *Proposition 3.1 means that the optimal investment level $u^0(s)$ (2.17) asymptotically has the impulse character: for the discount parameter $\lambda > 0$, or the obsolescence parameter $\sigma > 0$ tending to infinity, or the delay parameter $\alpha > 1$ tending to unit, the optimal investment level $u^0(s)$ (2.17) tends to zero for times $t_0 \leq s < t_a$ and it is impulsing to infinity for the final time $s = t_a$.*

4 Optimal Technological Trajectories

In this section we analyze properties of optimal technological trajectories. Substituting the optimal control plan $u^0(\cdot)$ (2.17) into the Cauchy formula (2.15) for technological trajectories $x(\cdot)$ we obtain the optimal technological trajectory $x^0(\cdot)$

$$x^0(s) = e^{-\sigma(s-t_0)} \left(x_0 + \frac{(e^{\sigma(t_a-t_0)}x_a - x_0)(e^{\rho(s-t_0)} - 1)}{(e^{\rho(t_a-t_0)} - 1)} \right) \tag{4.1}$$

Let us indicate properties of the optimal technological trajectory $x^0(\cdot)$ (4.1). We begin with indication of boundaries for its values.

Proposition 4.1 *The values of the optimal technological trajectory $x^0(\cdot)$ (4.1) are restricted by boundaries*

$$0 \leq x^0(s) \leq x_a, \quad t_0 \leq s \leq t_a \tag{4.2}$$

Proof. First of all let us note that since $x_a > x_0 \geq 0$, $t_a \geq s \geq t_0$ then from formula (4.1) the first inequality $x^0(s) \geq 0$ for $t_0 \leq s \leq t_a$ obviously follows.

Let us present optimal technological trajectory $x^0(\cdot)$ in the following form

$$\begin{aligned} x^0(s) &= ae^{(\rho-\sigma)(s-t_0)} + (x_0 - a)e^{-\sigma(s-t_0)} \\ a &= \frac{(e^{\sigma(t_a-t_0)}x_a - x_0)}{(e^{\rho(t_a-t_0)} - 1)} \end{aligned} \tag{4.3}$$

Calculating the first and second derivatives of function $x^0(s)$

$$\begin{aligned} \frac{dx^0(s)}{ds} &= a(\rho - \sigma)e^{(\rho-\sigma)(s-t_0)} - (x_0 - a)\sigma e^{-\sigma(s-t_0)} \\ \frac{d^2x^0(s)}{ds^2} &= a(\rho - \sigma)^2 e^{(\rho-\sigma)(s-t_0)} + (x_0 - a)\sigma^2 e^{-\sigma(s-t_0)} \end{aligned}$$

we obtain the following conclusions:

if $x_0 - a < 0$ then the first derivative $dx^0(s)/ds$ is positive and the optimal technological trajectory $x^0(s)$ is monotonically growing from the initial stage $x^0(t_0) = x_0$ till the commercialization stage $x^0(t_a) = x_a$;

in the opposite case $x_0 - a \geq 0$ the second derivative $d^2x^0(s)/ds^2$ is positive and starting from some time t_1, $t_0 \leq t_1 < t_a$ the first derivative $dx^0(s)/ds$ becomes positive and the convex trajectory $x^0(s)$ is monotonically growing from the level $x^0(t_1)$ till the final stage $x^0(t_a) = x_a$.

These conclusions imply the necessary second inequality $x^0(s) \leq x_a$, $t_0 \leq s \leq t_a$. □

We formulate now the monotonicity condition for the optimal technological trajectories $x^0(\cdot)$.

Proposition 4.2 *The monotonicity condition with respect to commercialization time t_a is valid for the optimal technological trajectories $x^0(\cdot)$*

$$x^0(s, t'_a) > x^0(s, t''_a), \quad t'_a < t''_a, \quad t_0 < s \leq \min\{t'_a, t''_a\} \tag{4.4}$$

Monotonicity condition (4.4) means that optimal technological trajectories for different commercialization times t_a don't intersect each other and thus form the field of characteristics.

Proof. Let us estimate derivative of optimal technological trajectories with respect to commercialization time t_a

$$\frac{\partial x^0}{\partial t_a} = \kappa(s)\frac{(\sigma x_a e^{\sigma(t_a-t_0)}(e^{\rho(t_a-t_0)}-1) - \rho e^{\rho(t_a-t_0)}(e^{\sigma(t_a-t_0)}x_a - x_0))}{(e^{\rho(t_a-t_0)}-1)^2} \leq$$

$$\leq x_a\kappa(s)\frac{(\sigma(e^{\rho\xi}-1) - \rho e^{(\rho-\sigma)\xi}(e^{\sigma\xi}-1))e^{\sigma\xi}}{(e^{\rho\xi}-1)^2}$$

$$\kappa(s) = e^{-\sigma(s-t_0)}(e^{\rho(s-t_0)}-1), \quad \xi = t_a - t_0$$

Consider the multiplier in the numerator

$$f(\xi) = -(\rho-\sigma)e^{\rho\xi} + \rho e^{(\rho-\sigma)\xi} - \sigma$$

which determines the sign of derivative $\partial x^0/\partial t_a$. Its derivative is expressed by formula

$$f'(\xi) = -\rho(\rho-\sigma)(e^{\rho\xi} - e^{(\rho-\sigma)\xi})$$

We obtain the following relations $f(0) = 0$, $f'(0) = 0$, and $f'(\xi) < 0$ for $\xi > 0$, which imply the negative values of numerator $f(\xi) < 0$ for $\xi > 0$, and consequently the negative sign of derivative $\partial x^0/\partial t_a < 0$. The last inequality provides monotonicity condition (4.4). □

We give now the estimates of technology rates $\dot{x}^0(\cdot)$.

Proposition 4.3 *At the commercialization time t_a the rate $\dot{x}^0(t_a)$ of the technological trajectory $x^0(\cdot)$ is positive.*

At the initial time t_0 the rate $\dot{x}^0(t_0)$ of the technological trajectory $x^0(\cdot)$ could be positive and negative. If $x_0 = 0$ or $\sigma = 0$ then the initial rate $\dot{x}^0(t_0)$ is positive. For the fixed positive initial stage $x_0 > 0$ and final stage $x_a > x_0$ the rate $\dot{x}^0(t_0)$ is positive for small innovation times $(t_a - t_0)$ but there exists a threshold $t_2 > t_0$ starting from which $t_a > t_2$ the rate $\dot{x}^0(t_0)$ is negative.

Proof. From the proof of Proposition 4.1 it is clear that the technology rate $\dot{x}^0(s)$ is strictly positive starting from time t_1, $t_1 \leq s < t_a$ and consequently is strictly positive at time t_a.

Let us calculate the first derivative $\dot{x}^0(\cdot)$ at the initial time t_0

$$\dot{x}^0(t_0) = a(\rho-\sigma) - (x_0 - a)\sigma = a\rho - x_0\sigma = \frac{(e^{\sigma(t_a-t_0)}x_a - x_0)\rho}{(e^{\rho(t_a-t_0)}-1)} - x_0\sigma \qquad (4.5)$$

It is clear that if $x_0 = 0$ or $\sigma = 0$ then the initial technology rate is positive

$$\dot{x}^0(t_0) = a\rho > 0$$

Transforming formula (4.5) we obtain the following relation

$$\dot{x}^0(t_0) = \frac{x_a((\rho-\sigma) + e^{\rho(t_a-t_0)})}{(e^{\rho(t_a-t_0)}-1)}\left(\frac{\rho e^{\sigma(t_a-t_0)}}{((\rho-\sigma)+\sigma e^{\rho(t_a-t_0)})} - \frac{x_0}{x_a}\right) \qquad (4.6)$$

Let us note that the first term

$$F(t_a) = \frac{\rho e^{\sigma(t_a-t_0)}}{((\rho-\sigma)+\sigma e^{\rho(t_a-t_0)})}$$

in (4.6) has the following properties. It equals to one $F(t_a) = 1$ when $t_a = t_0$ and it tends to zero $F(t_a) \to 0$ when the final time t_a tends to infinity $t_a \to +\infty$. Taking into account that the second term in (4.6) is less than one $x_0/x_a < 1$ we come to the conclusion that the initial rate $\dot{x}^0(t_0)$ changes the sign from positive to negative when the final time t_a varies from the initial time t_0 to infinity. □

Let us indicate the range of parameters α, λ, σ for which the optimal technological trajectories are convex.

Proposition 4.4 *If parameters α, λ, σ satisfy inequalities*

$$1 < \alpha \leq 2 + \frac{\lambda}{\sigma} \tag{4.7}$$

then the optimal technological trajectory $x^0(\cdot)$ is convex.

Proof. Let us calculate the second derivative of the optimal technological trajectory $x^0(\cdot)$

$$\ddot{x}^0(s) = -\sigma\dot{x}^0(s) + \dot{u}^0(s) = -\sigma(-\sigma + u^0(s)) + \dot{u}^0(s) = \sigma^2 x^0(s) - \sigma u^0(s) + \dot{u}^0(s)$$

Under condition (4.7) we have the chain of inequalities

$$-\sigma u^0(s) + \dot{u}^0(s) = -\sigma K e^{\frac{(\lambda+\sigma)}{(\alpha-1)}s} + \frac{(\lambda+\sigma)}{(\alpha-1)} K e^{\frac{(\lambda+\sigma)}{(\alpha-1)}s} =$$
$$K e^{\frac{(\lambda+\sigma)}{(\alpha-1)}s}\left(-\sigma + \frac{(\lambda+\sigma)}{(\alpha-1)}\right) = \frac{(\lambda - (\alpha-2)\sigma)}{(\alpha-1)} K e^{\frac{(\lambda+\sigma)}{(\alpha-1)}s} \geq 0$$

It is clear that if $-\sigma u^0(s) + \dot{u}^0(s) \geq 0$ then the second derivative is nonnegative $\ddot{x}^0(s) \geq 0$ since $x^0(s) \geq 0$, $s \in [t_0, t_a]$ and the optimal technological trajectory $x^0(\cdot)$ is convex. □

Remark 4.1 *Propositions 4.1-4.4 indicate the range of the model parameters for which the optimal technological scenario $x^0(\cdot)$ is a convex trajectory with the growth properties.*

5 The Value Function and Optimal Feedback for Technological Dynamics

We will pass now to analysis of the value function $(t, x) \to w(t, x)$, $(t, x) = (t_0, x_0)$

$$w = w(t, x, t_a, x_a, \alpha, \lambda, \sigma) = \int_t^{t_a} e^{-\lambda s}(u^0(s))^\alpha ds =$$
$$K^\alpha(t, x, t_a, x_a, \alpha, \lambda, \sigma)\int_t^{t_a} e^{-\lambda s} e^{\frac{\alpha(\lambda+\sigma)}{(\alpha-1)}s} ds =$$
$$\left(\frac{(\alpha\sigma+\lambda)}{(\alpha-1)}\right)^{(\alpha-1)} \frac{e^{-\lambda t}(e^{\sigma(t_a-t)}x_a - x)^\alpha}{(e^{\frac{(\alpha\sigma+\lambda)}{(\alpha-1)}(t_a-t)} - 1)^{(\alpha-1)}} = \rho^{(\alpha-1)}\frac{e^{-\lambda t}(e^{\sigma(t_a-t)}x_a - x)^\alpha}{(e^{\rho(t_a-t)} - 1)^{(\alpha-1)}} =$$
$$\rho^{(\alpha-1)}\frac{e^{-\lambda t_a}(x_a - x e^{-\sigma(t_a-t)})^\alpha}{(1 - e^{-\rho(t_a-t)})^{(\alpha-1)}}, \quad \rho = \frac{(\alpha\sigma+\lambda)}{(\alpha-1)} \tag{5.1}$$

Let us indicate properties of the value function $w(\cdot)$ with respect to the optimization parameter – the commercialization time t_a.

Proposition 5.1 *For the fixed parameters α, λ, σ, initial condition (t, x) and the commercialization technology level x_a, $x_a > x$ the value function $w(\cdot)$ (5.1) has the following properties as function $t_a \to w(t_a)$, $w(t_a) = w(t, x, t_a, x_a, \alpha, \lambda, \sigma)$ of the commercialization time t_a:*

it grows to infinity when the commercialization time t_a tends to the initial time t

$$w(t_a) \to +\infty, \quad t_a \downarrow t \tag{5.2}$$

it decreases to zero with the exponential rate $-\lambda$ when the commercialization time t_a tends to infinity

$$w(t_a) \to 0, \quad t_a \to +\infty, \quad \lim_{t_a \to +\infty} e^{\lambda t_a} w(t_a) = w_a < +\infty \tag{5.3}$$

Proof. Let us note that the numerator in relation (5.1) is strictly separated from zero $(e^{\sigma(t_a - t)} x_a - x) \geq x_a - x > 0$ since $x_a > x$. Hence denominator $(e^{\rho(t_a - t)} - 1)$ tends to zero when $t_a \to t$ and all other multipliers in relation (5.1) are strictly separated from zero, then the value function $w(t_a)$ grows to infinity while the commercialization time t_a tends to the initial time t.

