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GENERALIZED X-Y FUNCTIONS, THE LINEAR MATRIX INEQUALITY, AND TRIANGULAR FACTORIZATION FOR LINEAR CONTROL PROBLEMS

J. Casti

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Preface

In continuation of earlier IIASA work on the reduction of analytic and computational complexity for high-dimensional control processes, this Memorandum details the relationship of the earlier results for autonomous (constant coefficient) systems to the triangular factorization of a certain associated matrix. As a result of adopting this viewpoint, it is seen how the structure of time-dependent problems may be exploited to yield low-dimensional computational algorithms similar to those for constant systems.

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Abstract

The relationship between the linear matrix inequality (LMI), generalized X-Y functions, and triangular factorization is examined within the framework of the classical linear-quadratic-gaussian problem. It is shown that the generalized X-Y functions arise naturally as components within the factors of the matrix forming the LMI when that matrix is decomposed into its symmetric triangular factors. This viewpoint enables us to propose a low-dimensional computational algorithm for time-dependent problems which reduces to the generalized X-Y situation for constant systems.

In addition to the basic factorization results, we also briefly touch upon several related topics including the infinite-interval (regulator) problem, singular control problems, canonical forms, and numerical considerations.

1. Introduction

We consider the problem of minimizing the quadratic form

$$J = \int_t^T [(x,Qx) + 2(x,Su) + (u,Ru)]ds \quad ,$$

over vector functions u(s), where x(s) and u(s) are related by the linear differential equation

$$\frac{dx}{ds} = Fx + Gu \quad , \qquad x(t) = c \quad .$$

Here x and u are n- and m-dimensional vector functions, respectively, while Q, S, R, F, G are real, time-varying matrix functions of appropriate sizes with $Q = Q'$, $R = R'$. At the outset we make no assumptions on the definiteness of Q and R. The functions $u(s)$ are assumed to belong to the class

$$\mathscr{U} = \{u : (-\infty, T) \to R^m,\ u(s) \in L_2(\alpha, T) \text{ for all } \alpha \leq T\} \quad .$$

We further assume throughout that the pair (F,G) is controllable and that F, G, Q, R, S are as smooth as may be required for the needs at hand.

By making the assumption that the optimal control law $u(s)$ is linear feedback, i.e.,

$$u(s) = -K(s)x(s) \quad ,$$

for some $m \times n$ matrix function K, a reasonably straightforward integration by parts shows that the problem of minimizing J over all admissible u is equivalent to the minimization of

$$\mathrm{tr}\left\{[I \quad -K'] \begin{bmatrix} F'P + PF + Q + \dot{P} & PG + S \\ G'P + S' & R \end{bmatrix} \begin{bmatrix} I \\ -K \end{bmatrix} W \right\} \tag{1}$$

over matrices $K(s)$, $P(s)$, for all positive semidefinite matrix functions $W(s)$. The symbol "tr" denotes the matrix trace operation (details of this derivation are found in [1]). It is well known [2], that J has a bounded infimum and $x(s) \to 0$ as $s \to -\infty$ if and only if there exists a real, symmetric solution $P(t)$ to the linear matrix inequality

$$M(P) = \begin{bmatrix} F'P(t) + P(t)F + Q + \dot{P} & P(t)G + S \\ G'P(t) + S' & R \end{bmatrix} \geq 0 \tag{2}$$

for all $t \leq T$.

It is a fairly easy exercise to verify that (1) is minimized by the choice

$$RK = G'P + S' \quad , \tag{3}$$

with P satisfying the matrix Riccati differential equation

$$\frac{-dP}{dt} = Q + PF + F'P - (PG+S)R^{-1}(PG+S)' \quad , \tag{4}$$

$$P(T) = 0 \quad . \tag{5}$$

Note that for (4) to be valid, we must impose the additional restriction that R(s) is invertible on $t \leq s \leq T$. To avoid unnecessary complications with the main ideas of this paper, for the time being, we shall assume R(s) is invertible. The case of singular R will be discussed in a later section.

To compute the optimal feedback law K, we see that the above approach requires the solution of the $n \times n$ matrix Riccati equation (4), subject to the initial condition (5). As long as P(t) is such that (2) is satisfied, the problem has a unique solution given by the feedback control $u = -K(t)x(t)$, with K computed from (3).

