RM-76-25

SPATIAL ZERO POPULATION GROWTH*

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April 1976

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*Prepared for presentation at the Annual Meeting of the Population Association of America, April 29-May 1, 1976, Montreal, Canada. (Parts of this paper first appeared in preliminary form as a section of RR-75-24 which is now out of print.)

Preface

Interest in human settlement systems and policies has been a critical part of urban-related work at IIASA since its inception. Recently this interest has given rise to a concentrated research effort focusing on migration dynamics and settlement patterns. Four sub-tasks form the core of this research effort:

I. the study of spatial population dynamics;

II. the definition and elaboration of a new research area called demometrics and its application to migration analysis and spatial population forecasting;

III. the analysis and design of migration and settlement policy;

IV. a comparative study of national migration and settlement patterns and policies.

This paper, the seventh in the spatial population dynamics series, examines the spatial evolution of multiregional population systems that converge to a zero rate of growth. It considers how stabilization of a national population might affect migration and local growth, demonstrating that where people choose to live in the future presents issues and problems that are potentially as serious as those posed by the number of children they choose to have. Related papers in the spatial population dynamics series, and other publications of the migration and settlement study, are listed on the back page of this report.

April 1976

Summary

Increasing concern about the sizes and growth rates of national populations has generated a vast literature dealing with the socioeconomic and environmental consequences of a reduction of fertility to replacement levels and the consequent evolution of national populations to a zero growth condition called stationarity. But where people choose to live in the future presents issues and problems that are potentially as serious as those posed by the number of children they choose to have. Yet the spatial implications of reduced fertility have received very little attention and we are, in consequence, ill-equipped to develop adequate responses to questions about the ways in which stabilization of a national population is likely to affect migration and local growth.

This paper considers some of the redistributional consequences of an immediate reduction of fertility levels to bare replacement levels. It adopts the mathematical apparatus that has been used by demographers to analyze the evolution of national populations to zero growth and extends it by introducing the spatial impact of internal migration. Such an extension shows that stabilization of the regional populations in a multiregional system will alter the relative contributions of natural increase and migration to regional growth. Regional age compositions will also be affected, and in ways that are strongly influenced by the age patterns of migration. Retirement havens, for example, will receive proportionately higher flows of inmigrants as a national population increases in average age, whereas destinations that previously attracted mostly younger migrants will receive proportionately fewer inmigrants. Finally, the redistributional effects of stabilization will depend in a very direct way on the redistribution of total births that is occasioned by fertility reduction. A numerical illustration using data for India vividly illustrates the staggering population congestion that lies ahead for that country's urban areas, even if fertility there were to decline to replacement level immediately.

"Sense of Helplessness"

by

Jonathan Power

International Herald Tribune

April 6, 1976, p. 4

Bombay — The doors of the jumbo jet swing open and the night air rushes in. The warm stench of industrial waste and human excrement overpowers the smell of jet exhaust. This is Bombay -- city of six million, industrial giant, metropolis of the western seaboard, ringed by forests of chemical plants, textile mills and engineering factories. Inside are the people, crushed together, man on man, woman on woman, child on child. Squeezed between humanity run the open sewers, full of the putrid outpourings of an overwrought civilization.

Bombay at the end of the 17th century had only 10,000 inhabitants. By 1872 it was 644,000. Today, the density of population is higher than Manhattan's and growing steadily. Only economic recession keeps the numbers down. For once the wheels of industrial society move, the people come in their hordes, leaving the economic insecurity of the villages for this city where they think there must be hope.

SPATIAL ZERO POPULATION GROWTH

Andrei Rogers

and

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Abstract

If age-specific death rates and replacement level birth rates both remain fixed, a population that is undisturbed by migration will ultimately evolve into a stationary "zero-growth" population. This paper explores the spatial consequences of such zero growth by examining how a sudden reduction of fertility to replacement levels affects the spatial evolution of a multiregional population whose constituent regional populations experience the redistributional effects of internal migration.

1. Introduction

The growing public concern about the sizes and growth rates of national populations has generated a vast literature on the social, economic, and environmental impacts of a reduction of fertility to replacement levels and the consequent evolution of national populations to a zero growth condition. But where people choose to live in the future presents issues and problems that are potentially as serious as those posed by the number of children they choose to have. Yet the spatial implications of reduced fertility have received relatively little attention and we are, in consequence, ill-equipped to develop adequate responses to questions such as the one recently posed by the Commission on Population Growth and the American Future[1]:

> "How would stabilization of the national population affect migration and local growth"? (C.P.G.A.F., 1972, p. 13).

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[1]A notable exception is the work of Peter Morrison, who concludes: "...demographic processes interact in subtle and often complex ways, and the mechanisms by which declining fertility would influence population redistribution are only partially understood. Their elucidation can furnish a clearer picture of how and under what circumstances population redistribution can be influenced by public policy," (Morrison, 1972, p. 547).

The Commission observes that zero growth for the nation will mean an average of zero growth for local areas. This, of course, still allows for the possibility of nonzero growth in particular localities. Thus spatial zero growth, like temporal zero growth, may be viewed either as a condition that ultimately prevails uniformly or one that exists only because of a fortuitous balancing of regional rates of positive growth, of zero growth, and of decline. Since no obvious advantages arise from the latter case, demographers quite naturally have viewed the attainment of temporal zero growth in the long-run in terms of an indefinite continuation of temporal zero growth in the short-run. We follow this tradition in this paper and view the attainment of spatial zero growth in the long-run in terms of temporal zero growth within each region of a closed multiregional population system whose long-run behavior is defined by the multiregional Lotka equation (Rogers, 1975, Ch. 4):

$$\{\underset{\sim}{B}(t)\} = \int_{\alpha}^{\beta} \underset{\sim}{m}(a)\,\underset{\sim}{\ell}(a)\,\{\underset{\sim}{B}(t-a)\}da \quad .$$

In consequence, we confine our attention here to the evolution of a particular subset of stationary populations, called spatial zero growth populations, i.e., stable multiregional populations that have a zero growth rate. Thus we augment the usual twin assumptions of a fixed mortality schedule and a fixed fertility schedule, set at replacement level, with the assumption of a fixed migration schedule. Multiregional populations subjected to such regional growth regimes ultimately assume a persisting zero rate of growth in every region and exhibit zero growth both over time and over space.

2. Characteristics of Spatial Zero Growth Populations

If age-specific death rates are fixed and replacement level birth rates remain unchanged, a population that is closed to migration will ultimately evolve into a stationary population. The characteristics of such a population are well known. The number of individuals at any age would remain fixed, and the total number of deaths would exactly equal the total number of births. Because mortality risks would be relatively low from just after birth through middle age, the age composition of such a population would be nearly rectangular until ages 50 or 60, tapering much more rapidly thereafter with the increase in death rates among the older population.

The maintenance of a stationary population requires that parents have only as many children as are needed to maintain a fixed number of births every year. This means, for example, that a 1000 women must on the average produce a 1000 baby girls during their lifetime. Moreover, since some girls will not survive to become mothers, those who do must have slightly more than 1000 daughters in order to compensate for those who don't. Hence the gross reproduction rate (GRR) must be greater than unity by an amount just sufficient to maintain a unit level of <u>net</u> reproduction. For example, about 97 to 98 percent of women in the United States today survive to the principal ages of childbearing. Consequently, those who do must have approximately 1.027 daughters, on the average, as they pass through the child-bearing ages. In other words, the GRR must be 1.027 in order for the NRR to be unity[2].