Let us consider in relation (5.1) presentation of the value function $w(\cdot)$ in the form

$$w = w(t, x, t_a, x_a, \alpha, \lambda, \sigma) = \rho^{(\alpha-1)} \frac{(x_a - x e^{-\sigma(t_a - t)})^{\alpha}}{(1 - e^{-\rho(t_a - t)})^{(\alpha-1)}} e^{-\lambda t_a} \tag{5.4}$$

which is obtained by carrying out exponentials $e^{\sigma(t_a - t)}$, $e^{\rho(t_a - t)}$ from the brackets. Multiplying relation (5.4) on the term $e^{\lambda t_a}$ and passing to the limit while the commercialization time t_a goes to infinity we obtain the necessary asymptotics

$$\lim_{t_a \to \infty} e^{\lambda t_a} w(t_a) = \lim_{t_a \to \infty} \rho^{(\alpha-1)} \frac{(x_a - x e^{-\sigma(t_a - t)})^{\alpha}}{(1 - e^{-\rho(t_a - t)})^{(\alpha-1)}} =$$
$$\rho^{(\alpha-1)} x_a^{\alpha} = w_a, \quad 0 < w_a < \infty$$

The last relations mean that the value function $w(t_a)$ decreases to zero with the exponential rate $-\lambda$ while the commercialization time t_a grows to infinity. □

Let us derive the optimal feedback for the R&D investment. To this end it is necessary to express the adjoint variable $\psi_1 = -\partial w / \partial x$ through the current position (t, x)

$$\psi_1 = -\partial w / \partial x = \alpha e^{-\lambda t} \rho^{(\alpha-1)} \frac{(e^{\sigma(t_a - t)} x_a - x)^{(\alpha-1)}}{(e^{\rho(t_a - t)} - 1)^{(\alpha-1)}} \tag{5.5}$$

Substituting expressions for price ψ_1 (5.5) and price $\psi_2 = 1$ to relation (2.11) of the structure of R&D investment we obtain the optimal investment feedback

$$u^0 = u^0(t, x, t_a, x_a, \alpha, \lambda, \sigma) = \rho \frac{(e^{\sigma(t_a - t)} x_a - x)}{(e^{\rho(t_a - t)} - 1)} \tag{5.6}$$

and the optimal feedback dynamics for the technology stock

$$\dot{x} = -\sigma x + \rho \frac{(e^{\sigma(t_a - t)} x_a - x)}{(e^{\rho(t_a - t)} - 1)} \tag{5.7}$$

Remark 5.1 *The optimal investment feedback has the quite clear sense: if the current technology stock $x = x(t)$ does not reach yet the commercialization level x_a, $x < x_a$, then the optimal R&D investment level u^0 increases proportionally to the difference $(e^{\sigma(t_a - t)} x_a - x)$ with the intensification coefficient $\rho / (e^{\rho(t_a - t)} - 1)$. This coefficient rapidly increases when time t approaches the commercialization time t_a and enforces the innovator to reach the commercialization technology level $x(t) \uparrow x_a$ with the optimal expenditure.*

Objective II. Selection of Optimal Scenario and Commercialization Time

6 The Profit Function of Innovation

Let us introduce the profit function of innovation and examine its properties. It is reasonable to use usual structure of profit from innovation as a balance between benefit from commercialization of new technologies and expenditure for creating new technologies. The benefit from commercialization of a new technology can be expressed by the amount of sales of goods in which this technology is embedded (see [Barzel, 1968]). The amount of sales S_0 with the commercialized technology may have different levels $S_0 = S_a$ or $S_0 = (S_a + S_b)$, $S_b > 0$ and depend on the difference between the commercialization time of the innovator t_a and the expected commercialization time t_b on the market. We define this effect of different levels of sales by the following construction of the benefit function

$$d = d(t_a, t_b, S_a, S_b, \lambda, \mu) = \int_{t_a}^{+\infty} S_0 e^{-(\lambda-\mu)s} ds = \tag{6.1}$$
$$\int_{t_a}^{+\infty} S_a e^{-(\lambda-\mu)s} ds + \max\{0, \int_{t_a}^{t_b} S_b e^{-(\lambda-\mu)s} ds\}$$
$$S_0 = S_0(s, t_a, t_b) = \begin{cases} S_a + S_b & \text{if } t_a \le s < t_b \\ S_a & \text{if } s \ge t_b \end{cases}$$

Parameter μ, $0 < \mu < \lambda$ in relation (6.1) stands for the rate of the discounted stream of the innovation benefits.

The structure (6.1) of the benefit function $d(\cdot)$ means that the earlier commercialization time t_a, $t_a < t_b$ gives the additional bonus sales S_b, $S_b > 0$ in comparison with the usual amount of sales S_a: in the period $[t_a, t_b)$ the total amount of sales $S_0 = S_a + S_b$ for the innovator is larger than the usual amount of sales $S_0 = S_a$ in the period $[t_b, +\infty)$ after the market commercialization time t_b. For the later commercialization time t_a, $t_a \ge t_b$ the innovator does not receive bonus sales and in the period $[t_a, +\infty)$ its benefit is determined by the usual amount of sales $S_0 = S_a$.

Calculating integrals in relation (6.1) we obtain the following form of the benefit function

$$d(t_a, t_b, S_a, S_b, \lambda, \mu) = \frac{1}{(\lambda-\mu)} S_a e^{-(\lambda-\mu)t_a} + \max\{0, \frac{1}{(\lambda-\mu)} S_b (e^{-(\lambda-\mu)t_a} - e^{-(\lambda-\mu)t_b})\} \tag{6.2}$$

Remark 6.1 *The benefit $d = d(t_a, t_b, S_a, S_b, \lambda, \mu)$ (6.2) is a convex and monotonically decreasing function with respect to the commercialization time t_a, $t_a > t_0$.*

Proof. The first term in relation (6.2) for the benefit function $d(\cdot)$ is an exponential function with the negative rate $-(\lambda - \mu) < 0$ and hence it is convex and monotonically decreasing.

The second term is a convex function since it is a function of the maximum type for two convex functions: the zero constant and the exponential function with the negative rate $-(\lambda - \mu) < 0$. It is clear that this maximum type function is a monotone nonincreasing one.

The sum of two terms with such convex and monotone properties is a convex and monotonically decreasing function. □

Let us pass to the second component – the expenditure for creating new technology or the cost function. It is very natural to consider the optimal integrated R&D investment $w(\cdot)$ in this role

$$w = w(t, x, t_a, x_a, \alpha, \lambda, \sigma) = \int_t^{t_a} e^{-\lambda s} r^0(s) ds =$$
$$\int_t^{t_a} e^{-\lambda s} (u^0(s))^\alpha ds = \rho^{(\alpha-1)} \frac{(x_a - x e^{-\sigma(t_a-t)})^\alpha}{(1 - e^{-\rho(t_a-t)})^{(\alpha-1)}} e^{-\lambda t_a} \tag{6.3}$$

We define the profit function $R(\cdot)$ of the innovation (the present value of the innovation) as the balance of the benefit function $d(\cdot)$ (6.1) and the cost function (the optimal investment expenditure) $w(\cdot)$ (6.3)

$$R(t, x, t_a, x_a, t_b, S_a, S_b, \alpha, \lambda, \mu, \sigma) = d(t_a, t_b, S_a, S_b, \lambda, \mu) - w(t, x, t_a, x_a, \alpha, \lambda, \sigma) =$$
$$\frac{1}{(\lambda-\mu)} S_a e^{-(\lambda-\mu)t_a} + \max\{0, \frac{1}{(\lambda-\mu)} S_b (e^{-(\lambda-\mu)t_a} - e^{-(\lambda-\mu)t_b})\} -$$
$$\rho^{(\alpha-1)} \frac{(x_a - x e^{-\sigma(t_a-t)})^\alpha}{(1 - e^{-\rho(t_a-t)})^{(\alpha-1)}} e^{-\lambda t_a}, \quad \rho = \frac{(\alpha\sigma + \lambda)}{(\alpha - 1)} \tag{6.4}$$

Let us examine properties of the profit function $R(\cdot)$ (6.4).

Proposition 6.1 *The profit function $t_a \to R(t_a)$ (6.4) tends to minus infinity $R(t_a) \to -\infty$ when the commercialization time t_a is close to the initial time t, $t_a \to t$. Then it increases and becomes positive $R(t_a) > 0$ while time t_a grows. For the large commercialization times $t_a \to +\infty$ the profit function declines to zero $R(t_a) \downarrow 0$.*

There exists an optimal commercialization time $t^m > t$ at which the profit function $R(t_a)$ reaches its positive maximum value $R^m = R(t^m) \geq R(t_a)$, $R^m > 0$ on the interval $t_a > t$.

Proof. The benefit component $d(t_a)$ (6.2) has finite values when the commercialization time t_a is close to the initial time t. The cost function $w(t_a)$ according to Proposition 5.1 tends to plus infinity while time t_a tends to the initial time t. Hence the profit function $R(t_a) = d(t_a) - w(t_a)$ has negative values which tend to minus infinity when the commercialization time t_a is close to the initial time t, $t_a \to t$.

Let us prove that starting from some time $t^p > t_0$ the profit function $R(t_a)$, $t_a > t^p$ is positive. Really let us fix time t^p for which the following inequalities take place

$$1 - e^{(\lambda-\mu)(t_a-t_b)} < 0, \quad 1 - e^{-\rho(t_a-t)} > \rho, \quad \frac{S_a}{(\lambda-\mu)} e^{\mu t_a} - x_a^\alpha > 0, \quad t_a > t^p$$

Basing on these inequalities we can derive the necessary conclusion on the strict positiveness of the profit function $R(t_a)$ (6.4) for $t_a > t^p$

$$R(t_a) = (\frac{S_a}{(\lambda-\mu)} e^{\mu t_a} + \max\{0, \frac{S_b}{(\lambda-\mu)} e^{\mu t_a} (1 - e^{(\lambda-\mu)(t_a-t_b)})\} -$$
$$\rho^{(\alpha-1)} \frac{(x_a - x e^{-\sigma(t_a-t)})^\alpha}{(1 - e^{-\rho(t_a-t)})^{(\alpha-1)}}) e^{-\lambda t_a} \geq$$
$$\left(\frac{S_a}{(\lambda-\mu)} e^{\mu t_a} - \rho^{(\alpha-1)} \frac{(x_a - x e^{-\sigma(t_a-t)})^\alpha}{(1 - e^{-\rho(t_a-t)})^{(\alpha-1)}} \right) e^{-\lambda t_a} >$$
$$\left(\frac{S_a}{(\lambda-\mu)} e^{\mu t_a} - x_a^\alpha \right) e^{-\lambda t_a} > 0, \quad t_a > t^p$$

Let us note that when the commercialization time t_a is growing to infinity then all terms in relation (6.4) for the profit function $R(t_a)$ tends to zero and hence the profit function $R(t_a)$ starting from some time t^d is strictly positive and declines to zero from above $R(t_a) \downarrow 0$ when $t_a \to +\infty$, $t_a > t^d$.

Finally we note that the profit function $R(t_a)$ (6.4) is continuous on the interval $t_a \in (t, +\infty)$. According to the indicated above properties for arbitrary $\varepsilon > 0$ there exists a closed interval $[t^l, t^r]$, $t < t^l < t^r < +\infty$ such that outside it the profit function has the following qualitative behavior: it is negative $R(t_a) < 0$ for small times $t_a \in (t, t^l)$ and its values are small and positive $0 < R(t_a) < \varepsilon$ for large times $t_a \in (t^r, +\infty)$. It is clear that the profit function $R(t_a)$ reaches its positive maximum $R^m = R(t^m) \geq R(t_a)$, $R^m > \varepsilon > 0$ at the optimal commercialization time t^m on the segment $[t^l, t^r]$. $\square$

Remark 6.2 *Proposition 6.1 means that for the small commercialization time t_a the cost of innovation is too high and the profit function has the negative values. For the large commercialization time t_a the profit function is positive but its values tend to zero when the commercialization time t_a tend to infinity.*

Let us pass to the problem of finding the maximum profit and selecting the optimal scenario. We present the profit $R(t_a)$ (6.4) as a function of the maximum type

$$R(t_a) = \max\{\frac{S_a}{(\lambda - \mu)} e^{-(\lambda-\mu)t_a} - w(t_a), \frac{(S_a + S_b)}{(\lambda - \mu)} e^{-(\lambda-\mu)t_a} - w(t_a) - r(t_b)\} \tag{6.5}$$

Here

$$r(t_b) = \frac{S_b}{(\lambda - \mu)} e^{-(\lambda-\mu)t_b} \tag{6.6}$$

Let us introduce the following notations for components of the profit function

$$RC(t_a, S) = \frac{S}{(\lambda - \mu)} e^{-(\lambda-\mu)t_a} - w(t_a) \tag{6.7}$$

with different levels of sales $S = S_1 = S_a$, $S = S_2 = (S_a + S_b)$, $0 < S_1 < S_2$. The maximum values and corresponding optimal times we denote by symbols

$$RC_i^m = \max_{t_a > t} RC(t_a, S_i) = RC(t_i^m, S_i), \quad t_i^m \in (t, +\infty), \quad i = 1, 2 \tag{6.8}$$

One can consider optimal times t_1^m, t_2^m and maximum profits RC_1^m, $(RC_2^m - r(t_b))$ which correspond to different levels of sales S_1, S_2 with the commercialized technology as two different scenarios Σ_1, Σ_2 of the technology development.

Remark 6.3 *The optimal times t_i^m for reaching the maximum values RC_i^m, $i = 1, 2$ (6.8) may be not unique. In this case it is necessary to consider the sets of all optimal times $T_i^m = \{t_i^m\}$ and introduce selectors of optimal times $t_i^m \in T_i^m$, $i = 1, 2$. For definiteness one can operate with selectors which have the smallest values.*

Let us indicate properties of optimal constructions: optimal values RC_i^m, optimal times t_i^m, $i = 1, 2$, for the profit function $R(\cdot)$ (6.5).

