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# The Reachability of Techno-Labor Homeostasis via Regulation of Investments in Labor and R&D: Mathematical Proofs

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# Abstract

The goal of this paper is to provide accurate proofs for Propositions 4.1 and 4.2 of Kryazhimskii et al. 2002, where these propositions play a central role in the analysis of a mathematical model of techno-labor development.

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## Introduction

Kryazhimskii, et. al., 2002, suggests a mathematical model of techno-labor development and discusses its application to the analysis of selected industry sectors of Japan. The analysis is based on two key propositions (Kryazhimskii, et. al., 2002, Propositions 4.1 and 4.2) which characterize the model's dynamics. The goal of this paper is to provide accurate mathematical proofs to these propositions.

In section 1 we introduce the model and analyze its vector field using appropriate transformations of state variables.

In section 2 we classify models's behaviors.

In sections 3 and 4 we formulate and prove the key propositions.

## 1 Model. Vector field analysis

The model we analyze in this paper was designed and discussed in detail in Kryazhimskii, et. al., 2002. Here, we do not comment it in substantial terms. We mention only that it describes the dynamics of an economy sector in the space of two variables, the accumulated technology stock (briefly, technologies), $T$, and capital accumulated in labor (briefly, welfare), $Z$.

The model has the form

$$\begin{cases} \dot{T} = \mu u T^{\alpha} Z^{\beta+\gamma} - \rho_T T, \\ \dot{Z} = \mu(1-u) T^{\alpha} Z^{\beta} - \rho_Z Z, \end{cases} \tag{1.1}$$

Here

$$\alpha,\ \beta,\ \gamma,\ \mu \in (0,1), \rho_T,\ \rho_Z \geq 0 \tag{1.2}$$

are fixed parameters (whose meaning is explained in detail in Kryazhimskii, et. al., 2002). The parameter $u \in (0,1)$ is called a *control*. Following Kryazhimskii, et. al., 2002, we call (1.1) the *techno-labor system*. The techno labor system operates on the time interval $[0,\infty)$. Its state space is the positive orthant $O^+$ in the 2-dimensional space: $O^+ = \{(Z,T) \in R^2 : Z > 0,\ T > 0\}$. Accordingly, any initial state of (1.1),

$$(Z(0), T(0)) = (Z_0, T_0), \tag{1.3}$$

is assumed to belong to $O^+$.

Theory of ordinary differential equations (see, e.g., Hartman, 1964) yields that for every initial state $(Z_0, T_0)$ and every control $u$ there exists the unique solution $t \mapsto (Z(t), T(t))$

of (1.1) which is defined on $[0,\infty)$ and satisfies the initial condition (1.3); moreover, $(Z(t),T(t)) \in O^+$ for every $t \geq 0$. Using a standard terminology of theory of ordinary differential equations, we call $t \mapsto (Z(t),T(t))$ the *solution of the Cauchy problem* (1.1), (1.3). Note that the techno-labor system (1.1) describes also the dynamics of production, $Y$, defined as $Y = T^\alpha Z^\beta$ (see Kryazhimskii et. al., 2002).

In the rest of this section, we analyse the vector field of system (1.1).

We denote by $G_Z(u)$ the set of all $(Z,T) \in O^+$, at which the vector field of system (1.1) has the zero projection onto the $Z$ axis, and by $G_T(u)$ the set of all $(Z,T) \in O^+$, at which this vector field has the zero projection onto the $T$ axis. Simple computations yield that $G_Z(u)$ is a curve on the $(Z,T)$ plane, whose equation is

$$T = \left(\frac{\rho Z}{\mu}\right)^{1/\alpha} \frac{1}{(1-u)^{1/\alpha}} Z^{(1-\beta)/\alpha}, \tag{1.4}$$

and $G_T(u)$ is the curve on the $(Z,T)$ plane, whose equation is

$$T = \left(\frac{\rho Z}{\mu}\right)^{1/(1-\alpha)} u^{1/(1-\alpha)} Z^{(\beta+\gamma)/(1-\alpha)}. \tag{1.5}$$

In what follows, we assume that $\alpha+\alpha\gamma+\beta \neq 1$ and consider two cases, case 1, *stagnation*,

$$\alpha + \alpha\gamma + \beta < 1. \tag{1.6}$$

and case 2, *progress*,

$$\alpha + \alpha\gamma + \beta > 1. \tag{1.7}$$

The definitions of cases 1 and 2 as stagnation and progress, respectively, are motivated by the structure of the vector field of system (1.1) in these cases; this structure is characterized in statements (ii) and (iii) of the next proposition formulated in Kryazhimskii, et. al., 2002, without proofs for graphical illustrations see Fig. 1.1 and Fig. 1.2.