If the coefficient matrices F, G, Q, S, R are constant, a major simplification in the foregoing solution procedure occurs when the matrices $(Q-SR^{-1}S')$ and $GR^{-1}G'$ have low rank. Specifically, if

$$\text{rank } (Q-SR^{-1}S') = p \quad , \tag{6}$$

$$\text{rank } (GR^{-1}G') = m \quad , \tag{7}$$

then it can be shown [3,4] that the optimal feedback gain matrix K may be computed from a system of $n(p+m)$ equations. These equations, termed "generalized X-Y functions" in [4], are formed by two matrix functions L and N, satisfying the equations

$$\dot{L} = (F'-N'R^{-1/2}G')L \quad , \qquad L(T) = Z \quad , \tag{8}$$

$$\dot{N}' = -LL'GR^{-1/2} \quad , \qquad N(T) = 0 \quad , \tag{9}$$

where $ZZ' = Q - SR^{-1}S'$. Thus, from the rank assumptions (6) and (7), plus the definition

$$N' = (PG+S)R^{-1/2} \quad ,$$

we see that L is an $n \times p$ matrix, while N' is of size $n \times m$.

From (3), we have the obvious but critical relation

$$K' = N'R^{-1/2} \quad , \tag{10}$$

connecting the low-dimensional function N with the optimal feedback rule K. In addition, the derivation of Eqs. (8)-(9) given in [3,4] shows that

$$L(t)L'(t) = -\dot{P}(t) \quad , \tag{11}$$

another important formula for our subsequent development.

The starting point of our investigation is the remark in [2] that "the basic importance of the LMI seems to be largely unappreciated. It would be interesting to see whether or not it can be exploited in computational algorithms, for example." The principle goals of this study are to underscore the central role of the LMI in least-squares problems, to demonstrate its connection with the low-dimensional functions L and N, and to show how it may be exploited to generate a feasible low-dimensional computational approach to time-varying problems.

Our approach is to work with a modified version of the matrix M(P) appearing in (2). We construct the matrix $\mathcal{M}(P)$ by deleting the $\dot{P}$ term from M. This gives

$$\mathcal{M}(P) = \begin{bmatrix} F'P(t) + P(t)F + Q & P(t)G + S \\ G'P(t) + S' & R \end{bmatrix} . \quad (12)$$

We then factor $\mathcal{M}$ into its symmetric triangular components and relate the entries appearing in the triangular arrays to the original problem, as well as to the functions L and N. Finally, we show how the factorization procedure may be streamlined to generate so-called "fast" [5] algorithms for computing K in the case of time-varying F, G, Q, R, S.

2. Basic Results

We begin our analysis by factoring the (n+m) × (n+m) matrix $\mathcal{M}(P)$ into upper and lower triangular components. Thus, we have

$$\begin{bmatrix} \phi & \Gamma \\ 0 & \psi \end{bmatrix} \begin{bmatrix} \phi' & 0 \\ \Gamma' & \psi' \end{bmatrix} = \begin{bmatrix} F'P + PF + Q & PG + S \\ G'P + S' & R \end{bmatrix} . \quad (13)$$

Such a factorization implies the following relations:

$$\phi\phi' + \Gamma\Gamma' = F'P + PF + Q \quad , \quad (14)$$

$$\psi\Gamma' = G'P + S' \quad , \quad (15)$$

$$\psi\psi' = R \quad . \quad (16)$$

Formulas (14)-(16) hold for any fixed symmetric matrix P. Equations (15) and (16) immediately give

$$\psi(t) = R^{1/2}(t)V'(t) \quad , \quad (17)$$

$$R^{1/2}V'\Gamma'(t) = G'P + S' \quad , \quad (18)$$

where V is an arbitrary orthogonal matrix which, for convenience, we choose equal to the identity I. Since R is assumed invertible for all t, Eq.(14) shows that ϕ satisfies

$$\phi(t)\phi'(t) = Q + F'P + PF - (PG+S)R^{-1}(PG+S)' \quad . \tag{19}$$

Recalling the basic results (3) and (4), we see that the connections between the optimal feedback law K, the solution of the matrix Riccati equation (4), and the functions ϕ, Γ, ψ, are

$$K(t) = R^{-1/2}\Gamma'(t) \quad , \tag{20}$$

$$-\dot{P}(t) = \phi(t)\phi'(t) \quad . \tag{21}$$

To see the relationship between the factors Γ, ϕ, and ψ and the low-dimensional functions L and N, we compare Eqs.(20), (21) with Eqs.(10) and (11). This comparison, together with the proof of Eqs.(8)-(9) given in [4], immediately suggests