The net reproduction rate, like the total fertility rate and the gross reproduction rate, summarizes the fertility experience of a population of all ages during a single year as if it were the experience of a single cohort that passed through all ages of childbearing. It is a hypothetical value that treats cross sectional data as if it were longitudinal data, in

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[2]Because there are usually about 105 baby boys born for every 100 baby girls, mothers in a stationary population of males and females would need to have a total rate of reproduction about three percent more than twice 1.027. In this way we obtain the total fertility rate of 2.11 used, for example, in the United States Census Bureau projections, (U.S. Bureau of the Census, 1972).

order to estimate the number of daughters that would be born per woman subjected to specified age-specific risks of fertility and mortality. A commonly used procedure for obtaining NRR is to multiply each female age-specific fertility rate, $m(a)$ say, by the corresponding probability of surviving from birth to that age $\ell(a)$, integrating the product over all ages of childbearing:

$$\text{NRR} = \int_{\alpha}^{\beta} \ell(a)m(a)da \equiv R(0) \quad .$$

Since NRR may be viewed as the zeroth moment of the net maternity function, it usually is denoted by $R(0)$, a notation which we shall now adopt.

Total births in a stationary <u>multiregional</u> population must, of course, equal total deaths. However, because of the redistributive effects of migration, total births in any particular region need not equal total deaths in that region. This can be readily demonstrated by means of the accounting identity connecting regional stable intrinsic rates:

$$r_j = b_j - d_j - o_j + i_j = b_j - d_j + n_j \quad .$$

Setting $r_j = 0$ gives the fundamental relationship that must hold in every region of a spatial zero growth population:

$$\hat{b}_j = \hat{d}_j - \hat{n}_j \quad ,$$

where the caret is introduced to designate a stationary population. Thus only if the net migration rate is zero will regional births equal regional deaths in a spatial zero growth population.

The maintenance of a spatial zero growth population requires that the total number of births in every region remain constant over time. Thus we may substitute the trial solution vector $\{\underset{\sim}{B}(t)\} = \{\hat{\underset{\sim}{Q}}\}$ into the multiregional Lotka equation to find

$$\{\hat{\underset{\sim}{Q}}\} = \left[\int_{\alpha}^{\beta} \hat{\underset{\sim}{m}}(a)\underset{\sim}{\ell}(a)da\right]\{\hat{\underset{\sim}{Q}}\} = \hat{\underset{\sim}{R}}(0)\{\hat{\underset{\sim}{Q}}\} \quad , \tag{1}$$

where carets are once again used to distinguish stationary population measures, and the element in the i^{th} row and the j^{th} column of $\hat{\underset{\sim}{R}}(0)$ is the stationary regional net reproduction rate <u>in region i</u> of women born in region j:

$$_j\hat{R}_i(0) = \int_{\alpha}^{\beta} {}_j\ell_i(a)\hat{m}_i(a)da \quad .$$

Equation (1) shows that for a spatial zero growth population to be maintained, the dominant characteristic root of the matrix $\hat{\underset{\sim}{R}}(0)$ must be unity. Consequently a reduction of fertility to replacement level may be interpreted as a reduction of the elements of $\underset{\sim}{m}(a)$ to a level that reduces the dominant characteristic root of a given net reproduction matrix $\underset{\sim}{R}(0)$ to unity. Such an operation transforms $\underset{\sim}{m}(a)$ to $\hat{\underset{\sim}{m}}(a)$ and $\underset{\sim}{R}(0)$ to $\hat{\underset{\sim}{R}}(0)$.

Stabilization of the regional populations in a multiregional system will alter the relative contributions of natural increase and migration to regional growth. Regional age compositions will also be affected, and in ways that are strongly influenced by the age patterns of migration. Retirement havens such as San Diego and Miami, for example, will receive proportionately higher flows of inmigrants as the national population increases in average age; destinations that previously attracted mostly younger migrants, on the other hand, will receive proportionately fewer inmigrants. Finally, as we demonstrate in the next section, the redistributional effects of stabilization depend in a very direct way on the redistribution of total births that is occasioned by the reduction in fertility.

3. Alternative Spatial Paths to Zero Population Growth

In his paper for the Commission on Population Growth and the American Future, Ansley Coale (1972) considers three alternative paths to a stationary population: (1) maintaining births constant at the levels recorded in 1970; (2) moving to a replacement level of fertility either immediately or in the very near future; and (3) reducing childbearing levels such that total population size is held fixed beginning immediately. He finds only slight differences between the first two alternatives and rejects the third as infeasible since it would require an immediate decline in the birth rate of almost 50 percent. We shall therefore confine our attention to Coale's second alternative path and will explore a few of its spatial ramifications.

Imagine a multiregional population system growing at some positive rate of growth, i.e., exhibiting a net reproduction matrix $\underset{\sim}{R}(0)$ with a dominant characteristic root $\lambda_1[\underset{\sim}{R}(0)]$ that is greater than unity. If the rate of childbearing in each region of this multiregional population system were immediately reduced such that every woman born in that region would now have a net reproduction rate of unity, then

$${}_i\hat{R}(0) = \sum_{j=1}^{m} {}_i\hat{R}_j(0) = 1$$

or, in matrix form,

$$\hat{\underset{\sim}{R}}(0)'\{\underset{\sim}{1}\} = \{\underset{\sim}{1}\} \quad , \tag{2}$$

where the prime denotes transposition.

Following the normal practice in single-region exercises of this kind, assume that the reduction of the fertility of each regional cohort of women is achieved by reducing each region's age-specific fertility rates by the same fixed proportion, γ_i say. Then

$${}_i\hat{R}(0) = \sum_{j=1}^{m} {}_i\hat{R}_j(0) = \sum_{j=1}^{m} \int_{\alpha}^{\beta} {}_i\ell_j(a)\gamma_j m_j(a)da = \sum_{j=1}^{m} \gamma_j \, {}_iR_j(0) = 1$$

and

$$\hat{\underset{\sim}{R}}(0) = \underset{\sim}{\gamma}\underset{\sim}{R}(0) \quad , \tag{3}$$

where $\underset{\sim}{\gamma}$ is a diagonal matrix of fertility adjustment factors. Substituting (3) into (2) gives

$$\underset{\sim}{R}(0)'\underset{\sim}{\gamma}\{\underset{\sim}{1}\} = \{\underset{\sim}{1}\} \quad ,$$

whence

$$\{\underset{\sim}{\gamma}\} = [\underset{\sim}{R}(0)']^{-1} \{\underset{\sim}{1}\} \quad . \tag{4}$$