**Proposition 1.1** Let $\alpha+\alpha\gamma+\beta \neq 1$ and $u \in (0,1)$ be an arbitrary control. The following statements hold true:

(i) the curves $G_Z(u)$ and $G_T(u)$ intersect at the unique point $(Z^*(u),T^*(u))$ defined as the solution of the algebraic system (1.4), (1.5), and $(Z^*(u),T^*(u))$ is the unique rest point of the techno-labor system (1.1) under control $u$;

(ii) if case 1 (1.6), stagnation, takes place, then at the rest point $(Z^*(u),T^*(u))$ the slope of $G_Z(u)$ on the $(Z,T)$ plain is greater than the slope of $G_T(u)$, implying that the vector field of the techno-labor system (1.1) has the form shown in Fig. 1.1;

(iii) if case 2 (1.7), progress, takes place, then at the rest point $(Z^*(u),T^*(u))$ the slope of $G_Z(u)$ on the $(Z,T)$ plain is smaller than the slope of $G_T(u)$, implying that the vector field of the techno-labor system (1.1) has the form shown in Fig. 1.2.

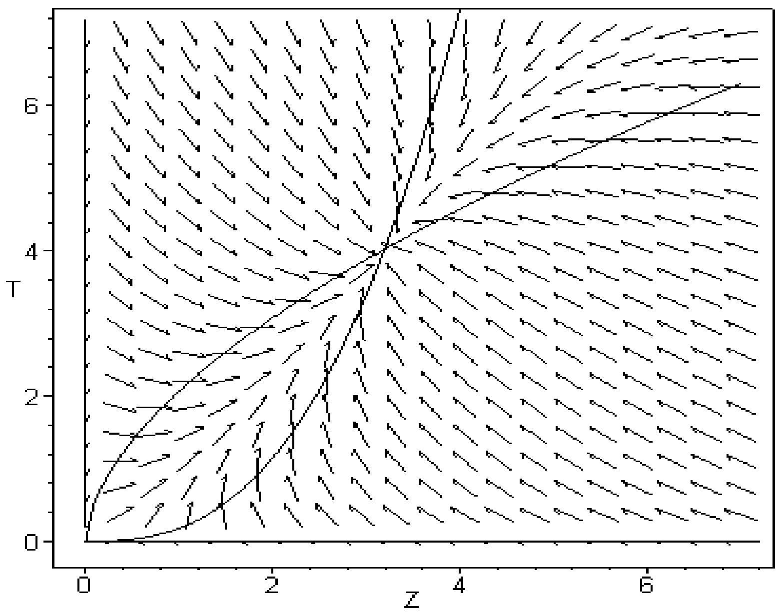


Fig. 1.1.
The vector field of the techno-labor system in case 1, stagnation.
The curve $G_Z(u)$ lies lower than $G_T(u)$ in a neighborhood of the origin
and higher than $G_T(u)$ in a neighborhood of infinity.

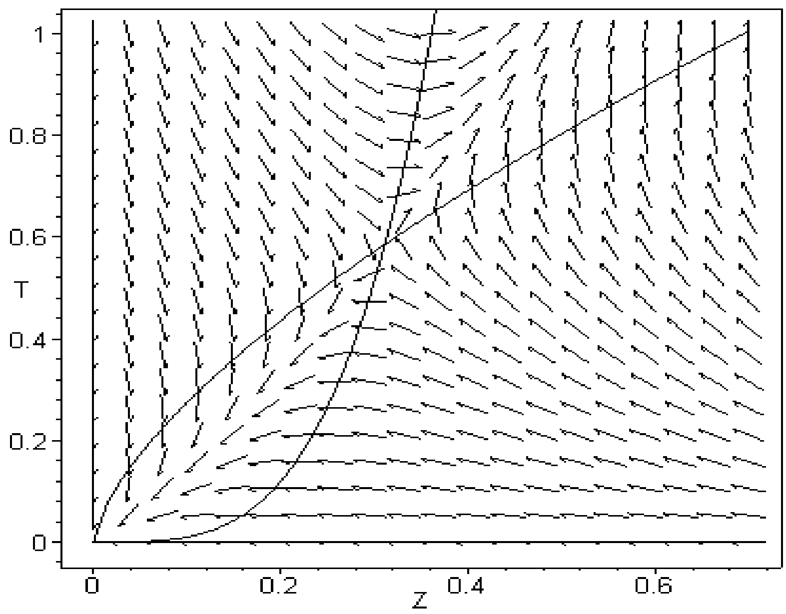


Fig. 1.2.
The vector field of the techno-labor system in case 2, progress.
The curve $G_Z(u)$ lies higher than $G_T(u)$ in a neighborhood of the origin
and lower than $G_T(u)$ in a neighborhood of infinity.