Proposition 6.2 *1) The optimal value R^m of the profit function $R(t_a)$ (6.4) attained at the optimal commercialization time t^m can be presented as maximum of optimal profits*

RC_1^m, $(RC_2^m - r(t_b))$ for two possible scenarios of technology development with optimal commercialization times t_1^m, t_2^m

$$R^m = \max_{t_a > t} R(t_a) = R(t^m) =$$
$$\max\{RC_1^m, RC_2^m - r(t_b)\} = \max\{RC(t_1^m, S_1), RC(t_2^m, S_2) - r(t_b)\} \qquad (6.9)$$

2) The optimal commercialization times t_1^m, t_2^m of two investment scenarios Σ_1, Σ_2 with different levels of sales S_1, S_2 are connected by inequalities

$$t < t_2^m < t_1^m \qquad (6.10)$$

Due to this fact the scenario Σ_2 with possible larger sales $S_2 > S_1$ and smaller commercialization time $t_2^m < t_1^m$ can be called "fast innovation" in comparison with the scenario Σ_1 of "slow innovation" with the normal level of sales S_1 and larger commercialization time t_1^m.

3) Components $RC(t_a, S_i)$, $i = 1, 2$ of the profit function $R(t_a)$ are connected by inequalities

$$\begin{array}{ll} RC(t_a, S_2) - r(t_b) \geq RC(t_a, S_1), & t < t_a \leq t_b \\ RC(t_a, S_2) - r(t_b) < RC(t_a, S_1), & t_a > t_b \end{array} \qquad (6.11)$$

Proof. Let us note first that components $RC(t_a, S_i)$, $i = 1, 2$ have the same properties as the profit function $R(t_a)$ (6.4): (i) they are continuous on the interval $(t, +\infty)$, (ii) decline to minus infinity when time t_a tend to initial time t, (iii) they increase and become positive while time t_a grows, (iv) for the large commercialization times $t_a \to +\infty$ the components decline to zero. Due to these properties components $RC(t_a, S_i)$ reach the positive maximum values $RC_i^m > 0$ (6.8) at the finite moments of time – optimal commercialization times $t_i^m \in (t, +\infty)$, $i = 1, 2$. Taking into account the maximum type structure of the profit function $R(t_a)$ (6.5) we derive for the optimal profit value R^m the formula (6.9) which selects the maximum value among two possible scenarios RC_1^m, $(RC_2^m - r(t_b))$.

Let us prove the second statement that scenario Σ_2 with larger sales $S_2 > S_1$ is faster $t_2^m < t_1^m$ than scenario Σ_1 with normal level S_1. It is clear that $t_1^m \neq t_2^m$ since

$$\frac{\partial}{\partial t_a}(RC(t_1^m, S_2)) = -S_2 e^{-(\lambda-\mu)t_a} - \frac{\partial w}{\partial t_a}(t_1^m) < -S_1 e^{-(\lambda-\mu)t_a} - \frac{\partial w}{\partial t_a}(t_1^m) =$$
$$\frac{\partial}{\partial t_a}(RC(t_1^m, S_1)) = 0$$

and component $RC(t_a, S_2)$ strictly declines at the maximum point t_1^m.

Assume the contrary $t_2^m > t_1^m$. According to definition of commercialization times t_i^m, $i = 1, 2$ we have two contradicting inequalities

$$RC(t_1^m, S_1) \geq RC(t_2^m, S_1)$$

and

$$RC(t_2^m, S_1) = RC(t_2^m, S_2) - \frac{S_b}{(\lambda-\mu)} e^{-(\lambda-\mu)t_2^m} \geq RC(t_1^m, S_2) - \frac{S_b}{(\lambda-\mu)} e^{-(\lambda-\mu)t_2^m} =$$
$$RC(t_1^m, S_1) + \frac{S_b}{(\lambda-\mu)}(e^{-(\lambda-\mu)t_1^m} - e^{-(\lambda-\mu)t_2^m}) > RC(t_1^m, S_1)$$

The contradiction implies the necessary inequality for commercialization times $t < t_2^m < t_1^m$.

The third statement obviously follows from formula

$$RC(t_a, S_2) - r(t_b) = RC(t_a, S_1) + r(t_a) - r(t_b) =$$
$$RC(t_a, S_1) + \frac{S_b}{(\lambda - \mu)}(e^{-(\lambda-\mu)t_a} - e^{-(\lambda-\mu)t_b})$$

and monotonicity of function $t_a \to (e^{-(\lambda-\mu)t_a} - e^{-(\lambda-\mu)t_b})$ which provide the proper inequalities in relation (6.11). □

Remark 6.4 *The optimal commercialization times t_i^m of two investment scenarios Σ_i, $i = 1, 2$ can be found as solutions of necessary optimality conditions*

$$\frac{\partial}{\partial t_a}(RC(t_i^m, S_i)) = -S_i e^{-(\lambda-\mu)t_i^m} - \frac{\partial w}{\partial t_a}(t_i^m) =$$
$$-S_i e^{-(\lambda-\mu)t_i^m} + \rho^{(\alpha-1)}\frac{(x_a - xe^{-\sigma(t_i^m - t)})^\alpha}{(1 - e^{-\rho(t_i^m - t)})^{(\alpha-1)}} e^{-\lambda t_i^m}(\lambda +$$
$$(\alpha - 1)\frac{\rho e^{-\rho(t_i^m - t)}}{(1 - e^{-\rho(t_i^m - t)})} - \alpha\frac{\sigma x e^{-\sigma(t_i^m - t)}}{(x_a - xe^{-\sigma(t_i^m - t)})}) = 0, \quad i = 1, 2 \tag{6.12}$$

or equivalent equations

$$-S_i e^{\mu t_i^m} + \rho^{(\alpha-1)}\frac{(x_a - xe^{-\sigma(t_i^m - t)})^\alpha}{(1 - e^{-\rho(t_i^m - t)})^{(\alpha-1)}}(\lambda +$$
$$(\alpha - 1)\frac{\rho e^{-\rho(t_i^m - t)}}{(1 - e^{-\rho(t_i^m - t)})} - \alpha\frac{\sigma x e^{-\sigma(t_i^m - t)}}{(x_a - xe^{-\sigma(t_i^m - t)})}) = 0, \quad i = 1, 2 \tag{6.13}$$

Relation (6.12) defines commercialization times t_i^m, $i = 1, 2$ as implicit functions of the current position – time t, technology stock $x = x(t)$, the commercialization technological level x_a and econometric parameters α, σ, λ.

7 Dynamical Optimality Principle for Investment Scenarios

In this section we address to the qualitative properties of the proposed solution. We focus on optimality properties and dynamic programming principle.

We consider first the optimality properties. Let us note that in the previous sections we constructed optimal solutions for two objectives. We show now that these two solutions solve the joint problem of maximizing the technology innovation

$$V^m = \sup_{t_a, u(\cdot)} V(t_a, u(\cdot), t_b, t, x, x_a, \alpha, \lambda, \mu, \sigma) \tag{7.1}$$
$$V = V(\cdot) = \frac{S_a}{(\lambda - \mu)} e^{-(\lambda-\mu)t_a} + \max\{0, \frac{S_b}{(\lambda - \mu)}(e^{-(\lambda-\mu)t_a} - e^{-(\lambda-\mu)t_b})\} -$$
$$- \int_t^{t_a} e^{-\lambda} u^\alpha(s) ds$$
$$\dot{x}(s) = -\sigma x(s) + u(s), \quad x(t) = x, \quad x(t_a) = x_a$$

Proposition 7.1 *The optimal value R^m (6.9) of the profit function $R(t_a)$ (6.4) in the two-level problem with the consequently optimized objectives coincides with the total maximum V_m (7.1) of the technology innovation $R^m = V^m$.*

Proof. Let us suppose the contrary $R^m < V^m$. Consider two possibilities. Assume first that the total maximum has the infinite value $V^m = +\infty$. In this case according to definition of supremum (7.1) for any parameter $\varepsilon > 0$ there exist time t_ε, and investment control $u_\varepsilon(\cdot)$ such that the strict inequality $V(t_\varepsilon, u_\varepsilon(\cdot)) > R^m + 1/\varepsilon$ takes place. For the fixed commercialization time $t_a = t_\varepsilon$ and corresponding optimal investment plan $u^0(\cdot)$ (2.17) we have the conflicting chain of relations

$$R^m = R(t^m) \geq R(t_\varepsilon) = V(t_\varepsilon, u^0(\cdot)) \geq V(t_\varepsilon, u_\varepsilon(\cdot)) > R^m + \frac{1}{\varepsilon}$$

In the second case assume that the total maximum value is finite $0 < V^m < +\infty$. According to definition of supremum (7.1) we have two relations

$$\begin{array}{ll} 1) & V^m \geq V(t_a, u(\cdot)) \quad \forall \quad t_a, u(\cdot) \\ 2) & \forall \quad \varepsilon > 0 \quad \exists \quad t_\varepsilon, u_\varepsilon(\cdot) \quad V(t_\varepsilon, u_\varepsilon(\cdot)) > V^m - \varepsilon \end{array}$$

Defining the optimal investment plan $u^0(\cdot)$ (2.17) for the commercialization time $t_a = t_\varepsilon$ we obtain the chain of inequalities

$$R^m = R(t^m) \geq R(t_\varepsilon) = V(t_\varepsilon, u^0(\cdot)) \geq V(t_\varepsilon, u_\varepsilon(\cdot)) > V^m - \varepsilon$$

which contradicts to assumption $R^m < V^m$.

So we conclude with the proper relation $R^m = V^m$. □

Let us prove that the dynamic programming principle is valid for the optimal value R^m (6.9) of the profit function $R(t_a)$ (6.4). To this end we fix the market commercialization time t_b, the levels of sales S_a, S_b with the commercialized technology and econometric coefficients α, λ, μ, σ. For the initial position (t, x) of the technology stock we denote by the symbol R^m the optimal value of innovation

$$R^m = R^m(t, x) = \max_{t_a > t} R(t_a, t, x) \tag{7.2}$$

by the symbol T^m – the set of optimal innovation times

$$T^m = T^m(t, x) = \{t^m : \ R^m(t, x) = R(t^m, t, x)\} \tag{7.3}$$

by the symbol U^0 – the set of optimal innovation plans and by X^0 – the set of optimal technological trajectories

$$\begin{aligned} & U^0 = U^0(\tau, t, x) = \{u^0(s), \quad t \leq \tau \leq s \leq t^m : \\ & w(t, x, t^m, x_a) = \min_{u(\cdot)} \int_t^{t^m} e^{-\lambda s} u^\alpha(s) ds = \int_t^{t^m} e^{-\lambda s} (u^0(s))^\alpha ds\} \end{aligned} \tag{7.4}$$

$$\begin{aligned} & X^0 = X^0(\tau, t, x) = \{x^0(s), \quad t \leq \tau \leq s \leq t^m : \\ & \dot{x}^0(s) = -\sigma x^0(s) + u^0(s), \quad x^0(t) = x, \quad x^0(t^m) = x_a\} \end{aligned} \tag{7.5}$$

Finally denote by the symbol $w(t, x, \tau, \xi)$ the optimal cost of innovation from level x to level ξ, $x \leq \xi$ on interval $[t, \tau]$, $t \leq \tau$

$$w(t, x, \tau, \xi) = \min_{u(\cdot)} \int_t^\tau e^{-\lambda s} u^\alpha(s) ds = \int_t^\tau e^{-\lambda s} (u^0)^\alpha(s) ds \tag{7.6}$$

$$\dot{x}^0(s) = -\sigma x^0(s) + u^0(s), \quad t \leq s \leq \tau \leq t^m, \quad x^0(t) = x, \quad x^0(\tau) = \xi$$

Proposition 7.2 *The optimal constructions of innovation T^m, U^0, X^0, R^m satisfy the dynamic programming principle in the following form*

$$\begin{aligned} &T^m(\tau,\xi) \subseteq T^m(t,x), \quad U^0(\tau,\tau,\xi) \subseteq U^0(\tau,t,x) \\ &X^0(\tau,\tau,\xi) \subseteq X^0(\tau,t,x), \quad \xi = x^0(\tau,t,x), \quad t \le \tau \le t^m \end{aligned} \tag{7.7}$$

$$\begin{aligned} &R^m(\tau,\xi) - R^m(t,x) = w(t,x,t^m,x_a) - w(\tau,\xi,t^m,x_a) = w(t,x,\tau,\xi) \\ &\xi = x^0(\tau,t,x), \quad t \le \tau \le t^m \end{aligned} \tag{7.8}$$

Relations (7.7) of the dynamic programming principle mean that along the optimal trajectory (τ,ξ), $\xi = x^0(\tau,t,x)$, which starts at the initial technological position (t,x), $x = x^0(t,t,x)$ the optimal innovation times $T^m(\tau,\xi)$, the optimal investment plans $U^0(\tau,\tau,\xi)$ and optimal trajectories $X^0(\tau,\tau,\xi)$ are identical to each other and can be determined at the initial technological position $(\tau,\xi) = (t,x)$.

The sense of dynamic programming condition (7.8) consists in the fact that along the optimal trajectory (τ,ξ), $\xi = x^0(\tau,t,x)$, the difference of optimal innovation profits $(R^m(\tau,\xi) - R^m(t,x))$ and equivalently the difference between values of optimal investment $(w(t,x,t^m,x_a) - w(\tau,\xi,t^m,x_a))$ constitutes the value of optimal innovation expenditure $w(t,x,\tau,\xi)$ for reaching the technology stock ξ at time τ starting from the initial technological position (t,x).