The adjustment factor γ_i may be re-expressed in a way that offers additional insights into its properties. According to (1)

$$\hat{Q}_i = \sum_{j=1}^{m} {}_j\hat{R}_i(0)\hat{Q}_j \quad .$$

Dividing both sides of the equation by $\hat{Q}_i$ gives

$$1 = \sum_{j=1}^{m} \frac{\hat{Q}_j}{\hat{Q}_i} \, {}_j\hat{R}_i(0) = \hat{R}_i(0) \quad , \text{ say,}$$

where $\hat{R}_i(0)$ may be defined to be the net reproduction rate of women living in region i (as distinguished from the net reproduction rate of those born in region i). But

$$\hat{R}_i(0) = \sum_{j=1}^{m} \frac{\hat{Q}_j}{\hat{Q}_i} \gamma_i \, {}_jR_i(0) = \gamma_i R_i(0) = 1 \quad ,$$

where we define $R_i(0) = \sum_{j=1}^{m} \frac{\hat{Q}_j}{\hat{Q}_i} \, {}_jR_i(0)$. Hence

$$\gamma_i = \frac{1}{R_i(0)} \quad ,$$

and

$$\{\underset{\sim}{R}(0)\} = \underset{\sim}{\gamma}^{-1} \{\underset{\sim}{1}\} = \hat{\underset{\sim}{Q}}^{-1} \underset{\sim}{R}(0) \{\hat{\underset{\sim}{Q}}\} \quad , \tag{5}$$

where $\{\hat{\underset{\sim}{Q}}\} = \hat{\underset{\sim}{Q}} \{\underset{\sim}{1}\}$ and $\hat{\underset{\sim}{Q}}$ is a diagonal matrix.

The vector $\{\hat{\underset{\sim}{Q}}\}$ in (5) is the characteristic vector associated with the unit dominant characteristic root of $\hat{\underset{\sim}{R}}(0)$ and denotes the total number of births in each region of a spatial zero growth population. The proportional allocation of total births that it defines is directly dependent on the transformation that is applied to change $\underset{\sim}{R}(0)$ to $\hat{\underset{\sim}{R}}(0)$, a particular example of which is given by (3). Since in a spatial zero growth population the regional stationary equivalent population $\hat{Y}_i$ is equal to the quotient $\hat{Q}_i/\hat{b}_i$, we see that the different ways in which $\underset{\sim}{R}(0)$ is transformed into $\hat{\underset{\sim}{R}}(0)$ become, in fact, alternative "spatial paths" to a stationary multiregional population.

A numerical illustration may be instructive at this point. The net reproduction behavior of the urban and rural female populations of the United States in 1968 is crudely approximated by the net reproduction matrix

$$\underset{\sim}{R}(0) = \begin{bmatrix} {}_uR_u(0) & {}_rR_u(0) \\ {}_uR_r(0) & {}_rR_r(0) \end{bmatrix} = \begin{bmatrix} 0.85 & 0.45 \\ 0.25 & 0.90 \end{bmatrix} \quad , \tag{6}$$

where, for example, ${}_rR_u(0) = 0.45$ denotes the net reproduction rate in urban areas of rural-born women. In other words, under the regime of growth observed in 1968, each woman born in rural areas will, on the average, replace herself in the succeeding generation by 1.35 daughters, one third of whom will be born in urban areas. Urban-born women, on the other hand, have a lower net reproduction rate, i.e., ${}_uR(0) = 1.10 < {}_rR(0)$, which when combined with the net reproduction rate of rural-born women gives the United States female population an overall net reproduction rate of $\lambda_1[\underset{\sim}{R}(0)] = 1.21$, where $\lambda_1[\underset{\sim}{R}(0)]$ is the dominant characteristic root of the net reproduction matrix $\underset{\sim}{R}(0)$.

About 73 percent of the 1968 United States female population lived in urban areas. A projection to stable growth under the 1968 growth regime reduces that allocation to approximately 2/3 of the stable population and yields an intrinsic growth rate of slightly under one percent per annum. What would be the spatial allocation under a similar projection, but one in which fertility was immediately reduced to a level of one daughter per urban- or rural-born woman? To obtain an estimate of the regional shares in the stationary population that would evolve out of such a projection we need first to derive the fertility adjustment factors γ_u and γ_r, respectively. Calculations carried out using (4) give $\gamma_u = 1$ and $\gamma_r = 3/5$, whence

$$\hat{\underset{\sim}{R}}(0) = \begin{bmatrix} 0.85 & 0.45 \\ 0.15 & 0.55 \end{bmatrix} \tag{7}$$

Note that both groups of women now exhibit unit rates of net reproduction, and observe that the dominant characteristic root of $\hat{\underset{\sim}{R}}(0)$ is unity.

The characteristic vector associated with the unit dominant characteristic root of $\hat{\underset{\sim}{R}}(0)$ indicates that 3/4 of the total births in the spatial zero growth population will occur in urban areas. Since $\hat{Q}_i = \hat{b}_i \hat{Y}_i$,

$$\frac{\hat{Y}_i}{\hat{Y}_j} = \frac{\hat{Q}_i}{\hat{Q}_j} \cdot \frac{\hat{b}_j}{\hat{b}_i} , \tag{8}$$

a result that equates the ratio of stationary regional shares to the corresponding stationary birth ratio times the <u>reciprocal</u> of the corresponding ratio of regional intrinsic birth rates. Since the stationary birth ratio of urban to rural births is given by $\hat{\underset{\sim}{R}}(0)$ to be 3 (i.e., 3/4 to 1/4) and because the ratio of rural to urban intrinsic birth rates is likely to be close to unity (it comes out to be 1.07 in the projection) we conclude that about 3/4 of the spatial zero growth population will reside in urban areas[3].

We have observed earlier that the proportional allocation of total births in a spatial zero growth population depends directly on the transformation by which $\underset{\sim}{R}(0)$ is changed to $\hat{\underset{\sim}{R}}(0)$. Alternative transformations are in effect alternative spatial paths to such a population inasmuch as they lead to alternative spatial allocations of the total multiregional population. This can be easily illustrated by considering an alternative fertility reduction program: one which reduces the aggregate net reproduction rate to unity by reducing each regional fertility schedule by the same proportion, γ say. That is, let

$$\hat{\hat{\underset{\sim}{R}}}(0) = \gamma \underset{\sim}{R}(0) \quad , \text{ where} \tag{9}$$

$$\gamma = \frac{1}{\lambda_1 [\underset{\sim}{R}(0)]} \quad .$$

In the context of our numerical illustration this means that the fertility of urban-born women would now be reduced to below replacement levels whereas that of rural-born women would be permitted to exceed replacement fertility levels. That is,

$$\hat{\hat{\underset{\sim}{R}}}(0) = \frac{1}{1.211} \underset{\sim}{R}(0) = \begin{bmatrix} 0.70 & 0.37 \\ 0.21 & 0.74 \end{bmatrix} \tag{10}$$

and ${}_u\hat{\hat{R}}(0) = 0.91 \quad , \quad {}_r\hat{\hat{R}}(0) = 1.11 \quad .$

The spatial implications of this alternative path to a spatial zero growth population are quite different, as can be seen by calculating the characteristic vector associated with the unit dominant characteristic root of $\hat{\hat{\underset{\sim}{R}}}(0)$. The characteristic vector in this case allocates approximately 55 percent of total multiregional births to urban areas. Since the ratio

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[3]This result of course refers to regional designations that existed in 1968. In light of the continuing urbanization of rural regions it is probably a conservative estimate.

of rural to urban intrinsic birth rates would now be somewhat higher than unity, however, we should expect a correspondingly higher concentration in urban areas than is indicated by this allocation of total births.

Table 1 summarizes the principal numerical results of this section, augmenting them with a hypothetical example whose numerical values more clearly illustrate the two alternative spatial paths described above.