Theorem 1. Assume F, G, S, Q, R are constant matrices of appropriate sizes. Further, let the rank conditions (6), (7) hold. Then the factors Γ and ϕ in the triangular decomposition of $\mathcal{M}$ are related to the functions L and N in Eqs.(8)-(9) by

$$\phi = LU' \quad , \tag{22}$$

$$\Gamma' = TN \quad , \tag{23}$$

where U(t) and T(t) are arbitrary $p \times p$, $m \times m$ orthogonal matrices, respectively.

Proof. Since $\dot{P}(t)$ is symmetric, there exists an orthogonal transformation V(t) such that

$$-\dot{P}(t) = V'(t)\left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right]V(t) \quad ,$$

where D is a $p \times p$ diagonal matrix with entries ± 1, if rank $\dot{P}(t) = p$. However,

$$-\dot{P}(t) = \phi\phi' = LL' \quad .$$

Thus,

$$\phi\phi' = V'\left[\begin{array}{c|c} D^{1/2} & 0 \\ \hline 0 & 0 \end{array}\right] U'U \left[\begin{array}{c|c} D^{1/2} & 0 \\ \hline 0 & 0 \end{array}\right] V \quad ,$$

where $U(t)$ is an arbitrary $n \times n$ orthogonal matrix. Hence, we see that the identification

$$L = V'\left[\begin{array}{c} D^{1/2} \\ \hline 0 \end{array}\right] \quad ,$$

which follows upon neglecting the last n-p columns of $\left[\begin{array}{c|c} D^{1/2} & 0 \\ \hline 0 & 0 \end{array}\right]$ gives the result (22).

Relation (23) is an easy consequence of the definition $N = (PG+S)R^{-1/2}$ and relation (15).

Remarks:

(1) Theorem 1 shows that the functions ϕ and Γ' arising from the factorization of $\mathcal{M}$ are equivalent, modulo the orthogonal group, to the generalized X-Y functions L and N. We need only be careful to note the convention that L is defined by deleting the irrelevant zero columns from the factorization of $\Omega - SR^{-1}S'$.

(2) Relation (16) shows that the factor ψ is independent of the matrix P. Thus, ψ can be produced once and for all at the beginning of any computational process, thereby requiring only the triangular decomposition of the first n rows of the matrix $\mathcal{M}$ at each stage of the process.

Actually, by (15) we see that it is possible to obtain Γ' by solving a triangular system of equations. Thus, the only real factorization necessary is that of determining ϕ which, by remark (1), requires only algebraic operations on the first p rows and n columns of $\mathcal{M}$ when the rank condition (6) is satisfied. In short, at most we need triangularly decompose a single $n \times n$ matrix at each step of the process.

Furthermore, if desired even the triangularization of ϕ may be averted by using the relationship

$$-\dot{P} = Q + PF + F'P - \Gamma\Gamma' \tag{24}$$

to update P. The point here, of course, is that if $\dot{P}(t)$ has low rank, then the partial factorization of $\mathcal{M}$ to produce ϕ may enable us to update P much more efficiently using the relation

$$-\dot{P} = \phi\phi' \quad , \tag{25}$$

in place of (24).

3. General Computational Procedure

Notwithstanding the theoretical interest associated with the LMI and the triangular decomposition of $\mathcal{M}$, the primary importance of the above results lie in their use for the development of efficient computational approaches to least-squares problems. In this section, we sketch the outline of a computational algorithm suitable for either time-varying or constant systems.