Proof. Let us prove the first statement. All others can be proved analogously.

Let $t_p^m \in T^m(\tau,\xi)$, $t^m \in T^m(t,x)$ be the optimal innovation times at the current position (τ,ξ) and the initial position (t,x) respectively.

Let us denote by the symbol $u_p^0(\cdot)$ the optimal innovation plan which minimizes the innovation expenditures on the time interval $[\tau, t_p^m]$

$$\int_\tau^{t_p^m} e^{-\lambda s}(u_p^0(s))^\alpha ds = \min_{u(\cdot)} \int_\tau^{t_p^m} e^{-\lambda s} u^\alpha(s) ds$$

By the symbol $x_p^0(\cdot)$ we denote the technological trajectory which is generated by the optimal innovation plan

$$\dot{x}_p^0(s) = -\sigma x_p^0(s) + u_p^0(s)$$

from the current technological position $\xi = x_p^0(\tau)$ and directed to the commercialization level $x_a = x_p^0(t_a)$.

We assume as usual that $u^0(\cdot)$ is the optimal innovation plan and $x^0(\cdot)$ is the corresponding technological trajectory with the optimal innovation time $t^m \in T^m(t,x)$ and the fixed boundary conditions $x^0(t) = x$, $x^0(t_a) = x_a$.

Let us remind that the dynamic programming principle takes place for the optimal innovation cost $w(\cdot)$

$$w(t,x,t^m,x_a) = w(t,x,\tau,\xi) - w(\tau,\xi,t^m,x_a) \tag{7.9}$$

and optimal investment plans $u^0(\cdot)$, $u_p^0(\cdot)$ are connected with these costs by relations

$$\begin{aligned} &w(t,x,\tau,\xi) = \int_t^\tau e^{-\lambda s}(u^0(s))^\alpha ds, \quad w(t,x,t^m,x_a) = \int_\tau^{t^m} e^{-\lambda s}(u^0(s))^\alpha ds \\ &\dot{x}^0(s) = -\sigma x^0(s) + u^0(s), \quad x^0(t) = x, \quad x^0(\tau) = \xi, \quad x^0(t^m) = x_a \end{aligned} \tag{7.10}$$

$$w(\tau,\xi,t_p^m,x_a)=\int_\tau^{t_p^m} e^{-\lambda s}(u_p^0(s))^\alpha ds \tag{7.11}$$
$$\dot{x}_p^0(s)=-\sigma x_p^0(s)+u_p^0(s),\quad x_p^0(\tau)=\xi,\quad x_p^0(t_p^m)=x_a$$

Let us suppose on the contrary to relations (7.7) of the dynamic programming principle that there exists better innovation time at the current position (τ,ξ) than at the initial position (t,x)

$$t_p^m\in T^m(\tau,\xi),\quad t_p^m\notin T^m(t,x) \tag{7.12}$$

and/or optimal innovation plans $u^0(\cdot)$, $u_p^0(\cdot)$ and corresponding optimal trajectories $x^0(\cdot)$, $x_p^0(\cdot)$ don't coincide essentially on the joint time interval

$$u^0(s)\neq u_p^0(s),\quad x^0(s)\neq x_p^0(s),\quad s\in A\subseteq[\tau,\min\{t^m,t_p^m\}],\quad mes(A)>0 \tag{7.13}$$

Here A is a measurable set and $mes(A)$ is its measure.

Relations (7.10), (7.12), (7.13) imply the strict inequality for the optimal innovation result R_p^m

$$R_p^m=R(t_p^m)=\max_{t_a}R(t_a,\tau,\xi)>R(t^m,\tau,\xi)=d(t^m)-w(\tau,\xi,t^m,x_a)=$$
$$d(t^m)-(w(t,x,\tau,\xi)+w(\tau,\xi,t^m,x_a))+w(t,x,\tau,\xi)=R^m+w(t,x,\tau,\xi) \tag{7.14}$$

Let us compose the new optimal innovation plan $u^*(\cdot)$ from innovation plans $u^0(\cdot)$, $u_p^0(\cdot)$

$$u^*(s)=\begin{cases} u^0(s) & \text{if } s\in[t,\tau) \\ u_p^0(s) & \text{if } s\in[\tau,t_p^m]\end{cases} \tag{7.15}$$

and the corresponding technological scenario $x^*(\cdot)$ from technological scenarios $x^0(\cdot)$, $x_p^0(\cdot)$

$$x^*(s)=\begin{cases} x^0(s) & \text{if } s\in[t,\tau] \\ x_p^0(s) & \text{if } s\in[\tau,t_p^m]\end{cases} \tag{7.16}$$

Taking into account relations (7.10), (7.11), (7.15), (7.16) we have the following chain of inequalities for the optimal innovation result R^m

$$R^m=R(t^m)=\max_{t_a}R(t_a,t,x)\geq R(t_p^m,t,x)=$$
$$d(t_p^m)-w(t,x,t_p^m,x_a)\geq d(t_p^m)-\int_t^{t_p^m}e^{-\lambda s}(u^*(s))^\alpha ds=$$
$$d(t_p^m)-\int_t^\tau e^{-\lambda s}(u^0(s))^\alpha ds-\int_\tau^{t_p^m}e^{-\lambda s}(u_p^0(s))^\alpha ds=$$
$$d(t_p^m)-w(t,x,\tau,\xi)-w(\tau,\xi,t_p^m,x_a)=R_p^m-w(t,x,\tau,\xi) \tag{7.17}$$

The last relation (7.17) contradicts to assumption (7.14) and hence the dynamic programming relations (7.7) are valid. The dynamic programming principle (7.8) for the optimal innovation value $R^m(\cdot)$ then follows from relations (7.7) for optimal sets T^m, U^0, X^0 and principle (7.9) for the optimal innovation cost $w(\cdot)$. □

Objective III. Assessment of the Market Potential Innovation

8 The Heavy Dynamics of the Market Innovation

In this section we consider the third problem of econometric assessing the market technology trajectories and prediction of the market commercialization time t_b. To some extent the competitive market environment can be treated as the second player which makes the technological innovation. The peculiarity of this player consists in the fact that the market environment is not homogeneous: it constitutes the large group of agents with different utilities, managers and resources. Therefore, it is reasonable not to model possible decision making rules of agents but exploit the current information about the market technological innovation for estimating the market commercialization time. Due to inertness of the market it is rather natural to describe its behavior by the "heavy" dynamics with the small resource for acceleration. In this case the current information about the market technological stock and its rate allows to predict the market innovation trajectory with the reasonable confidence level and to analyze sensitivity of the market commercialization time t_b with respect to uncertainties.

Let us describe the model of market dynamics in more details. We assume that the market of technological innovation consists of n agents. The growth of the technology stock of each agent y_i with respect to investment r_i is described by similar equations with the obsolescence effect

$$\dot{y}_i(s) = -\sigma y_i(s) + r_i(s), \quad s \geq t_0, \quad y_i(t_0) = y_i^0, \quad i = 1, ..., n \tag{8.1}$$

Introducing average market technology stock y, average market technology rate $\dot{y}$ and average market technology investment r_b by convolutions of stocks y_i, rates $\dot{y}_i$ and investments r_i with weight coefficients α_i

$$y = \sum_{i=1}^{n} \alpha_i y_i, \quad y_0 = \sum_{i=1}^{n} \alpha_i y_i^0, \quad \dot{y} = \sum_{i=1}^{n} \alpha_i \dot{y}_i, \quad r_b = \sum_{i=1}^{n} \alpha_i r_i \tag{8.2}$$

$$\sum_{i=1}^{n} \alpha_i = 1, \quad \alpha_i \geq 0, \quad i = 1, ..., n$$

we obtain the aggregated dynamics of the market innovation

$$\dot{y}(s) = -\sigma y(s) + r_b(s), \quad s \geq t_0, \quad y(t_0) = y_0 \tag{8.3}$$

Let us estimate the potential of the market environment to accelerate its innovation. We introduce the following notions :

$\dot{y}_i/y_i$ – technology rate of firm i;

r_i/y_i – R&D intensity of firm i;

$\dot{y}/y$ – technology rate of the market;

r_b/y – R&D intensity of the market.

One can bind technology rates and R&D intensities of firms with technology and R&D intensity of the market by relations

$$\frac{\dot{y}}{y} = \sum_{i=1}^{n} \beta_i \frac{\dot{y}_i}{y_i}, \quad \frac{r_b}{y} = \sum_{i=1}^{n} \beta_i \frac{r_i}{y_i} \tag{8.4}$$

$$\beta_i = \beta_i(y) = \frac{\alpha_i y_i}{y}, \quad \sum_{i=1}^{n} \beta_i = 1, \quad \beta_i \geq 0, \quad i = 1, ..., n$$

Basing on relations (8.4) one can present technology rate $\dot{y}/y$ and R&D intensity r_b/y of the market as the mean value of technology rates $\dot{y}_i/y_i$ and R&D intensities r_i/y_i of firms with variation given by relation

$$D = \sum_{i=1}^{n} \beta_i \left(\frac{\dot{y}_i}{y_i} - \frac{\dot{y}}{y} \right)^2 = \sum_{i=1}^{n} \beta_i \left(\frac{r_i}{y_i} - \frac{r_b}{y} \right)^2 \tag{8.5}$$

Introducing notations for R&D intensities

$$z = z(s) = \frac{r_b}{y}, \quad z_i = z_i(s) = \frac{r_i}{y_i}, \quad i = 1, ..., n$$

we rewrite investment dynamics (8.1), (8.3) of firms and the market in the form of quasi-linear differential equations

$$\dot{y}_i(s) = -\sigma y_i(s) + z_i(s) y_i(s) = (-\sigma + z_i(s)) y_i(s), \quad i = 1, ..., n \tag{8.6}$$

$$\dot{y}(s) = -\sigma y(s) + z(s) y(s) = (-\sigma + z(s)) y(s) \tag{8.7}$$

Investment dynamics equations (8.6), (8.7) are in consistency with the concept of exponential technological growth since solutions of these equations are given by the Cauchy exponential formulas

$$y_i(s) = y_i^0 e^{b_i(s)}, \quad b_i(s) = -\sigma(s - t_0) + \int_{t_0}^{s} z_i(\tau) d\tau, \quad i = 1, ..., n \tag{8.8}$$

$$y(s) = y_0 e^{b(s)}, \quad b(s) = -\sigma(s - t_0) + \int_{t_0}^{s} z(\tau) d\tau \tag{8.9}$$

Let us estimate the first derivatives of R&D intensities r_i/y_i, r_b/y which influences directly on the second derivative (acceleration) of the technology stocks $\ln y_i$, $\ln y$

$$\dot{z}_i = \left(\frac{r_i}{y_i} \right)' = \frac{\dot{r}_i y_i - r_i \dot{y}_i}{y_i^2} = \frac{r_i}{y_i} \left(\frac{\dot{r}_i}{r_i} - \frac{\dot{y}_i}{y_i} \right) = \frac{\ddot{y}_i y_i - (\dot{y}_i)^2}{y_i^2} = (\ln y_i)'' \tag{8.10}$$

$$\dot{z} = \left(\frac{r_b}{y} \right)' = \frac{\dot{r}_b y - r_b \dot{y}}{y^2} = \frac{r_b}{y} \left(\frac{\dot{r}_b}{r_b} - \frac{\dot{y}}{y} \right) = \frac{\ddot{y} y - (\dot{y})^2}{y^2} = (\ln y)'' \tag{8.11}$$

We express now the investment acceleration $\dot{z} = (\ln y)''$ of the market through the investment accelerations $\dot{z}_i = (\ln y_i)''$ of firms

$$\dot{z} = (\ln y)'' = \left(\frac{r_b}{y} \right)' = \left(\sum_{i=1}^{n} \beta_i(y) \frac{r_i}{y} \right)' = \sum_{i=1}^{n} \beta_i \left(\frac{r_i}{y_i} \right)' + \sum_{i=1}^{n} \beta_i' \frac{r_i}{y_i} =$$
$$\sum_{i=1}^{n} \beta_i \dot{z}_i + \sum_{i=1}^{n} \beta_i \left(\frac{\dot{y}_i}{y_i} - \frac{\dot{y}}{y} \right) \frac{r_i}{y_i} = \sum_{i=1}^{n} \beta_i (\ln y_i)'' + \sum_{i=1}^{n} \beta_i \left(\frac{r_i}{y_i} - \frac{r_b}{y} \right) \frac{r_i}{y_i} =$$
$$\sum_{i=1}^{n} \beta_i (\ln y_i)'' + \sum_{i=1}^{n} \beta_i \left(\frac{r_i}{y_i} - \frac{r_b}{y} \right)^2 = \sum_{i=1}^{n} \beta_i (\ln y_i)'' + \sum_{i=1}^{n} \beta_i \left(\frac{\dot{y}_i}{y_i} - \frac{\dot{y}}{y} \right)^2 \tag{8.12}$$

We will make a reasonable assumption about inertness of the market and assume that overwhelming majority of firms develop the technology according to the chosen scenario with the zero acceleration

$$(\ln y_j)'' = \left(\frac{r_j}{y_j} \right)' = \frac{r_j}{y_j} \left(\frac{\dot{r}_j}{r_j} - \frac{\dot{y}_j}{y_j} \right) = 0, \quad j = 1, ..., n^0, \quad 0 \le \frac{n - n^0}{n} \ll 1 \tag{8.13}$$

The last relation means that firms of this large group hold the strategy of proportional investment: the investment rate $\dot{r}_j/r_j$ should be proportional to the technology rate $\dot{y}_j/y_j$

$$\frac{\dot{r}_j}{r_j} = \frac{\dot{y}_j}{y_j}, \quad j = 1, ..., n^0 \tag{8.14}$$