Table 1: Examples of Two Alternative Spatial Paths to Zero Population Growth

I. Data: USA, 1968 (urban-rural)

$$\underset{\sim}{R}(0) = \begin{bmatrix} {}_u R_u(0) & {}_r R_u(0) \\ {}_u R_r(0) & {}_r R_r(0) \end{bmatrix} = \begin{bmatrix} 0.85 & 0.45 \\ 0.25 & 0.90 \end{bmatrix} \qquad \lambda_1[\underset{\sim}{R}(0)] = 1.211$$

Alternative A:

$$\{\underset{\sim}{\gamma}\} = [\underset{\sim}{R}(0)']^{-1}\{\underset{\sim}{1}\} = [1 \quad 3/5]$$

$$\hat{\underset{\sim}{R}}(0) = \underset{\sim}{\gamma}\underset{\sim}{R}(0) = \begin{bmatrix} 0.85 & 0.45 \\ 0.15 & 0.55 \end{bmatrix}$$

$$\lambda_1[\hat{\underset{\sim}{R}}(0)] = 1 \qquad \{\hat{\underset{\sim}{Q}}\} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Alternative B:

$$\gamma = \frac{1}{\lambda_1[\underset{\sim}{R}(0)]} = \frac{1}{1.211}$$

$$\hat{\hat{\underset{\sim}{R}}}(0) = \gamma\underset{\sim}{R}(0) = \begin{bmatrix} 0.70 & 0.37 \\ 0.21 & 0.74 \end{bmatrix}$$

$$\lambda_1[\hat{\hat{\underset{\sim}{R}}}(0)] = 1 \qquad \{\hat{\hat{\underset{\sim}{Q}}}\} = \begin{bmatrix} 1.22 \\ 1 \end{bmatrix}$$

II. Data: Hypothetical

$$\underset{\sim}{R}(0) = \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1 \end{bmatrix} \qquad \lambda_1[\underset{\sim}{R}(0)] = 1\,\frac{1}{4} = \frac{5}{4}$$

Alternative A: $\{\underset{\sim}{\gamma}\}' = [6/5 \quad 2/5]$

$$\hat{\underset{\sim}{R}}(0) = \underset{\sim}{\gamma}\underset{\sim}{R}(0) = \begin{bmatrix} 9/10 & 3/5 \\ 1/10 & 2/5 \end{bmatrix}$$

$$\lambda_1[\hat{\underset{\sim}{R}}(0)] = 1 \qquad \{\hat{\underset{\sim}{Q}}\} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$$

Alternative B: $\gamma = 4/5$

$$\hat{\hat{\underset{\sim}{R}}}(0) = \gamma\underset{\sim}{R}(0) = \begin{bmatrix} 3/5 & 2/5 \\ 1/5 & 4/5 \end{bmatrix}$$

$$\lambda_1[\hat{\hat{\underset{\sim}{R}}}(0)] = 1 \qquad \{\hat{\hat{\underset{\sim}{Q}}}\} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

4. The Momentum of Spatial Zero Population Growth

Differences between most observed population age compositions and those of stationary populations make it virtually impossible to attain zero growth in the near future. A closed population's birth rate and growth rate depend on its fertility schedule and its age composition. Consequently whether and how long a population continues to grow after achieving a net reproduction rate of unity depends on that population's age composition and its degree of divergence from that of a stationary population. The ratio by which the ultimate stationary population exceeds a current population is the "momentum" of that population, a quantity that recently has been given analytical content by Keyfitz (1971) who shows that the momentum of a population numbering K individuals and having an age composition close to stable may be approximated by the expression

$$\frac{\hat{Y}}{K} = \frac{b\ e(0)}{r\ \mu}\left(\frac{R(0) - 1}{R(0)}\right) , \tag{11}$$

where b is the birth rate, r the rate of growth, e(0) the expectation of life, and R(0) the net reproduction rate, all measured before the drop in fertility, and μ is the mean age of childbearing afterward[4]. The derivation assumes that the population is approximately stable before the decline in fertility so that b and r are intrinsic stable rates of the initial (nonstationary) regime of growth.

Straightforward population projection calculations may be used to obtain the future population that evolves from any

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[4]Observe that (11) also may be expressed as

$$\hat{Y} = e(0)\hat{Q} ,$$

where

$$\hat{Q} = \frac{bK}{r\mu}\left(\frac{R(0) - 1}{R(0)}\right) . \tag{12}$$

particular observed or hypothetical regime of growth. Therefore (11) is not needed to obtain a numerical estimate of an ultimate stationary population. However Keyfitz's simple momentum formula gives us an understanding of the population dynamics that are hidden in the arithmetical computations of a population projection. It identifies in an unambiguous way the five parameters of a current population that determine the size of the ultimate stationary population.

In order to evaluate the accuracy of Keyfitz's momentum formula we have carried out a two-region projection of the 1968 United States female population on the assumption that age-specific fertility rates in each region drop immediately to replacement levels. Table 2 shows that the ultimate total stationary multiregional population exceeds its 1968 level by about a third. Equation (11) estimates the momentum to be about the same[5].

$$\frac{\hat{Y}}{K(1968)} = \frac{0.01878}{0.00432} \cdot \frac{74.3}{26.3} \frac{(1.12-1)}{1.12} = 1.31 \quad .$$

A multiregional generalization of Keyfitz's momentum formula may be shown to be

$$\left\{\frac{\hat{Y}}{K}\right\} = \frac{1}{r} \underset{\sim}{e}(0) \; \underset{\sim}{R}(1)^{-1} \left[\underset{\sim}{R}(0) - \underset{\sim}{\Psi}(r)\right] \left\{\underset{\sim}{b}\right\} \quad , \tag{13}$$

where $\underset{\sim}{R}(1)$ is a matrix with elements ${}_jR_i(1) = {}_j\mu_i \cdot {}_jR_i(0)$, $\underset{\sim}{e}(0)$ is a matrix of regional expectations of life at birth, and where total multiregional stable births Q are allocated to regions according to the stationary proportions defined by the characteristic vector associated with the unit characteristic root of $\underset{\sim}{\hat{R}}(0)$. Evaluating (13) with the same two-region data that produced the

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[5]Unlike Keyfitz we do not use the observed birth rate but divide total stable births Q by the current population, i.e., b = Q/K(1968) = 1,920,961/102,276,992 = 0.01878. That is why our approximation is more accurate than similar ones reported by Keyfitz.

Table 2: Relations Under Stability and Spatial Zero Growth: United States Females, 1968*

Variables	1. West	2. Rest of U.S.	Total
K(1968)	17,264,114	85,012,878	102,276,992
r		0.00432	
Y	26,989,870	94,302,612	121,292,482
Q	410,412	1,510,549	1,920,961
$\hat{r}$		0.00000	
$\hat{Y}$	31,013,997	103,625,370	134,639,366
$\hat{Q}$	406,374	1,399,361	1,805,735

*Spatial zero growth projection carried out under "Alternative A," i.e., one baby girl per woman.

above momentum of 1.31 gives:

$$\begin{bmatrix} \dfrac{\hat{Y}_1}{K_1(1968)} \\ \dfrac{\hat{Y}_2}{K_2(1968)} \end{bmatrix} = \frac{1}{0.00432} \begin{bmatrix} 52.39 & 6.95 \\ 23.10 & 67.34 \end{bmatrix} \begin{bmatrix} 0.0509 & -0.0045 \\ -0.0161 & 0.0382 \end{bmatrix} \begin{bmatrix} 0.0833 & 0.0098 \\ 0.0350 & 0.1108 \end{bmatrix}$$

$$\times \begin{bmatrix} 0.02377 \\ 0.01777 \end{bmatrix} = \begin{bmatrix} 1.74 \\ 1.23 \end{bmatrix} . \qquad (14)$$

A comparison of these regional momenta with those found by the population projection summarized in Table 2 reveals that the quality of approximation afforded by (12) is adequate (1.74 and 1.23 as approximations of 1.80 and 1.22, respectively).