The steps of our algorithm are the following:

0. Factor $R(t)$ into its triangular components $\psi(t)$, $\psi'(t)$.
1. Compute $\Gamma'(t)$ by solving the triangular system (15).
2. Determine $K(t) = \psi^{-1}(t)\Gamma'(t)$.
3. Triangularly factor the first n rows of $\mathcal{M}$ to determine ϕ (note that the factorization will terminate after p rows if $\dot{P}(t)$ has rank p).
4. Let $t \to t - \Delta$ and determine $P(t-\Delta)$ using the relation

$$-\dot{P}(t) = \phi(t)\phi'(t) \quad .$$

5. Go to step 1.

The preceding algorithm makes clear the tremendous advantages offered by the generalized X-Y functions L and N for constant systems. In the-time varying case it is necessary to go through the matrix function P in order to obtain the desired quantity Γ which, in essence, characterizes the optimal control law. However, Theorem 1 shows that for time-invariant systems, the L and N equations enable us to produce (modulo an inessential orthogonal transformation) the factors ϕ and Γ directly, totally bypassing the matrix P.

4. Singular Control

The formulas derived for the optimal feedback law K in section 2 were based upon the assumption that the control weighting matrix R was positive-definite, hence nonsingular. However, in some cases of practical interest R may be only positive semidefinite, or even indefinite if, for example, the LQG problem arises as an approximation to a system with non-quadratic costs. Thus, it is of interest to re-examine the factorization results from this point of view.

Upon reviewing the steps leading to Eq.(2) for K, we see that the invertibility of R is not invoked for the factorization of $\mathcal{M}$. Thus, Eqs.(14)-(18) remain valid for any symmetric R. From Eq.(3) we know that the optimal feedback law K must satisfy the relation

$$\begin{aligned} RK &= G'P + S' \\ &= R^{1/2}\Gamma' \\ &= \psi\Gamma' \end{aligned} \tag{26}$$

modulo an inessential orthogonal transformation. Equation (26) is a set of m^2 equations in the mn unknown components of K. In general, this system will be underdetermined, therby giving an m(n-m)-parameter family of solutions for the optimal feedback law K.

5. Algebraic Considerations

Since the factorization of $\mathcal{M}$ lies at the heart of our discussion, it is desirable to choose a coordinate system in which $\mathcal{M}$ assumes the simplest possible form. Regarding the original problem as being completely specified by the five matrices $\Sigma = (F,G,Q,R,S)$, we introduce the so-called "feedback" group $\mathcal{G}$ of transformations consisting of

$\mathcal{G}$:
(I) nonsingular coordinate changes T in the state space,
(II) nonsingular coordinate changes V in the control space,
(III) application of an arbitrary feedback law L.

Under the group $\mathcal{G}$, the system Σ transforms as follows:

$$\Sigma \xrightarrow{(I)} (TFT^{-1}, TG, T'^{-1}QT^{-1}, R, T'^{-1}S) = \Sigma_I \quad ,$$

$$\Sigma \xrightarrow{(II)} (F, GV^{-1}, Q, V'^{-1}RV^{-1}, SV^{-1}) = \Sigma_{II} \quad ,$$

$$\Sigma \xrightarrow{(III)} (F-GL, G, Q+L'RL-L'S'-SL, R, S-L'R) = \Sigma_{III} \quad .$$

It is a fairly routine exercise to see that, under the feedback group, it is possible to reduce $\mathcal{M}$ to the form

$$\mathcal{M}_{\mathcal{G}}(P) = \begin{bmatrix} T'^{-1}(Q+PF+F'P)T^{-1} & 0 \\ 0 & V^{-1}{}'RV^{-1} \end{bmatrix} .$$

This reduction involves choosing L such that $RL' = (PG+S)'$. It is important to note that, in general, the transformations

T, V, and L will need to be time-varying since P is time-varying. However, the situation is materially improved if R is constant since then we may choose V to diagonalize R, leaving only the factorization of the term $T'^{-1}(Q+PF+F'P)T^{-1}$ for each t. This operation may also be "trivialized" at the expense of computing the orthogonal matrix T which diagonalizes the symmetric matrix Q + PF + F'P, for each t.

6. The Regulator Problems and Spectral Factors

In the event the original problem is over the semi-infinite interval $(-\infty, T)$, the matrices M of Eq. (2) and $\mathscr{M}$ of Eq. (12) coincide. It is evident that in this case $\dot{P} = 0$ which implies $\phi = 0$. Thus, the factorization of $\mathscr{M}$ degenerates into the case

$$\begin{bmatrix} \Gamma \\ \psi \end{bmatrix} [\Gamma' \quad \psi'] = \mathscr{M}(P) \quad ,$$

i.e.

$$\Gamma\Gamma' = F'P + PF + Q \quad , \tag{27}$$

$$\psi\Gamma' = G'P + S' \quad , \tag{28}$$

$$\psi\psi' = R \quad , \tag{29}$$

where P is any solution of the algebraic Riccati equation.