We assume that in the rest small group there could be firms with the positive or negative investment accelerations

$$(\ln y_k)'' > 0, \quad k = 1, ..., n^+, \quad (\ln y_l)'' < 0, \quad l = 1, ..., n^- \tag{8.15}$$

Due to the previous assumptions one can suppose that the mean value of firms' accelerations $(\ln y_i)''$ is finite and small with respect to the absolute value of technology rate $\dot{y}/y$

$$|M| = |\sum_{i=1}^{n} \beta_i (\ln y_i)''| \ll R \tag{8.16}$$

Here parameter R is the minimal value of technology rate $\dot{y}/y$

$$R = \min_{s \geq t_0} |\frac{\dot{y}(s)}{y(s)}| \tag{8.17}$$

Finally let us assume that technology rates $\dot{y}_i/y_i$ for the major part of firms are close to each other and consequently differ slightly from the average technology rate $\dot{y}/y$, and, therefore, the variance D (8.5) of technology rates $\dot{y}_i/y_i$ is also small in comparison with the absolute value of technology rate

$$D = \sum_{i=1}^{n} \beta_i \left(\frac{\dot{y}_i}{y_i} - \frac{\dot{y}}{y} \right)^2 \ll R \tag{8.18}$$

Taking into account all previous assumptions we arrive to the "heavy" technological dynamics of the market environment

$$\begin{aligned} \dot{y}(s) &= -\sigma y(s) + z(s) y(s) \\ \dot{z}(s) &= v(s), \quad |v(s)| \leq v_0, \quad v_0 \ll R \end{aligned} \tag{8.19}$$

Introducing change of variables

$$\begin{aligned} & p = \ln y \quad \Longleftrightarrow \quad y = e^p \\ & q = z - \sigma = \frac{\dot{y}}{y} \quad \Longleftrightarrow \quad z = q + \sigma = \frac{\dot{y}}{y} + \sigma \end{aligned} \tag{8.20}$$

we present the "heavy" motion of the market environment in the form of the "crocodile" dynamics known in the differential games theory (see [Isaacs, 1965])

$$\begin{aligned} & \dot{p}(s) = q(s) \\ & \dot{q}(s) = v(s), \quad |v(s)| \leq v_0, \quad v_0 \ll R \end{aligned} \tag{8.21}$$

Later we will specify more precisely condition $v_0 \ll R$ of the market inertness.

Let us note that the "light" motion (2.2) of the competitive innovator can be presented as the known "boy" dynamics (see [Isaacs, 1965])

$$\begin{aligned} & \dot{X}(s) = U(s), \quad |U(s)| \leq U_0 < 1 - \sigma \\ & X = X(s) = \ln x(s), \quad U = U(s) = -\sigma + \frac{u(s)}{x(s)} \end{aligned} \tag{8.22}$$

We can consider the described dynamic model of innovation as a game between two qualitatively different players: one of them – the innovator ("boy"), has the "light" dynamics which provides high possibilities for acceleration even for small levels of technology rates; the second player is the weakly controllable (weakly dynamical) "heavy" market ("crocodile") which may have large rates of technological growth but due to its nonhomogeneity and inertness has small acceleration.

Let us note that another type of the problem statement in the game with "heavy" and "light" dynamics was considered for optimizing navigation noise by the "toreador" in the pursuit-evasion process (see [Ivanov, Tarasyev,Ushakov, Khripunov, 1993]).

The inertness property of the "heavy" market dynamics allows to estimate the market commercialization time t_b basing on information about the current technology stock $y = y(s)$ or equivalently of its logarithm $p = p(s) = \ln y(s)$ and the current technology rate $q = q(s) = \dot{y}(s)/y(s)$. The innovator can use this estimate for choosing a scenario of innovation and guaranteed optimization of its own commercialization time t_a.

9 The Market Commercialization Time

Let us estimate the commercialization time of the market t_b basing on the "heavy" dynamics (8.21), and the current information about the technology stock $y = y(t)$ and its rate $\dot{y} = \dot{y}(t)$. We fix acceleration v, $|v| \leq v_0$ and derive solutions for technology trajectories of the "heavy" dynamics (8.21)

$$q(s) = v(s-t) + q(t) = v(s-t) + \frac{\dot{y}}{y}, \quad s \geq t \tag{9.1}$$

$$p(s) = \frac{v}{2}(s-t)^2 + q(t)(s-t) + p(t) = \frac{v}{2}(s-t)^2 + \frac{\dot{y}}{y}(s-t) + \ln y \tag{9.2}$$

We assume that the market starts commercialization of the new technology when its technological stock $y = y(s)$ reaches the necessary technological level y_b, $0 \leq y(s) \leq y_b$. It means that the commercialization time t_b of the market can be found from dynamics (9.2) as the first passage time of the commercialization level y_b

$$p(t_b) = \ln y_b \quad \Longleftrightarrow \quad \frac{v}{2}(t_b-t)^2 + \frac{\dot{y}}{y}(t_b-t) + (\ln y - \ln y_b) = 0 \tag{9.3}$$

and expressed by formula

$$t_b - t = \frac{(-\dot{y}/y + ((\dot{y}/y)^2 + 2v(\ln y_b - \ln y))^{1/2})}{v} = \frac{2(\ln y_b - \ln y)}{(\dot{y}/y + ((\dot{y}/y)^2 + 2v(\ln y_b - \ln y))^{1/2})} \tag{9.4}$$

In particular, for the zero acceleration $v = 0$ the market commercialization time t_b^0 can be assessed by relation

$$t_b^0 - t = \left(\frac{\dot{y}}{y}\right)^{-1} (\ln y_b - \ln y) \tag{9.5}$$

Analogously for the minimum acceleration $v = -v_0$ the market commercialization time t_b^1 is given by formula

$$t_b^1 - t = \frac{(-\dot{y}/y + ((\dot{y}/y)^2 - 2v_0(\ln y_b - \ln y))^{1/2})}{v_0} = \frac{2(\ln y_b - \ln y)}{(\dot{y}/y + ((\dot{y}/y)^2 - 2v_0(\ln y_b - \ln y))^{1/2})} \tag{9.6}$$

For the maximum acceleration $v = v_0$ the market commercialization time t_b^2 is determined by equation

$$t_b^2 - t = \frac{(-\dot{y}/y + ((\dot{y}/y)^2 + 2v_0(\ln y_b - \ln y))^{1/2})}{v_0} =$$
$$\frac{2(\ln y_b - \ln y)}{(\dot{y}/y + ((\dot{y}/y)^2 + 2v_0(\ln y_b - \ln y))^{1/2})} \tag{9.7}$$

Let us introduce notations for extremal trajectories generated by extremal accelerations $v = \pm v_0$ and the zero acceleration $v = 0$. Denote by the symbol $(y_1(s), \dot{y}_1(s))$ the trajectory of the market dynamics (8.19) or (8.21) generated by the minimum acceleration $v = -v_0$, by the symbol $(y_2(s), \dot{y}_2(s))$ – the trajectory generated by the maximum acceleration $v = v_0$, by the symbol $(y_0(s), \dot{y}_0(s))$ – the trajectory generated by the zero acceleration $v = 0$. Analogously by the symbol $(y(s), \dot{y}(s))$ we denote the market trajectory of dynamics (8.19) or (8.21) with an arbitrary measurable acceleration $v = v(s)$, $|v(s)| \leq v_0$ and by the symbol t_b – the corresponding commercialization time $y(t_b) = y_b$. We assume that all trajectories start from the same initial position $(y, \dot{y}) = (y(t), \dot{y}(t))$.

Remark 9.1 *According to the comparison theorem the following relations take place for the market trajectories $(y(s), \dot{y}(s))$, $(y_1(s), \dot{y}_1(s))$, $(y_2(s), \dot{y}_2(s))$*

$$y_1(s) \leq y(s) \leq y_2(s), \quad \dot{y}_1(s) \leq \dot{y}(s) \leq \dot{y}_2(s), \quad s \geq t \tag{9.8}$$
$$y_1(t) = y_2(t) = y(t) = y, \quad \dot{y}_1(t) = \dot{y}_2(t) = \dot{y}(t) = \dot{y}$$

In particular, the similar inequalities are fulfilled for the normal market trajectory $(y_0(s), \dot{y}_0(s))$ generated by the zero acceleration $v = 0$

$$y_1(s) \leq y_0(s) \leq y_2(s), \quad \dot{y}_1(s) \leq \dot{y}_0(s) \leq \dot{y}_2(s), \quad s \geq t \tag{9.9}$$
$$y_1(t) = y_2(t) = y_0(t) = y, \quad \dot{y}_1(t) = \dot{y}_2(t) = \dot{y}_0(t) = \dot{y}$$

Consequently the analogous relations are valid for the market commercialization times t_b, t_b^1, t_b^2

$$t_b^2 \leq t_b \leq t_b^1 \tag{9.10}$$

and for the normal commercialization time t_b^0

$$t_b^2 \leq t_b^0 \leq t_b^1 \tag{9.11}$$

Let us estimate the maximum and minimum commercialization times t_b^1, t_b^2 from above and below. One can see that the following inequalities hold for them

$$\left(\frac{\dot{y}}{y}\right)^{-1} (\ln y_b - \ln y) = t_b^0 - t \leq t_b^1 - t =$$
$$\frac{2(\ln y_b - \ln y)}{(\dot{y}/y + ((\dot{y}/y)^2 - 2v_0(\ln y_b - \ln y))^{1/2})} \leq 2\left(\frac{\dot{y}}{y}\right)^{-1} (\ln y_b - \ln y) \tag{9.12}$$

$$\frac{1}{2}\left(\frac{\dot{y}}{y}\right)^{-1} (\ln y_b - \ln y) \leq \frac{2(\ln y_b - \ln y)}{(\dot{y}/y + ((\dot{y}/y)^2 + 2v_0(\ln y_b - \ln y))^{1/2})} =$$
$$t_b^2 - t \leq t_b^0 - t = \left(\frac{\dot{y}}{y}\right)^{-1} (\ln y_b - \ln y) \tag{9.13}$$

Basing on inequalities (9.12), (9.13) one can assess sensitivity of maximum and minimum commercialization times t_b^1, t_b^2 with respect to the absolute value of acceleration v_0.

Proposition 9.1 *The deviation of maximum and minimum commercialization times t_b^1, t_b^2 from the normal time t_b^0 is evaluated by linear functions of acceleration v_0*

$$0 \le t_b^1 - t_b^0 \le 2v_0 \left(\frac{\dot{y}}{y}\right)^{-3} (\ln y_b - \ln y)^2 \tag{9.14}$$

$$0 \le t_b^0 - t_b^2 \le \frac{1}{2} v_0 \left(\frac{\dot{y}}{y}\right)^{-3} (\ln y_b - \ln y)^2 \tag{9.15}$$

Proof. Substituting extremal values of acceleration $v = -v_0$, $v = v_0$ into equation (9.3), subtracting from them the analogous relation for zero acceleration $v = 0$ and taking into account estimates (9.12), (9.13) we obtain the necessary chain of inequalities

$$0 \le t_b^1 - t_b^0 = \frac{1}{2} v_0 \left(\frac{\dot{y}}{y}\right)^{-1} (t_b^1 - t)^2 \le 2v_0 \left(\frac{\dot{y}}{y}\right)^{-3} (\ln y_b - \ln y)^2$$

$$0 \le t_b^0 - t_b^2 = \frac{1}{2} v_0 \left(\frac{\dot{y}}{y}\right)^{-1} (t_b^2 - t)^2 \le \frac{1}{2} v_0 \left(\frac{\dot{y}}{y}\right)^{-3} (\ln y_b - \ln y)^2$$

□

Remark 9.2 *The linear assessment of sensitivity (9.14), (9.15) for commercialization times t_b^1, t_b^2 means that its accuracy is directly proportional to the absolute value of acceleration v_0. According to the assumption of the market inertness this value is relatively small $v_0 \ll R$. Therefore, one can use the normal trajectory $(y_0(s), \dot{y}_0(s))$ of the market with the zero acceleration $v = 0$, its commercialization time t_b^0 and sensitivity estimates (9.14), (9.15) for prediction of the market commercialization time t_b, $t_b^2 \le t_b \le t_b^1$.*

Let us note that coefficient $(\dot{y}/y)^{-3}(\ln y_b - \ln y)^2$ shows dependence of sensitivity estimates (9.14), (9.15) with respect to the current situation $(y, \dot{y})$ of the market trajectory: the greater is the technology rate $\dot{y}/y$ and/or the closer is the current technology stock y to the commercialization level y_b, the smaller is this coefficient and sensitivity estimates (9.14), (9.15) are more accurate.

Finally let us indicate sensitivity of the market commercialization time $t_b = t_b(v)$ (9.4) with respect to nonnegative accelerations $v \ge 0$.