Equation (13) is not as practically useful as its single-region counterpart because it is much more difficult to come up with accurate guesses or estimates of the values taken on by the many parameters. Thus a more effective procedure may be to first estimate the ultimate size of the total stationary multiregional population, $\hat{Y}$, using Keyfitz's formula and then rely on (8) to allocate that total to the various regions of the multiregional system. But such a procedure requires estimates of the stationary birth rate ratios, and these are hard to come by. A more feasible alternative is to first estimate the ultimate size of the total stationary births, $\hat{Q}$, by means of (12); next, distribute that total among the various regions according to the allocation defined by the characteristic vector associated with the unit root of $\hat{\underset{\sim}{R}}(0)$; and then premultiply the resulting vector by $\underset{\sim}{e}(0)$ to find $\{\hat{\underset{\sim}{Y}}\}$, i.e.,

$$\{\hat{\underset{\sim}{Y}}\} = \underset{\sim}{e}(0)\{\hat{\underset{\sim}{Q}}\} \quad , \qquad (15)$$

where $\{\hat{\underset{\sim}{Q}}\}$ is scaled to sum to $\hat{Q}$, the latter coming from the single-region calculation in (12).

Returning to the example used in (14) we may establish that

$$\begin{bmatrix} \hat{Y}_1 \\ \hat{Y}_2 \end{bmatrix} = \begin{bmatrix} 52.39 & 6.95 \\ 23.10 & 67.34 \end{bmatrix} \begin{bmatrix} 406,374 \\ 1,399,361 \end{bmatrix} = \begin{bmatrix} 31,013,997 \\ 103,625,370 \end{bmatrix}$$

and with this result obtain the same regional momenta as before.

Equation (15) shows that the geographical distribution of a spatial zero growth population depends very fundamentally on three matrices: $\underset{\sim}{e}(0)$, $\underset{\sim}{R}(0)$, and $\underset{\sim}{\gamma}$. The first describes the multiregional levels of mortality and migration; the second sets out the multiregional net reproduction patterns before the decline in fertility; and the third defines the particular "spatial path" by which fertility is reduced. The product $\underset{\sim}{\gamma}\underset{\sim}{R}(0)$ gives $\underset{\sim}{\hat{R}}(0)$, whose characteristic vector associated with the unit root and scaled to sum to $\hat{Q}$ is $\{\underset{\sim}{\hat{Q}}\}$.

Equation (15) also may be used to dramatically underscore our earlier assertion that "where people choose to live in the future presents issues and problems that are potentially as serious as those posed by the number of children they choose to have." Consider, for example, the projection to zero growth of India's population that was recently carried out by Norman Ryder on the basis of the following assumptions:

> "To simplify the task of projecting the population of India, we make the following assumptions: it is a stable population with a growth rate $r = +0.025$ and survival functions corresponding to those labelled "West, level 13" (for which the female and male expectations of life at birth are 50 and 47.114, respectively) in the Coale/Demeny collection; the mean age of (gross) maternity $m = 29$; the ratio of male to female births $k = 1.05$; and the current population size is 600 million." (Ryder, 1974, p. 6)

From these assumptions it follows that the initial number of female births per annum $B(t) = 12.156$ million, $R(0) = 2.019$, and $\mu = 28.672$. Applying (12), Ryder finds a $\hat{Q}$ of 8.558 million and a zero growth population of 851 million. He then shows that if India's survival level rises to $e(0) = 70$ years for females and $e(0) = 66.023$ for males, and

> "if replacement level fertility takes 40 years to achieve and the mean age of gross reproduction declines from 29 to 27, the ultimate female birth cohort size will be...15.029 million. Given that value,...the consequent ultimate population size is 2.094 billion." (Ryder, 1974, p. 7)

Ryder concludes that "the thought of a population of 2.1 billion for India is staggering" and goes on to examine in what respects the components of his projection may be modifiable.

There is no question but that a 2.1 billion population for India is staggering. What is even more mind-boggling, however, is that approximately 70 to 80 percent of this total is likely to be found in that nation's already teeming and over-congested urban areas (the current figure is 20 percent). To show this, we need only to introduce a few additional assumptions and then apply Equation (15). Specifically, assume that life expectancy is 55 years in urban areas and 45 in rural areas, with the migration pattern being such that

$$\underset{\sim}{e}(0) = \begin{bmatrix} {}_ue_u(0) & {}_re_u(0) \\ {}_ue_r(0) & {}_re_r(0) \end{bmatrix} = \begin{bmatrix} 50 & 20 \\ 5 & 25 \end{bmatrix} .$$

Assume, further, that the spatial pattern of net reproduction after the drop to replacement fertility is that of the U.S. example in Table 1. Then, under Alternative A,

$$\{\underset{\sim}{\hat{Q}}\} = \hat{Q} \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix}$$

and

$$\{\underset{\sim}{\hat{Y}}\} = \begin{bmatrix} \hat{Y}_u \\ \hat{Y}_r \end{bmatrix} = \hat{Q} \begin{bmatrix} 50 & 20 \\ 5 & 25 \end{bmatrix} \times \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix} = \hat{Y} \begin{bmatrix} 0.80 \\ 0.20 \end{bmatrix} \text{ million.}$$

Thus, under our assumptions, a fertility reduction of the first kind, puts roughly 80 percent of India's ultimate zero growth population into urban areas. A similar calculation using the second scheme of fertility reduction (i.e., Alternative B) gives about 72 percent for the same figure.

Table 3 summarizes the above numerical calculations. Note that the "momentum" for urban areas varies from 5 to 14, depending on the time that it takes for the fertility decline to occur and on the spatial pattern of that fertility reduction.