Equations (27)-(29) suggest the following linear successive approximation scheme for determining the optimal steady-state gain function $K(\infty)$:

(i) Guess P_n.

(ii) Determine Γ_n' as the solution of

$$\psi\Gamma_n' = G'P_n + S' \quad .$$

(iii) Obtain P_{n+1} as the solution of the equation

$$F'P_{n+1} + P_{n+1}F = \Gamma_n\Gamma_n' - Q \quad .$$

As before, step (ii) of the algorithm involves a triangular system of equations, while step (iii) is a standard Lyapunov matrix equation for which efficient algorithms exist [6].

It should also be clear from the above considerations that the factorization of $\mathscr{M}$ has direct bearing upon the "inverse" problem of optimal control. In fact, several important results have already been obtained in [7] using similar techniques.

An important technique in the analysis of the regulator problem is the "spectral" decomposition of the function

$$H(\bar{s},s) = R + S(Is-F)^{-1}G + G'(I\bar{s}-F')^{-1}S' + G'(I\bar{s}-F')^{-1}Q(Is-F)^{-1}G \quad , \tag{30}$$

where s is a complex variable. The function $H(\bar{s},s)$ plays an important role in many problems of applied mathematics, arising from the Laplace transform of the original LQG problem. The pseudo-spectral factors of H are defined to be any rational matrices W(s), W'(-s) satisfying

$$H(-s,s) = W'(-s)W(s) \quad . \tag{31}$$

It is of interest to examine the connections between these factors of H and the elements Γ and ψ.

We first note the following result:

Theorem 2. Let P be any symmetric matrix satisfying $\mathscr{M}(P) \geq 0$ and let $\mathscr{M}$ be factored as

$$\mathscr{M}(P) = \begin{bmatrix} \Gamma \\ \\ \psi \end{bmatrix} [\Gamma' \quad \psi'] \quad . \tag{32}$$

(Here we neglect the irrelevant zero columns of the triangular factors.) Then the pseudo-spectral factors W(s), W'(-s) are related to Γ and ψ as

$$W(s) = \psi' + \Gamma'(Is-F)^{-1}G \quad .$$

Proof. Premultiply by $[G'(-Is-F')^{-1} \quad I]$, post-multiply by $\begin{bmatrix}(Is-F)^{-1}G \\ I\end{bmatrix}$ and use the definition of $H(-s,s)$ given in Eq.(30).

Remark:

In [2] it is shown that use of the solutions P^* of the algebraic Riccati equation give $\mathcal{M}$ smallest possible rank which, in turn, yields the lowest rank $W(s)$, i.e., if $R > 0$ and q is the number of rows in Γ' and ψ', then $q = \text{rank } R = m$ if and only if P is a solution of the algebraic Riccati equation. In this case, $W(s)$ is a square $m \times m$ matrix with an inverse analytic in $\text{Re } s > 0$. By a result of Youla [11], such a $W(s)$ is unique and the matrix

$$W^+(s) = \psi'(I+R^{-1}(G'P^*+S')(Is-F)^{-1}G)$$

yields the so-called "spectral factorization" of $H(-s,s)$.

7. Infinite-Dimensional Problems

Upon replacing the defining matrices F, G, Q, R, S by operators acting in suitable Hilbert spaces, the results presented above become applicable to the so-called "distributed parameter" control problems. This observation opens the way for a factorization approach to systems governed by partial differential equations and differential-delay equations. Some related results giving generalized X-Y functions for these situations may be found in [8].

The passage to the infinite-dimensional setting also calls to mind the deep and beautiful theory of triangular factorization of operators developed by Gohberg-Krein [9], Schumitzky-McNabb [10], and others. The connection between the triangular factors and the generalized X-Y functions for time-invariant operators given in Theorem 1, should now motivate a re-examination of the above work for further extension to the non-self-adjoint case.

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