Remark 9.3 *The sensitivity of the market commercialization time $t_b = t_b(v)$ (9.4) with respect to accelerations v can be expressed by the first derivative*

$$t_b'(v) = \frac{-2(\ln y_b - \ln y)^2((\dot{y}/y)^2 + 2v(\ln y_b - \ln y))^{-1/2}}{(\dot{y}/y + ((\dot{y}/y)^2 + 2v(\ln y_b - \ln y))^{1/2})^2} \tag{9.16}$$

Its absolute value for nonnegative accelerations $v \ge 0$ reaches maximum at the zero acceleration $v = 0$ due to its monotonic decrease

$$\max_{v \ge 0} |t_b'(v)| = |t_b'(0)| = \frac{1}{2} \left(\frac{\dot{y}}{y}\right)^{-3} (\ln y_b - \ln y)^2 \tag{9.17}$$

For nonnegative accelerations $v_2 > v_1 \ge 0$ according to the Lagrange mean value theorem the following chain of inequalities take place

$$|t_b(v_1) - t_b(v_2)| = t_b(v_1) - t_b(v_2) = t_b'(\bar{v})(v_1 - v_2) = |t_b'(\bar{v})|(v_2 - v_1) \le$$
$$|t_b'(v_1)|(v_2 - v_1) \le \frac{1}{2} \left(\frac{\dot{y}}{y}\right)^{-3} (\ln y_b - \ln y)^2 (v_2 - v_1), \quad 0 \le v_1 < \bar{v} < v_2 \tag{9.18}$$

The last inequality and monotonic decrease of the absolute value of derivative $|t_b'(v_1)|$ show that estimate (9.18) is improved with growth of acceleration v_1.

10 Guaranteed Strategy of Technological Innovation

In this section we construct the optimal guaranteed strategy of technological innovation and describe the saddle point equilibrium in the game between the "light" innovator and the "heavy" market. The special attention will be paid to the questions on sensitivity of the optimal profit result and robust properties of optimal strategies for scenarios selection.

Let us introduce the basic element of the guaranteed strategy: the threshold time t_b^* of the market innovation which separates two innovation scenarios (6.8) of innovator: the slow scenario Σ_1 with the optimal innovation time t_1^m, and the quick scenario Σ_2 with the optimal innovation time t_2^m, $t_2^m < t_1^m$. We define it as the innovation time of the market which equalizes profits of slow scenario RC_1^m and quick scenario $(RC_2^m - r(t_b^*))$ of innovator (see (6.8), (6.9))

$$R^m = RC_1^m = RC_2^m - r(t_b^*) \tag{10.1}$$

Let us indicate properties of the threshold time t_b^*.

Proposition 10.1 *The threshold time t_b^* of the market innovation exists, is unique and separates commercialization times t_1^m, t_2^m of slow scenario Σ_1 and quick scenario Σ_2 of innovator*

$$t_2^m < t_b^* < t_1^m \tag{10.2}$$

Proof. Let us consider the auxiliary function

$$f(t_b) = r(t_b) - (RC_2^m - RC_1^m) = \frac{S_b}{(\lambda-\mu)} e^{-(\lambda-\mu)t_b} - \left(\frac{(S_a+S_b)}{(\lambda-\mu)} e^{-(\lambda-\mu)t_2^m} - \frac{S_a}{(\lambda-\mu)} e^{-(\lambda-\mu)t_1^m} + w(t_1^m) - w(t_2^m) \right) \tag{10.3}$$

Exponential function $f(t_b)$ with the negative exponent $-(\lambda-\mu)$ monotonically decreases for $t_b > t$. Let us calculate values of function $f(t_b)$ at points t_1^m, t_2^m

$$f(t_1^m) = r(t_1^m) - (RC_2^m - RC_1^m) = -\left(\frac{(S_a+S_b)}{(\lambda-\mu)} e^{-(\lambda-\mu)t_2^m} - w(t_2^m) \right) + \left(\frac{(S_a+S_b)}{(\lambda-\mu)} e^{-(\lambda-\mu)t_1^m} - w(t_1^m) \right) = -RC(t_2^m, S_2) + RC(t_1^m, S_2) = -RC_2^m + RC(t_1^m, S_2) < 0$$

$$f(t_2^m) = r(t_2^m) - (RC_2^m - RC_1^m) = -\left(\frac{S_a}{(\lambda-\mu)} e^{-(\lambda-\mu)t_2^m} - w(t_2^m) \right) + \left(\frac{S_a}{(\lambda-\mu)} e^{-(\lambda-\mu)t_1^m} - w(t_1^m) \right) = -RC(t_2^m, S_1) + RC(t_1^m, S_1) = -RC(t_2^m, S_1) + RC_1^m > 0$$

The last inequalities and the strict monotonicity of function $f(t_b)$ imply the existence of the unique root t_b^* of equation

$$f(t_b^*) = r(t_b^*) - (RC_2^m - RC_1^m) = 0, \quad t_2^m < t_b^* < t_1^m \tag{10.4}$$

According to definition this root t_b^* is the required threshold time (10.1). □

Let us note that equation (10.4) implies the following expression for the threshold time

$$t_b^* = \frac{1}{(\lambda-\mu)}(\ln \frac{S_b}{(\lambda-\mu)} -$$
$$\ln(\frac{(S_a+S_b)}{(\lambda-\mu)}e^{-(\lambda-\mu)t_2^m} - \frac{S_a}{(\lambda-\mu)}e^{-(\lambda-\mu)t_1^m} + w(t_1^m) - w(t_2^m))) \tag{10.5}$$

We calculate now minimax and maximin of the profit function $R = R(t_a, t_b)$ (6.4), (6.5) with respect to the innovator and the market commercialization times t_a, t_b. We show that they coincide with each other and thus there exists a saddle point equilibrium in the game of the "light" innovator and the "heavy" market. The values V of the profit function calculated at the saddle commercialization times for the current game positions $(t, x(t), y(t), \dot{y}(t))$ generate the value function of the game $(t, x, y, \dot{y}) \to V(t, x, y, \dot{y})$. The value function $V(\cdot)$ contains the full relevant information for constructing the guaranteed optimal strategy of innovator. The guaranteed optimal strategy generates the dynamical process of assessing market, selecting scenarios and optimizing investment. In this process the innovator can guarantee the saddle equilibrium value of the profit function.

Let us calculate first the upper estimate of equilibrium – minimax of the profit function $R = R(t_a, t_b)$ (6.4), (6.5)

$$V^* = \min_{t_b \in [t_b^2, t_b^1]} \max_{t_a > t}\{\max\{\frac{S_1}{(\lambda-\mu)}e^{-(\lambda-\mu)t_a},$$
$$\frac{S_2}{(\lambda-\mu)}e^{-(\lambda-\mu)t_a} - \frac{S_b}{(\lambda-\mu)}e^{-(\lambda-\mu)t_b}\} - w(t_a)\} =$$
$$\min_{t_b \in [t_b^2, t_b^1]} \max\{\max_{t_a > t}\{\frac{S_1}{(\lambda-\mu)}e^{-(\lambda-\mu)t_a} - w(t_a)\},$$
$$\max_{t_a > t}\{\frac{S_2}{(\lambda-\mu)}e^{-(\lambda-\mu)t_a} - w(t_a)\} - \frac{S_b}{(\lambda-\mu)}e^{-(\lambda-\mu)t_b}\} =$$
$$\min_{t_b \in [t_b^2, t_b^1]} \max\{RC_1^m, RC_2^m - \frac{S_b}{(\lambda-\mu)}e^{-(\lambda-\mu)t_b}\} \tag{10.6}$$

Let us note that the maximum type function in the last relation can be presented in the piecewise smooth form

$$g(t_b) = \max\{RC_1^m, RC_2^m - \frac{S_b}{(\lambda-\mu)}e^{-(\lambda-\mu)t_b}\} = \begin{cases} RC_1^m & \text{if} \quad t < t_b \leq t_b^* \\ RC_2^m - r(t_b) & \text{if} \quad t_b \geq t_b^* \end{cases}$$

It is clear that function $g(t_b)$ is continuous, piecewise smooth and monotonically non-decreasing. Therefore, its maximum over t_b on the closed interval $[t_b^2, t_b^1]$ is reached at the left edge t_b^2 and the minimax value V^* is determined by formula

$$V^* = \max_{t_b \in [t_b^1, t_b^2]} g(t_b) = g(t_b^2) = \begin{cases} RC_1^m & \text{if} \quad t < t_b^2 \leq t_b^* \\ RC_2^m - r(t_b^2) & \text{if} \quad t_b^2 \geq t_b^* \end{cases} \tag{10.7}$$

We compute now the lower or guaranteed estimate of equilibrium – maximin of the profit function $R = R(t_a, t_b)$ (6.4), (6.5)

$$V_* = \max_{t_a > t} \min_{t_b \in [t_b^2, t_b^1]}\{\max\{\frac{S_1}{(\lambda-\mu)}e^{-(\lambda-\mu)t_a},$$
$$\frac{S_2}{(\lambda-\mu)}e^{-(\lambda-\mu)t_a} - \frac{S_b}{(\lambda-\mu)}e^{-(\lambda-\mu)t_b}\} - w(t_a)\} \tag{10.8}$$

The maximum type function in the last relation can be presented in the piecewise smooth form

$$h(t_a, t_b) = \max\{\frac{S_1}{(\lambda - \mu)} e^{-(\lambda-\mu)t_a}, \frac{S_2}{(\lambda - \mu)} e^{-(\lambda-\mu)t_a} - \frac{S_b}{(\lambda - \mu)} e^{-(\lambda-\mu)t_b}\} - w(t_a) =$$

$$= \begin{cases} RC(t_a, S_1) & \text{if} \quad t < t_b \leq t_a \\ RC(t_a, S_2) - r(t_b) & \text{if} \quad t_b \geq t_a \end{cases}$$

For the fixed time $t_a > t$ function $t_b \rightarrow h(t_a, t_b)$ is continuous, piecewise smooth and monotonically nondecreasing. Hence, its maximum over t_b on the closed interval $[t_b^2, t_b^1]$ is reached at the left edge t_b^2

$$H(t_a) = \max_{t_b \in [t_b^1, t_b^2]} h(t_a, t_b) = h(t_a, t_b^2) = \begin{cases} RC(t_a, S_1) & \text{if} \quad t < t_b^2 \leq t_a \\ RC(t_a, S_2) - r(t_b^2) & \text{if} \quad t_b^2 \geq t_a \end{cases}$$

Finally we can derive expression for maximin V_* of the profit function $R = R(t_a, t_b)$

$$V_* = \max_{t_a > t} H(t_a) = \max_{t_a > t} \max\{RC(t_a, S_1), RC(t_a, S_2) - r(t_b^2)\} =$$

$$= \max\{RC_1^m, RC_2^m - r(t_b^2)\} = \begin{cases} RC_1^m & \text{if} \quad t < t_b^2 \leq t_b^* \\ RC_2^m - r(t_b^2) & \text{if} \quad t_b^2 \geq t_b^* \end{cases} \tag{10.9}$$

Coincidence of expressions (10.7), (10.9) for maximin and minimax of the profit function $R(t_a, t_b)$ shows that in the game of the "light" innovator and the "heavy" market there exists the saddle equilibrium. We formulate this result in the following statement.

Proposition 10.2 *There exists the saddle equilibrium in the game of innovator and market. Its value V is determined by identity of maximin and minimax expressions*

$$V = V^* = V_* = \min_{t_b \in [t_b^2, t_b^1]} \max_{t_a > t} R(t_a, t_b) = \max_{t_a > t} \min_{t_b \in [t_b^2, t_b^1]} R(t_a, t_b) = R(t_a^e, t_b^e) =$$

$$= \begin{cases} RC_1^m & \textit{if} \quad t < t_b^2 \leq t_b^* \\ RC_2^m - r(t_b^2) & \textit{if} \quad t_b^2 \geq t_b^* \end{cases} \tag{10.10}$$

The equilibrium pair (t_a^e, t_b^e) of commercialization times which realizes the external maximum and minimum in maximin and minimax operators is determined by relations

$$t_a^e = \begin{cases} t_1^m & \textit{if} \quad t < t_b^2 < t_b^* \\ t_2^m & \textit{if} \quad t_b^2 \geq t_b^* \end{cases} \tag{10.11}$$

$$t_b^e = t_b^2 \tag{10.12}$$

Remark 10.1 *The value V (10.10) of the game between innovator and market as well as optimal strategy t_a^e (10.11) of selecting innovation scenarios depend on the current position of the innovation process $(t, x, y, \dot{y})$. The optimal feedback for dynamical selection of innovation scenarios $t_a^e = t_a^e(t, x, y, \dot{y})$ (10.11) together with the optimal investment plan $u^0 = u^0(t, x, t_a^e)$ (5.6) generate technological trajectories of innovator dynamics (2.2) which provide the optimal profit $V = V(t, x, y, \dot{y})$ even in presence of unfavorable technological development t_b^e (10.12) of the market.*

Let us indicate implementation of optimal strategy $t_a^e(\cdot)$ (10.11), $u^0(\cdot)$ (2.17) in trajectories $x^0(\cdot)$ of technological dynamics of innovator (1.1) operating in the market environment $(y(\cdot), \dot{y}(\cdot))$ (1.7).

Proposition 10.3 *Optimal strategy of selecting scenarios $t_a^e(\cdot)$ (10.11) and optimal investment plan $u^0(\cdot)$ (2.17) generate technological trajectories $x^0(\cdot)$ (1.1) which switch innovation scenarios no more than one time for any dynamic behavior $(y(\cdot), \dot{y}(\cdot))$ (1.7) of the market.*

Proof. Consider two cases. In the first case assume that at the current game position $(t, x, y, \dot{y})$ the fast scenario of innovation is more preferable than the slow one

$$t_b^2(t, y, \dot{y}) \geq t_b^*(t, x) \tag{10.13}$$

In this case the technological trajectory $x_2^0(\cdot)$, $x_2^0(t) = x$ is developing in the direction of the commercialization time t_2^m, $t_2^m < t_1^m$ (10.11) according to the optimal investment plan $u^0(\cdot)$ (2.17).