Table 3: Spatial Zero Population Growth in India

I. The Single-Region (Nonspatial) Model

Assumptions

Mortality: Females, e(0) = 50 ; Males, e(0) = 47.114 (Coale and Demeny, 1966)

Fertility: r = 0.025 ; mean age of gross maternity, m = 29 ; k = 1.05

Current Population: K(t) = 600 million

Females

Current Births: B(t) = 12.156 million

Net Reproduction Rate: R(0) = 2.019

Mean Age of Childbearing in Stationary Population:

$$\mu = R(1)R(0)^{-1} = 28.672$$

Equation (12)

$$\hat{Q} = \frac{12.156}{0.025(28.672)} \left(\frac{2.019-1}{2.019}\right) = 8.558 \text{ million}$$

$$\hat{Y} = [50 + 1.05(47.114)]\ 8.558 = 851 \text{ million females } \underline{\text{and}} \text{ males}$$

"Immediate Decline" Momentum	$= \frac{851}{600} = 1.42$
"Gradual Decline" Momentum	$= \frac{2,094}{600} = 3.49$

II. The Multiregional (Spatial) Model

Assumptions

Mortality and Migration: $\underset{\sim}{e}(0) = \begin{bmatrix} 50 & 20 \\ 5 & 25 \end{bmatrix}$

Net Reproduction After Drop to Replacement Level Fertility (Table 1):

Table 3 (Continued): Spatial Zero Population Growth in India

Alternative A

$$\underset{\sim}{\hat{R}}(0) = \begin{bmatrix} 0.85 & 0.45 \\ 0.15 & 0.55 \end{bmatrix}$$

Equation (15)

$$\left\{\underset{\sim}{\hat{Y}}\right\} = \begin{bmatrix} 50 & 20 \\ 5 & 25 \end{bmatrix} \times \hat{Q} \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix} = \hat{Y} \begin{bmatrix} 0.80 \\ 0.20 \end{bmatrix}$$

"Immediate Decline" Urban Momentum $= 1.42 \times \frac{0.80}{0.20} = 5.68$

"Gradual Decline" Urban Momentum $= 3.49 \times \frac{0.80}{0.20} = 13.96$

Alternative B

$$\underset{\sim}{\hat{R}}(0) = \begin{bmatrix} 0.70 & 0.37 \\ 0.21 & 0.74 \end{bmatrix}$$

$$\left\{\underset{\sim}{\hat{Y}}\right\} = \begin{bmatrix} 50 & 20 \\ 5 & 25 \end{bmatrix} \times \hat{Q} \begin{bmatrix} 0.55 \\ 0.45 \end{bmatrix} = \hat{Y} \begin{bmatrix} 0.72 \\ 0.28 \end{bmatrix}$$

"Immediate Decline" Urban Momentum $= 1.42 \times \frac{0.72}{0.20} = 5.11$

"Gradual Decline" Urban Momentum $= 3.49 \times \frac{0.72}{0.20} = 12.56$

5. The Spatial Reproductive Value and Zero Population Growth

The concept of reproductive value, as developed by R.A. Fisher (1929), revolves around the notion of regarding the offspring of a child as the repayment of a debt. Specifically, if the birth of a baby is viewed as a loan of a life and if the future offspring of this child are viewed as the subsequent repayment of this loan, suitably discounted at the annual rate r and compounded momently, then the present value of the repayment may be taken to be

$$\int_{\alpha}^{\beta} e^{-ra}\ell(a)m(a)da \quad .$$

Equating the loan with the discounted repayment gives

$$1 = \int_{\alpha}^{\beta} e^{-ra}\ell(a)m(a)da \quad ,$$

which is recognizable as the characteristic equation used to solve for r, the intrinsic rate of growth. Thus, as Keyfitz points out:

> "the equation can now be seen in a new light: the equating of loan and discounted repayment is what determines r, r being interpretable either as the rate of interest of an average loan or as Lotka's intrinsic rate of natural increase." (Keyfitz, 1975, p. 588)

In the above cited paper, Keyfitz considers how much of the debt is outstanding by the time the child has reached age x. He defines this quantity to be v(x), the reproductive value at age x, where

$$v(x) = \int_{x}^{\beta} e^{-r(a-x)} \frac{\ell(a)}{\ell(x)} m(a)da \quad , \qquad (16)$$

and v(0) is scaled to equal unity. He then goes on to show how the notion of reproductive value may be used to derive the momentum equation previously set out in (11).

Keyfitz begins by observing that if fertility were to drop immediately to replacement level, the ultimate stationary number of births in the resulting zero growth population would be

$$\hat{Q} = \frac{1}{\mu} \int_0^\beta K(x)\hat{V}(x)dx = \frac{\hat{V}}{\mu} \quad ,$$

a quantity he evaluates numerically by means of the approximation

$$\hat{Q} = R(0)R(1)^{-1} \sum_{x=0}^{\beta-5} K(x)\hat{V}(x) = R(0)R(1)^{-1}\hat{V} \quad ,$$

in which

$$\hat{V}(x) = \frac{5}{2}\hat{F}(x) + 5\sum_{t=5}^{\beta-x-5} \hat{F}(x+t)L(x+t)L(x)^{-1} \quad ,$$

and

$$\hat{F}(x) = \gamma F(x) \quad .$$

He then notes that the corresponding ultimate stationary population, calculated as

$$\hat{Y} = e(0)\hat{Q} = e(0)R(0)R(1)^{-1}\hat{V}, \tag{17}$$

gives the same result as a full population projection with the new $\hat{F}(x)$. Finally, he demonstrates that if $K(x)$, the initial age-disaggregated population, can be taken to be stable, then the even simpler form for $\hat{Y}$ given earlier in (12) holds true.

Keyfitz's arguments have their multiregional counterparts. To develop these it is convenient first to reexpress (16) for arbitrary values of $v(0)$, namely:

$$v(x) = v(0) \int_x^{\beta} e^{-r(a-x)} \ell(a)\ell(x)^{-1} m(a)da \quad ,$$

where

$$v(0) = v(0) \int_0^{\beta} e^{-ra} \ell(a)m(a)da = v(0)\psi(r) \quad .$$

This form of the equation immediately suggests the multiregional analog

$$\{\underset{\sim}{v}(x)\}' = \{\underset{\sim}{v}(0)\}' \int_x^{\beta} e^{-r(a-x)} \underset{\sim}{m}(a)\underset{\sim}{\ell}(a)\underset{\sim}{\ell}(x)^{-1} da \quad , \tag{18}$$

where

$$\{\underset{\sim}{v}(0)\}' = \{\underset{\sim}{v}(0)\}' \int_0^{\beta} e^{-ra} \underset{\sim}{m}(a)\underset{\sim}{\ell}(a)da = \{\underset{\sim}{v}(0)\}'\underset{\sim}{\psi}(r) \quad . \tag{19}$$

Equations (18) and (19) have the following demographic interpretation. If lives are loaned to regions according to the (row) vector $\{\underset{\sim}{v}(0)\}'$ then the amount of "debt" outstanding x years later is given by the (row) vector $\{\underset{\sim}{v}(x)\}'$, the regional expected number of subsequent offspring discounted back to age x. The elements of this vector, therefore, may be viewed as regional (or spatial) reproductive values at age x.

Observe that the vector $\{\underset{\sim}{v}(0)\}'$ is the <u>left</u> characteristic vector associated with the unit characteristic root of $\underset{\sim}{\psi}(r)$. Since the elements of this vector are determined only up to a scalar multiple, it becomes convenient to adopt a scaling that ensures consistency with the single-region reproductive value, namely that $v(0) = 1$. Hence $\{\underset{\sim}{v}(0)\}'$ will henceforth be scaled such that its elements add up to unity, and $\{\underset{\sim}{v}_1(0)\}'$ will denote the same vector with an arbitrary scaling. An analogous distinction will be made between $\{\underset{\sim}{Q}\}$ and $\{\underset{\sim}{Q}_1\}$, with the former being scaled to sum to Q and the latter assuming an arbitrary scaling.