Let us fix the new current positions of innovator (τ, ξ), $\xi = x_2^0(\tau)$, $t \leq \tau \leq t_2^m$ and the market environment $(\eta, \dot{\eta}) = (y(\tau), \dot{y}(\tau))$. Position $(\eta, \dot{\eta})$ is generated by some realization of acceleration $v = v(s)$, $|v(s)| \leq v_0$, $t \leq s < \tau$ in dynamics (1.7), and all trajectories of the market environment which start from position $(\eta, \dot{\eta})$ are generated by accelerations $v = v(s)$, $s \geq \tau$ with the same restriction $|v(s)| \leq v_0$. Then according to Remark 9.1 the extremal commercialization times $t_b^2(\cdot)$ of the market environment satisfy inequalities

$$\begin{aligned} & t_b^2(t, y, \dot{y}) \leq t_b^2(\tau, \eta, \dot{\eta}), \quad t \leq \tau \leq t_2^m \\ & y(t) = y, \quad \dot{y}(t) = \dot{y}, \quad y(\tau) = \eta, \quad \dot{y}(\tau) = \dot{\eta} \end{aligned} \tag{10.14}$$

Let us estimate dynamics of the threshold time $t_b^*(\cdot)$. By definition we have

$$\begin{aligned} & r(t_b^*(t, x)) = RC_2^m(t, x) - RC_1^m(t, x) \\ & r(t_b^*(\tau, \xi)) = RC_2^m(\tau, \xi) - RC_1^m(\tau, \xi), \quad t \leq \tau \leq t_2^m \end{aligned} \tag{10.15}$$

Taking into account the dynamic programming principle (7.8) and definition of maximum profit values $RC_i^m(\cdot)$, $i = 1, 2$ we have the following chain of relations

$$\begin{aligned} & r(t_b^*(\tau, \xi)) = RC_2^m(\tau, \xi) - RC_1^m(\tau, \xi) = \\ & RC_2^m(\tau, \xi) - w(t, x, \tau, \xi) - (RC_1^m(\tau, \xi) - w(t, x, \tau, \xi)) = \\ & RC_2^m(t, x) - (RC_1^m(\tau, \xi) - w(t, x, \tau, \xi)) \geq \\ & RC_2^m(t, x) - RC_1^m(t, x) = r(t_b^*(t, x)) \end{aligned} \tag{10.16}$$

Since function $r(t_b) = (S_b/(\lambda - \mu))e^{-(\lambda-\mu)t_b}$ is monotonically decreasing with respect to time t_b, then dynamics of threshold time $t_b^*(\cdot)$ also satisfies the monotonicity condition

$$t_b^*(\tau, \xi) \leq t_b^*(t, x), \quad \xi = x_2^0(\tau), \quad t \leq \tau \leq t_2^m \tag{10.17}$$

Combining relations (10.14), (10.17) of the dynamic programming principle with the condition of the fast scenario (10.13) we obtain the inequality

$$t_b^2(\tau, \eta, \dot{\eta}) \geq t_b^2(t, y, \dot{y}) \geq t_b^*(t, x) \geq t_b^*(\tau, \xi) \tag{10.18}$$

which provides the condition of the fast scenario at the new game position $(\tau, \xi, \eta, \dot{\eta})$. Due to this condition the technological trajectory $x_2^0(\cdot)$, $x_2^0(\tau) = \xi$ will be developing in the same direction of the commercialization time t_2^m, $t_2^m < t_1^m$ (10.11) according to the optimal investment plan $u^0(\cdot)$ (2.17) without switching to the slow scenario t_1^m.

In the second case the innovator chooses the slow scenario of innovation at the current game position $(t, x, y, \dot{y})$

$$t_b^2(t, y, \dot{y}) < t_b^*(t, x) \tag{10.19}$$

and the technological trajectory $x_1^0(\cdot)$, $x_1^0(t) = x$ is developing in the direction of the commercialization time t_1^m, $t_1^m > t_2^m$ (10.11) according to the optimal investment plan $u^0(\cdot)$ (2.17).

Let us fix the new current positions of innovator (τ, ξ), $\xi = x_1^0(\tau)$, $t \leq \tau \leq t_2^m$ and the market environment $(\eta, \dot{\eta}) = (y(\tau), \dot{y}(\tau))$.

The extremal commercialization times $t_b^2(\cdot)$ of the market environment satisfy the same inequalities (10.14) as in the first case.

According to the dynamic programming principle (7.8) and definition of maximum profit values $RC_i^m(\cdot)$, $i = 1, 2$ one can obtain the following chain of relations for the dynamics of the threshold times

$$\begin{aligned}
&r(t_b^*(\tau,\xi)) = RC_2^m(\tau,\xi) - RC_1^m(\tau,\xi) = \\
&(RC_2^m(\tau,\xi) - w(t,x,\tau,\xi)) - (RC_1^m(\tau,\xi) - w(t,x,\tau,\xi)) = \\
&(RC_2^m(\tau,\xi) - w(t,x,\tau,\xi)) - RC_1^m(t,x) \leq \\
&RC_2^m(t,x) - RC_1^m(t,x) = r(t_b^*(t,x)) \qquad (10.20)
\end{aligned}$$

The monotonicity decreasing property of function $t_b \to r(t_b)$ implies the monotonicity condition of the threshold times $t_b^*(\cdot)$

$$t_b^*(\tau,\xi) \geq t_b^*(t,x), \quad \xi = x_1^0(\tau), \quad t \leq \tau \leq t_1^m \qquad (10.21)$$

Comparing relations (10.14) and (10.21) of the dynamic programming principle we find out that two situations may take place. The first situation is similar to the slow scenario condition (10.19)

$$t_b^2(\tau,\eta,\dot{\eta}) < t_b^*(\tau,\xi) \qquad (10.22)$$

and the technological trajectory $x_1^0(\cdot)$, $x_1^0(\tau) = \xi$ will be developing in the same direction of the commercialization time t_1^m, $t_1^m > t_2^m$ (10.11) according to the optimal investment plan $u^0(\cdot)$ (2.17) without switching to the fast scenario t_2^m.

The second situation is equivalent to the fast scenario condition (10.18)

$$t_b^2(\tau,\eta,\dot{\eta}) \geq t_b^*(\tau,\xi) \qquad (10.23)$$

and the innovator switches the slow scenario t_1^m to the fast scenario t_2^m (10.11) in the technological trajectory $x_1^0(\cdot)$, $x_1^0(\tau) = \xi$ to the technological trajectory $x_2^0(\cdot)$, $x_2^0(\tau) = \xi$. After this switching the innovator finds himself in the position similar to the first case and will be developing along the technological trajectory $x_2^0(\cdot)$, $x_2^0(\tau) = \xi$ into the direction of the fast commercialization time t_2^m according to the optimal investment plan $u^0(\cdot)$ (2.17) without switching scenarios. Thus only one switch of scenarios is possible in implementation of optimal strategy $t_a^e(\cdot)$ (10.11). □

Let us discuss the robustness properties of optimal strategy $t_a^e(\cdot)$ (10.11), $u^0(\cdot)$ (2.17) with respect to parameter v_0, $v_0 \geq 0$ which restricts the acceleration potential of the market. The optimal strategy $t_a^e(\cdot)$ (10.11) depends on parameter v_0 which directly influences on procedure of scenarios selection. In principle, the maximum acceleration v_0 is an uncertain parameter, and its accurate estimation in practice is scarcely possible. Therefore, the key role in estimation of uncertainty belongs to the sensitivity and robustness properties of the maximum result $R^m(\cdot)$ (6.9) of the profit function $R(\cdot)$ (6.4), and the optimal strategy $t_a^e(\cdot)$ (10.11) for selecting innovation scenarios. Assume that lower and upper estimates v_1, v_2 for parameter v_0 are obtained empirically

$$0 \leq v_1 \leq v_0 \leq v_2 \qquad (10.24)$$

In practice the innovator can use in all formulae for optimal results $R(\cdot)$ (6.4) and strategies $t_a^e(\cdot)$ (10.11) the low bound v_1 for the maximum acceleration $v_0 = v_1$. But in realization of optimal innovation trajectories the innovator can meet larger accelerations $v_0 = v_2$. The question is how optimal strategies $t_a^e(\cdot)$ (10.11) with the low level assessment of acceleration v_1 will react on more energetic development of market with higher acceleration v_2? The answer on this question is the following. If market is inert - the value of maximum accelerations v_2 is small enough, then optimal strategy $t_a^e(\cdot)$ (10.11) provides the small difference between results for different levels of acceleration v_1, v_2. Let us specify the notion of the market inertness. We make more precise condition $v_0 \ll R$ of the "heavy" market dynamics (8.21)

$$0 \le v_0 = v_1 \le v_2 \le \frac{(\sqrt{5}-1)}{2} \frac{R^2}{(\ln y_b - \ln y)} \tag{10.25}$$

Let us note that constant $(\sqrt{5}-1)/2$ introduces the idea of "golden section" into estimate (10.25).

For formulating the robustness result we introduce the following notations. By the symbol $R^m(v_1)$ (6.9) we denote the optimal value of the the profit function $R(\cdot)$ (6.4) in presence of the market with the low level of acceleration v_1. Accordingly the symbol $t_a^e(v_1)$ stands for the optimal strategy t_a^e (10.11) oriented on level v_1. Assuming that in realization of optimal strategy $t_a^e(v_1)$ the innovator can meet higher acceleration level v_2 of the market we denote by the symbol $R(v_2)$ the value of the profit function $R(\cdot)$ (6.4) which the innovator obtains on the corresponding trajectories.

The following statement indicates the robustness property of the optimal strategy $t_a^e(\cdot)$ (10.11) by showing the measure of sensitivity between values $R^m(v_1)$, $R(v_2)$ of the profit function $R(\cdot)$ (6.4).

Proposition 10.4 *Under condition (10.25) of the market inertness the measure of sensitivity between values $R^m(v_1)$, $R(v_2)$ of the profit function $R(\cdot)$ (6.4) provided by the optimal strategy $t_a^e(v_1)$ for different levels of the market acceleration v_1, v_2 is determined by the following estimate*

$$|R(v_2) - R^m(v_1)| \le \frac{1}{2} S_b e^{-(\lambda-\mu)t} \left(\frac{\dot{y}}{y}\right)^{-3} (\ln y_b - \ln y)^2 (v_2 - v_1) \tag{10.26}$$

Proof. Assume first that the dynamical system (2.2), (8.21) of technological innovation is located at initial position $(t, x, y, \dot{y})$ and optimal strategy $t_a^e(v_1)$ (10.11) prescribes selection of the slow scenario t_a^1. Assuming that the innovator will hold the slow scenario till the commercialization time $s = t_a^1$ we come to the conclusion that the value of its profit in this case does not depend on the acceleration level v_0 and equals to $R^m(v_1) = RC_1^m$ either for the low acceleration level $v_0 = v_1$, or for the high acceleration level $v_0 = v_2$.

Let us consider the second possibility when at initial position $(t, x, y, \dot{y})$ optimal strategy $t_a^e(v_1)$ (10.11) prescribes selection of the fast scenario t_a^2. If the innovator will not change this scenario along the optimal trajectory till the commercialization time $s = t_a^2$ then his profit will be equal either to the value $R^m(v_1) = RC_2^m - r(t_b^2(v_1))$ in the case of the low level v_1, or to the value $R(v_2) = RC_2^m - r(t_b^2(v_2))$ in the case of the high level v_2 of the market acceleration. Comparing values $R^m(v_1)$, $R(v_2)$ and taking into account inequality (9.18) we obtain the following estimate

$$|R(v_2) - R^m(v_1)| = R^m(v_1) - R(v_2) = r(t_b^2(t, v_2)) - r(t_b^2(t, v_1)) =$$

$$\frac{S_b}{(\lambda-\mu)}(e^{-(\lambda-\mu)t_b^2(t,v_2)} - e^{-(\lambda-\mu)t_b^2(t,v_1)}) = S_b e^{-(\lambda-\mu)\theta}(t_b^2(t,v_1) - t_b^2(t,v_2)) =$$

$$S_b e^{-(\lambda-\mu)\theta}(t_b^2)'(\bar{v})(v_1 - v_2) \leq \frac{1}{2} S_b e^{-(\lambda-\mu)t}\left(\frac{\dot{y}}{y}\right)^{-3}(\ln y_b - \ln y)^2(v_2 - v_1) \quad (10.27)$$

Here

$$0 \leq t_b^2(v_2) < \theta < t_b^2(v_1), \quad 0 \leq v_1 < \bar{v} < v_2$$

Let us note that inequality (10.27) obtained for trajectories of the fixed fast scenario t_a^2 is similar to the necessary sensitivity estimate (10.26). Let us prove using relations of the dynamic programming principle (7.8) and condition of the market inertness (10.25) that estimate (10.27) will be valid for trajectories with possible switches of scenarios. Let us consider two steps in switching scenarios along technological trajectories. Other switches can be treated analogously. Assume that behavior of the market trajectory $(y(s), \dot{y}(s))$, $s \geq t$ is such that optimal strategy $t_a^e(v_1)$ (10.11) switches the fast scenario $t_a^2(t,x)$ for the slow scenario $t_a^1(\tau_1, \xi_1)$ at position $(\tau_1, \xi_1, y(\tau_1), \dot{y}(\tau_1))$, $t < \tau_1 < t_a^2(t,x)$, $x(\tau_1) = \xi_1$, and further switches the slow scenario $t_a^1(\tau_1, \xi_1)$ for the fast scenario $t_a^2(\tau_2, \xi_2)$ at position $(\tau_2, \xi_2, y(\tau_2), \dot{y}(\tau_2))$, $\tau_1 < \tau_2 < t_a^1(\tau, \xi_1)$, $x(\tau_2) = \xi_2$. Such development is possible, for example, if observation shows first the high level of the market acceleration and then indicates the market deceleration. Taking into account that the real level of the market acceleration can be higher and equal to v_2 we indicate dynamics of innovator's profit in the two-step procedure of switching scenarios as follows. At position $(t, x, y, \dot{y})$ the guaranteed level of profit is determined by relation

$$f_0 = RC_2^m(t,x) - r(t_b^2(t,v_2))$$

at position $(\tau_1, \xi_1, y(\tau_1), \dot{y}(\tau_1))$ after the first switch of scenarios – by relation

$$f_1 = RC_1^m(\tau_1, \xi_1) - w(t, x, \tau_1, \xi_1)$$

and at position $(\tau_2, \xi_2, y(\tau_2), \dot{y}(\tau_2))$ after the second switch of scenarios – by relation

$$f_2 = RC_2^m(\tau_2, \xi_2) - r(t_b^2(\tau_2, v_2)) - w(t, x, \tau_1, \xi_1) - w(\tau_1, \xi_1, \tau_2, \xi_2)$$

Let us remind that symbols $w(t, x, \tau_1, \xi_1)$, $w(\tau_1, \xi_1, \tau_2, \xi_2)$ stand here for optimal investment expenditures of transferring the technological trajectory $x(\cdot)$ from position (t, x), $x(t) = x$ to position (τ_1, ξ_1), $x(\tau_1) = \xi_1$ and further to position (τ_2, ξ_2), $x(\tau_2) = \xi_2$.

We see that in the first switch at position (τ_1, ξ_1) the profit does not decrease $f_1 \geq f_0$ since according to the structure of the optimal strategy $t_a^e(v_1)$ (10.11) and the dynamic programming principle (7.8), (9.10) we have the following chain of inequalities

$$\begin{aligned} f_1 &= RC_1^m(\tau_1, \xi_1) - w(t, x, \tau_1, \xi_1) \geq RC_2^m(\tau_1, \xi_1) - r(t_b^2(\tau_1, v_1)) - w(t, x, \tau_1, \xi_1) = \\ & RC_2^m(t, x) - r(t_b^2(\tau_1, v_1)) \geq RC_2^m(t, x) - r(t_b^2(\tau_1, v_2)) \geq \\ & RC_2^m(t, x) - r(t_b^2(t, v_2)) = f_0 \end{aligned} \quad (10.28)$$

In the second switch at position (τ_2, ξ_2) the profit may decrease $f_2 \leq f_1$ since the structure of the optimal strategy $t_a^e(v_1)$ (10.11) and the dynamic programming principle (7.8) provides relations

$$\begin{aligned} f_2 &= RC_2^m(\tau_2, \xi_2) - r(t_b^2(\tau_2, v_2)) - w(t, x, \tau_1, \xi_1) - w(\tau_1, \xi_1, \tau_2, \xi_2) = \\ & RC_1^m(\tau_2, \xi_2) + r(t_b^2(\tau_2, v_1)) - r(t_b^2(\tau_2, v_2)) - w(t, x, \tau_1, \xi_1) - w(\tau_1, \xi_1, \tau_2, \xi_2) = \\ & RC_1^m(\tau_1, \xi_1) + r(t_b^2(\tau_2, v_1)) - r(t_b^2(\tau_2, v_2)) - w(t, x, \tau_1, \xi_1) = \\ & f_1 + r(t_b^2(\tau_2, v_1)) - r(t_b^2(\tau_2, v_2)) \leq f_1 \end{aligned} \quad (10.29)$$

According to relations (10.28), (10.29) in the first switch from the fast scenario to the slow scenario the innovator sacrifices the fast development for nondecrease of profit $f_1 \geq f_0$. In the second switch he selects the fast scenario under risky condition $f_2 \leq f_1$. This risky condition turns into equality $f_2 = f_1$ and provides the chain of dynamic optimality principle $f_0 \leq f_1 = f_2$ if level of the market acceleration is assessed properly $v_1 = v_2$. In case of more energetic market $v_2 > v_1$ it is necessary to check condition of the profit increase $f_2 \geq f_0$ in the two-step switching procedure despite of the intermediate decrease of profit $f_2 \leq f_1$. We will do this under condition (10.25) of the market inertness and the dynamic programming principle (7.8). Let us estimate the difference $(f_2 - f_0)$. We have according to the Lagrange mean value theorem the following chain of relations

$$\begin{aligned} & f_2 - f_0 = (f_2 - f_1) + (f_1 - f_0) = (r(t_b^2(\tau_2, v_1)) - r(t_b^2(\tau_2, v_2))) + (f_1 - f_0) \geq \\ & -(r(t_b^2(\tau_2, v_2)) - r(t_b^2(\tau_2, v_1))) + (r(t_b^2(t, v_2)) - r(t_b^2(\tau_1, v_1))) \geq \\ & (r(t_b^2(\tau_1, v_2)) - r(t_b^2(\tau_1, v_1))) - (r(t_b^2(\tau_2, v_2)) - r(t_b^2(\tau_2, v_1))) = \\ & r''_{\tau v}(\bar{\tau}, \bar{v})(\tau_1 - \tau_2)(v_2 - v_1) = -r''_{\tau v}(\bar{\tau}, \bar{v})(\tau_2 - \tau_1)(v_2 - v_1) \end{aligned} \tag{10.30}$$

Here

$$\tau_1 < \bar{\tau} < \tau_2, \quad v_1 < \bar{v} < v_2$$

It is remained to prove that the second derivative $r''_{\tau v}$ of the composite function $r(t_b^2(\tau, v))$ (6.6), (9.7) is nonpositive. For the first and second derivatives of the composite function $r(t_b^2(\tau, v))$ we have the following relations

$$r'_\tau = -S_b e^{-(\lambda-\mu)t_b^2}(t_b^2)'_\tau \tag{10.31}$$

$$r''_{\tau v} = -S_b e^{-(\lambda-\mu)t_b^2}((t_b^2)''_{\tau v} - (\lambda - \mu)(t_b^2)'_\tau (t_b^2)'_v) \tag{10.32}$$

Let us calculate the first derivatives of function $t_b^2(\tau, v)$

$$\begin{aligned} & (t_b^2)'_\tau = 1 - \frac{\dot{q}}{v} + \frac{(q\dot{q} - qv)}{v(q^2 + 2v(\ln y_b - p))^{1/2}} = \\ & (1 - \frac{\dot{q}}{v})(1 - \frac{q}{(q^2 + 2v(\ln y_b - p))^{1/2}}) \geq 0, \quad -v \leq \dot{q} \leq v \end{aligned} \tag{10.33}$$

$$\begin{aligned} & (t_b^2)'_v = -\frac{2(\ln y_b - p)(q^2 + 2v(\ln y_b - p))^{1/2}}{(q + (q^2 + 2v(\ln y_b - p))^{1/2})^2} = \\ & -\frac{1}{2}(t_b^2 - \tau)^2 (q^2 + 2v(\ln y_b - p))^{-1/2} \leq 0 \end{aligned} \tag{10.34}$$

Taking into account signs (10.33), (10.34) of the first derivatives we deduce that the second term $-(\lambda - \mu)(t_b^2)'_\tau (t_b^2)'_v$ in relation (10.32) for the second derivative $r''_{\tau v}$ is nonnegative.

Let us calculate the second derivative $(t_b^2)''_{\tau v}$

$$(t_b^2)''_{\tau v} = \frac{\dot{q}}{v^2}(1 - \frac{q}{(q^2 + 2v(\ln y_b - p))^{1/2}}) + (1 - \frac{\dot{q}}{v})\frac{q(\ln y_b - p)}{(q^2 + 2v(\ln y_b - p))^{3/2}} \tag{10.35}$$

Taking into account relation

$$\ln y_b - p = \frac{v}{2}(t_b^2 - \tau)^2 + q(t_b^2 - \tau)$$

we rewrite expression in denominators of the second derivative $(t_b^2)''_{\tau v}$ (10.35) as follows

$$q^2 + 2v(\ln y_b - p) = q^2 + 2v(\frac{v}{2}(t_b^2 - \tau)^2 + q(t_b^2 - \tau)) = (q + v(t_b^2 - \tau))^2$$

Substituting these expressions into formula (10.35) we obtain the following chain of relations

$$(t_b^2)''_{\tau v} = \frac{\dot{q}(t_b^2 - \tau)}{v(q + v(t_b^2 - \tau))} + (1 - \frac{\dot{q}}{v})\frac{q(t_b^2 - \tau)(v(t_b^2 - \tau)/2 + q)}{(q + v(t_b^2 - \tau))^3} =$$
$$\frac{(t_b^2 - \tau)}{(q + v(t_b^2 - \tau))}\left(\frac{\dot{q}}{v} + (1 - \frac{\dot{q}}{v})\frac{q(v(t_b^2 - \tau)/2 + q)}{(q + v(t_b^2 - \tau))^2}\right) =$$
$$\frac{(t_b^2 - \tau)}{(q + v(t_b^2 - \tau))^3}(q(q + \frac{v}{2}(t_b^2 - \tau)) + \dot{q}(t_b^2 - \tau)(v(t_b^2 - \tau) + \frac{3}{2}q)) \qquad (10.36)$$

Since acceleration $\dot{q}$ satisfies restrictions $-v \leq \dot{q} \leq v$ then minimum of the last expression (10.36) for the second derivative $(t_b^2)''_{\tau v}$ is reached at the negative value - deceleration $\dot{q} = -v$, and we can write the estimate

$$(t_b^2)''_{\tau v} \geq \frac{(t_b^2 - \tau)}{(q + v(t_b^2 - \tau))^3}(q^2 - qv(t_b^2 - \tau) - v^2(t_b^2 - \tau)^2) \qquad (10.37)$$

It is clear that the last expression is nonnegative if acceleration level v satisfies inequalities

$$0 \leq v \leq \frac{(\sqrt{5} - 1)}{2}\frac{q}{(t_b^2 - \tau)} \qquad (10.38)$$

with the "golden section" constant $(\sqrt{5} - 1)/2$.

Let us note that inequalities (10.38) are fulfilled if condition (10.25) holds. It means that condition (10.25) implies the necessary nonnegative sign of the second derivative

$$(t_b^2)''_{\tau v} \geq 0 \qquad (10.39)$$

Combining relation for the difference $(f_2 - f_0)$ (10.30) with formula for the second derivative $r''_{\tau v}$ (10.32) and signs of the first and second derivatives $(t_b^2)'_\tau$ (10.33), $(t_b^2)'_v$ (10.34), $(t_b^2)''_{\tau v}$ (10.39) we come to the conclusion that optimal strategy $t_a^e(v_1)$ (10.11) provides the dynamic programming property $(f_2 - f_0) \geq 0$ even in the case it meets the high level v_2, $v_2 \geq v_1 \geq 0$ of the market acceleration. □

Finally let us make an accent on the case of zero acceleration $v_1 = 0$ in the optimal strategy $t_a^e(v_1)$ (10.11).

Remark 10.2 *In the case when the optimal strategy* $t_a^e(v_1)$ *(10.11) exploits the zero acceleration* $v_1 = 0$ *the expected market commercialization time* $t_b^2(v_1)$ *coincides with the normal acceleration time* t_b^0 *which is calculated by the simple formula* $t_b^0 = (\dot{y}/y)^{-1}(\ln y_b - \ln y)$ *without any information about acceleration* v*. This strategy can be naturally interpreted as the look-ahead behavior based only on the information about the current market position of technology stock* y *and technology rate* $\dot{y}/y$*.*

Conclusion

In this paper we propose a dynamic model of optimal innovation policy which is constructed on the feedback principle. The innovator can make a decision on selection of the innovation scenario optimizing the commercialization time and the investment level in the dynamic procedure of evaluating market technology stock and technology rate. It is assumed that the innovator can energetically react on the situation at the market of technologies. The dynamics of the technology stock of the innovator is modeled by a pattern of a firm's R&D investment with the time-delay and obsolescence effects. Its trajectories can be controlled by the level of R&D investment which affects directly on the technology rate (the first derivative) of the innovator and in this sense represent the "light" dynamics. Since the market environment constitutes the large group of agents with different interests it is reasonable to describe its trajectories by the "heavy" dynamics in which the acceleration (the second derivative) is a small quantity. The dynamic econometric information about the market technology stock (the state) and the market technology rate (the first derivative) under expectation of the small technology acceleration (the second derivative) allows to estimate the ensemble of the market trajectories, to predict the market commercialization time and analyze its sensitivity. The profit function of the innovator is presented by the balance of its benefit from the commercialization of the new technology and the expenditures to the technology investment. The benefit part depends on the possible amount of sales which supposes two alternatives: the usual level of sales and the bonus level of sales due to the earlier technology innovation and market overtaking. Analyzing the market commercialization time in the dynamic identification process the innovator may select the "fast" or "slow" innovation scenario. For the selected scenario the innovator can optimize its investment level minimizing expenditures using the dynamic optimality principles. The distinctive feature of the model consists in the dynamic feedback interaction of three problems: (i) identification of the market technology trajectory; (ii) selection of the innovation scenario and optimization of the commercialization time; (iii) feedback optimization of the investment level. In this game interaction between the innovator and the market the optimal feedback strategy of innovation is constructed as a part of the saddle type equilibrium. Sensitivity and robustness properties of the optimal profit result and the optimal strategy for scenarios selection are studied.

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