Using the multiregional notion of reproductive value, one can establish that the multiregional version of (17) is

$$\{\hat{\underset{\sim}{Y}}\} = \underset{\sim}{e}(0) \frac{\{\hat{\underset{\sim}{V}}_1(0)\}'\{\hat{\underset{\sim}{V}}\}}{\{\hat{\underset{\sim}{v}}_1(0)\}'\hat{\underset{\sim}{R}}(1)\hat{\underset{\sim}{R}}(0)^{-1}\{\hat{\underset{\sim}{Q}}_1\}} \{\hat{\underset{\sim}{Q}}_1\} \quad , \tag{20}$$

where

$$\{\hat{\underset{\sim}{V}}\} = \sum_{x=0}^{\beta-5} \hat{\underset{\sim}{V}}(x)\{\underset{\sim}{K}(x)\} \quad ,$$

$$\hat{\underset{\sim}{V}}(x) = \frac{5}{2}\hat{\underset{\sim}{F}}(x) + 5\sum_{t=5}^{\beta-x-5} \hat{\underset{\sim}{F}}(x+t)\underset{\sim}{L}(x+t)\underset{\sim}{L}(x)^{-1} \quad ,$$

and

$$\hat{\underset{\sim}{F}}(x) = \underset{\sim}{\gamma}\underset{\sim}{F}(x) \quad .$$

As in the single-region case, if the initial population can be taken to be stable then $\{\hat{\underset{\sim}{Y}}\}$ may be expressed in the somewhat simpler form of (13).

Table 4 illustrates the numerical evaluation of (20) with the two-region example of Table 2, in this instance assuming that fertility reduction is achieved via Alternative B. Note that under this particular "spatial path," the West receives a slightly smaller allocation of the total U.S. female population.

Table 4: The Spatial Reproductive Value and Spatial Zero Population Growth Calculations: Alternative B for United States Females, 1968

$$\{\hat{\underset{\sim}{V}}\} = \begin{bmatrix} 8,541,377 \\ 39,015,836 \end{bmatrix} \qquad \underset{\sim}{e}(0) = \begin{bmatrix} 52.39 & 6.95 \\ 23.10 & 67.34 \end{bmatrix}$$

$$\{\hat{\underset{\sim}{v}}_1(0)\}' = [1 \quad 1.0447] \qquad \{\underset{\sim}{Q}_1\} = \begin{bmatrix} 1 \\ 3.6879 \end{bmatrix}$$

$$\underset{\sim}{R}(1) = \begin{bmatrix} 20.401 & 2.419 \\ 8.613 & 27.183 \end{bmatrix} \qquad \underset{\sim}{R}(0)^{-1} = \begin{bmatrix} 1.305 & -0.112 \\ -0.395 & 1.000 \end{bmatrix}$$

$$\underset{\sim}{\gamma} = \gamma \underset{\sim}{I} \text{ where } \gamma = \frac{1}{\lambda_1[\underset{\sim}{R}(0)]} = \frac{1}{1.1201}$$

$$\{\hat{\underset{\sim}{Y}}\} = \begin{bmatrix} 52.39 & 6.95 \\ 23.10 & 67.34 \end{bmatrix} \frac{49,301,220}{127.717} \begin{bmatrix} 1 \\ 3.6879 \end{bmatrix} = \begin{bmatrix} 30,118,924 \\ 104,789,706 \end{bmatrix}$$

<u>Alternative A (from Table 2)</u>:

$$\{\hat{\underset{\sim}{Y}}\} = \begin{bmatrix} 31,013,997 \\ 103,625,370 \end{bmatrix}$$

6. Concluding Remarks

In this paper we have argued that where people choose to live may present issues and problems that are potentially as serious as those posed by the number of children they choose to have. This troublesome feature of spatial population dynamics appears even in zero growth populations, where the redistributional consequences of an immediate reduction of fertility levels can be of considerable importance.

With respect to methodological issues, this paper has demonstrated that the mathematical apparatus commonly used by demographers to examine the evolution of national populations to zero growth may be extended for application in spatial population analysis. The principal role in this extension is played by the characteristic matrix $\underset{\sim}{\psi}(r)$ and its right and left characteristic vectors, $\{\underset{\sim}{Q}\}$ and $\{\underset{\sim}{v}(0)\}'$, respectively. The former defines the regional allocation of stable equivalent births; the latter gives the regional distribution of the national reproductive value at birth. This distinction is hidden in the single-region model, where stable equivalent births and the reproductive value at birth are cancelled out in each of their respective definitional equations to give $1 = \psi(r)$.

APPENDIX

Summary of Results

Nonspatial (1-region) | Spatial (m-regions)

1. Introduction

Nonspatial (1-region)

The Lotka Model:

$$B(t) = \int_{\alpha}^{\beta} B(t-a)\,\ell(a)\; m(a)da$$

Stable Growth: $B(t) = Qe^{rt}$

$$Q = Q \int_{\alpha}^{\beta} e^{-ra}\; \ell(a)m(a)da \;=\; Q\,\psi(r)$$

r is such that $\psi(r) = 1$

$r = b - d$

$c(a) = be^{-ra}\ell(a)$

$K(t) \doteq \left(\frac{Q}{b}\right) e^{rt} = Ye^{rt}$

where

$Q = bY$

Spatial (m-regions)

The Lotka Model:

$$\left\{\underset{\sim}{B}(t)\right\} = \int_{\alpha}^{\beta} \underset{\sim}{m}(a)\underset{\sim}{\ell}(a)\left\{\underset{\sim}{B}(t-a)\right\} da$$

Stable Growth: $\left\{\underset{\sim}{B}(t)\right\} = \left\{\underset{\sim}{Q}\right\}e^{rt}$

$$\left\{\underset{\sim}{Q}\right\} = \left[\int_{\alpha}^{\beta} e^{-ra}\underset{\sim}{m}(a)\underset{\sim}{\ell}(a)da\right]\left\{\underset{\sim}{Q}\right\} = \underset{\sim}{\psi}(r)\left\{\underset{\sim}{Q}\right\}$$

r is such that $\lambda_1\left[\underset{\sim}{\psi}(r)\right] = 1$

$r_j = b_j - d_j - o_j + i_j = b_j - d_j + n_j$

$\underset{\sim}{c}(a) = \underset{\sim}{b}e^{-ra}\underset{\sim}{\ell}(a)$

$\left\{\underset{\sim}{K}(t)\right\} \doteq \left\{\underset{\sim}{Y}\right\}e^{rt}$

where

$\left\{\underset{\sim}{Q}\right\} = \underset{\sim}{b}\left\{\underset{\sim}{Y}\right\}$

2. Spatial Zero Population Growth

$B(t) = \hat{Q}$; $\hat{r} = 0$ and $\hat{b} = \hat{d}$

$$\boxed{\hat{Q} = \hat{Q}\int_\alpha^\beta \ell(a)\ \hat{m}(a)da = \hat{Q}\hat{R}(0)}$$

where $\hat{m}(a) = \gamma m(a)$,

$$\hat{R}(0) = \int_\alpha^\beta \ell(a)\gamma m(a)da = \gamma R(0)$$

and

$$R(0) = \int_\alpha^\beta \ell(a)m(a)da$$

$\left\{\underset{\sim}{B}(t)\right\} = \left\{\underset{\sim}{\hat{Q}}\right\}$: $\hat{r}_j = 0$ and $\hat{b}_j = \hat{d}_j - \hat{n}_j$

$$\boxed{\left\{\underset{\sim}{\hat{Q}}\right\} = \left[\int_\alpha^\beta \underset{\sim}{\hat{m}}(a)\underset{\sim}{\ell}(a)da\right]\left\{\underset{\sim}{\hat{Q}}\right\} = \underset{\sim}{\hat{R}}(0)\left\{\underset{\sim}{\hat{Q}}\right\}}$$

where $\underset{\sim}{\hat{m}}(a) = \underset{\sim}{\gamma}\underset{\sim}{m}(a)$,

$$\underset{\sim}{\hat{R}}(0) = \int_\alpha^\beta \underset{\sim}{\gamma}\underset{\sim}{m}(a)\underset{\sim}{\ell}(a)da = \underset{\sim}{\gamma}\underset{\sim}{R}(0)$$

and

$$\underset{\sim}{R}(0) = \int_\alpha^\beta \underset{\sim}{m}(a)\underset{\sim}{\ell}(a)da$$

3. Alternative Paths to SZPG

$\hat{R}(0) = 1$ for $\boxed{\gamma = \frac{1}{R(0)}}$

$\lambda_1\left[\underset{\sim}{\hat{R}}(0)\right] = 1$ for feasible $\underset{\sim}{\gamma}$

e.g.

$$\boxed{\begin{aligned}\left\{\underset{\sim}{\gamma}\right\} &= \left[\underset{\sim}{R}(0)'\right]^{-1}\left\{\underset{\sim}{1}\right\} \\ \text{or}\quad \gamma &= \frac{1}{\lambda_1[\underset{\sim}{R}(0)]}\end{aligned}}$$

(Table 1)

4. Momentum of ZPG

$$\frac{\hat{Y}}{K} = \frac{b\,e(0)}{r\mu}\left(\frac{R(0)-1}{R(0)}\right) = \frac{1}{r}\,e(0)R(1)^{-1}\left[R(0)-1\right]b$$

since

$$\mu = \frac{R(1)}{R(0)}$$

$$\left\{\frac{\hat{Y}}{K}\right\} = \frac{1}{r}\,\underset{\sim}{e}(0)\,\underset{\sim}{R}(1)^{-1}\left[\underset{\sim}{R}(0)-\underset{\sim}{\psi}(r)\right]\left\{\underset{\sim}{b}\right\}$$

(Table 2)

$$\hat{Y} = e(0)\hat{Q}$$

where

$$\hat{Q} = \frac{bK}{r\mu}\left(\frac{R(0)-1}{R(0)}\right)$$

$$\left\{\hat{\underset{\sim}{Y}}\right\} = \underset{\sim}{e}(0)\left\{\hat{\underset{\sim}{Q}}\right\}$$

where $\left\{\hat{\underset{\sim}{Q}}\right\}$ is the right characteristic vector of $\hat{\underset{\sim}{R}}(0)$, scaled to sum to $\hat{Q}$.

(Table 3)

5. Reproductive Value and ZPG

Definition:

$$v(x) = \int_x^{\beta} e^{-r(a-x)}\frac{\ell(a)m(a)da}{\ell(x)}$$

where
$v(0) = 1$.

Definition:

$$\left\{\underset{\sim}{v}(x)\right\}' = \left\{\underset{\sim}{v}(0)\right\}'\left[\int_x^{\beta} e^{-r(a-x)}\underset{\sim}{m}(a)\underset{\sim}{\ell}(a)\underset{\sim}{\ell}(x)^{-1}da\right]$$

where
$\left\{\underset{\sim}{v}(0)\right\}'$ is the left characteristic vector of the characteristic matrix $\underset{\sim}{\psi}(r)$, scaled to sum to unity.

ZPG: Numerical Evaluation

$$\hat{Y} = e(0)\,\frac{\hat{V}}{\mu} = e(0)R(0)\,R(1)^{-1}\hat{V}$$

where

$$\hat{V} = \sum_{x=0}^{\beta-5} K(x)\hat{V}(x) ,$$

$$\hat{V}(x) = \frac{5}{2}\,\hat{F}(x)+5\sum_{t=5}^{\beta-x-5}\hat{F}(x+t)L(x+t)L(x)^{-1} ,$$

and

$$\hat{F}(x) = \gamma F(x)$$

SZPG: Numerical Evaluation

$$\left\{\hat{\underset{\sim}{Y}}\right\} = \underset{\sim}{e}(0)\,\frac{\{\hat{\underset{\sim}{v}}_1(0)\}'\{\hat{\underset{\sim}{V}}\}}{\{\hat{\underset{\sim}{v}}_1(0)\}'\hat{\underset{\sim}{R}}(1)\hat{\underset{\sim}{R}}(0)^{-1}\{\hat{\underset{\sim}{Q}}_1\}}\left\{\hat{\underset{\sim}{Q}}_1\right\}$$

where

$$\left\{\hat{\underset{\sim}{V}}\right\} = \sum_{x=0}^{\beta-5}\hat{\underset{\sim}{V}}(x)\left\{\underset{\sim}{K}(x)\right\} ,$$

$$\hat{\underset{\sim}{V}}(x) = \frac{5}{2}\,\hat{\underset{\sim}{F}}(x)+5\sum_{t=5}^{\beta-x-5}\hat{\underset{\sim}{F}}(x+t)\underset{\sim}{L}(x+t)\underset{\sim}{L}(x)^{-1} ,$$

and

$$\hat{\underset{\sim}{F}}(x) = \underset{\sim}{\gamma}\underset{\sim}{F}(x) ,$$

6. Conclusion

The key roles are played by the right and left characteristic vectors associated with the unit characteristic root of $\underset{\sim}{\psi}(r)$. The right vector is $\{\underset{\sim}{Q}\}$, the vector of stable equivalent births. The left vector is $\{\underset{\sim}{v}(0)\}'$, the vector of reproductive values at birth. That is, let

$$\underset{\sim}{\psi}(r) = \int_{\alpha}^{\beta} e^{-ra}\, \underset{\sim}{m}(a)\underset{\sim}{\ell}(a)da \quad ,$$

then

$$\boxed{\begin{aligned} \{\underset{\sim}{Q}\} &= \underset{\sim}{\psi}(r)\{\underset{\sim}{Q}\} \\ \{\underset{\sim}{v}(0)\}' &= \{\underset{\sim}{v}(0)\}'\, \underset{\sim}{\psi}(r) \end{aligned}} \quad .$$

At spatial zero population growth r=0, whence

$$\hat{\underset{\sim}{\psi}}(0) = \int_{\alpha}^{\beta} \hat{\underset{\sim}{m}}(a)\underset{\sim}{\ell}(a)da = \hat{\underset{\sim}{R}}(0) \quad ,$$

and

$$\boxed{\begin{aligned} \{\hat{\underset{\sim}{Q}}\} &= \hat{\underset{\sim}{R}}(0)\{\hat{\underset{\sim}{Q}}\} \\ \{\hat{\underset{\sim}{v}}(0)\}' &= \{\hat{\underset{\sim}{v}}(0)\}'\hat{\underset{\sim}{R}}(0) \end{aligned}} \quad .$$

Note that in the single-region (nonspatial) model, stable equivalent births and the reproductive value at birth are cancelled out in each of the two definitional equations, thereby giving the identical result

$$1 = \psi(r) \quad .$$

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