Fig. 1.1 and Fig. 1.2 show that in each of cases 1 and 2 system (1.1) exhibits 4 different behaviors within 4 "angle" areas in the $(Z, T)$ plain, which are determined by curves $G_Z(u)$ and $G_T(u)$; we call these angle areas the *north-east, south-east, south-west and north-west angles* (for control $u$) according to their locations and denote them $G_{ZT}^{++}(u)$, $G_{ZT}^{+-}(u)$, $G_{ZT}^{--}(u)$, $G_{ZT}^{-+}(u)$, respectively. We assume that the north-west and south-east angles, $G_{ZT}^{-+}(u)$, $G_{ZT}^{+-}(u)$, are closed, i.e., contain their boundaries, and the north-east and south-west angles, $G_{ZT}^{++}(u)$, $G_{ZT}^{--}(u)$, are open, i.e., do not contain their boundaries.

In cases 1 and 2 the upper and lower boundaries of the north-east, south-east, north-west and south-west angles are parts of different curves. For example, in case 1 (1.6) the upper boundary of the north-east angle $G_{ZT}^{++}(u)$ is part of curve $G_Z(u)$ which is located above the rest point $(Z^*(u), T^*(u))$ (including this point), whereas in case 2 this part of curve $G_Z(u)$ is the lower boundary of $G_{ZT}^{++}(u)$.

We prove Proposition 1.1. using the logarithmic variables

$$\tau = \ln(T), \quad z = \ln(Z). \tag{1.8}$$

Dividing the first equation in (1.1) by $T$ and second by $Z$, we get

$$\frac{d\ln(T)}{dt} = \mu u T^{\alpha-1} Z^{\beta+\gamma} - \rho_T.$$

and

$$\frac{d\ln(Z)}{dt} = \mu u T^{\alpha} Z^{\beta-1} - \rho_Z.$$

Consequently, in the $(\tau, z)$ variables system (1.1) takes the form

$$\begin{cases} \dot{\tau} = \mu u e^{(\alpha-1)\tau+(\beta+\gamma)z} - \rho_T, \\ \dot{z} = \mu(1-u)e^{\alpha\tau+(\beta-1)z} - \rho_Z. \end{cases} \tag{1.9}$$

**Proof of Proposition 1.1**. In the $(z, \tau)$ variables the curve $G_T(u)$ where $\dot{T} = 0$ or, equivalently, $\dot{\tau} = 0$ has the equation

$$\mu u e^{(\alpha-1)\tau+(\beta+\gamma)z} = \rho_T.$$

Dividing by $\mu u$ and taking the logarithm, we find

$$(\alpha - 1)\tau + (\beta + \gamma)z = \ln(\frac{\rho_T}{\mu u})$$

and, finally,

$$\tau = \frac{\beta+\gamma}{1-\alpha} z + \frac{\ln(\frac{\mu u}{\rho_T})}{1-\alpha}. \tag{1.10}$$

This equation represents a straight line $G_\tau(u)$, the image of $G_T(u)$ on the $(z, \tau)$ plane:

$$G_\tau(u) = \left\{ (\tau, z) : \tau = \frac{\beta+\gamma}{1-\alpha} z + \frac{\ln(\frac{\mu u}{\rho_T})}{1-\alpha} \right\}. \tag{1.11}$$

Similarly, in the $(\tau, z)$ variables the curve $G_Z(u)$ where $\dot{Z} = 0$ or, equivalently, $\dot{z} = 0$ has the equation

$$\mu(1-u)e^{\alpha\tau+(\beta-1)z} = \rho_Z.$$

Dividing by $\mu(1-u)$ and taking the logarithm, we find

$$\alpha\tau + (\beta - 1)z = \ln(\frac{\rho_Z}{\mu(1-u)});$$

and, finally,

$$\tau = \frac{1-\beta}{\alpha}z + \frac{\ln(\frac{\rho_Z}{\mu(1-u)})}{\alpha}. \tag{1.12}$$

This equation represents a straight line $G_z(u)$, the image of $G_Z(u)$ on the $(z,\tau)$ plane:

$$G_z(u) = \left\{(\tau, z) : \tau = \frac{1-\beta}{\alpha}z + \frac{\ln(\frac{\rho_Z}{\mu(1-u)})}{\alpha}\right\}. \tag{1.13}$$

Let us prove statement (i). Due to (1.11) and (1.13) the rest points of system (1.9) are the solutions of the algebraic equation

$$\left\{\begin{array}{l} \tau = \frac{\beta+\gamma}{1-\alpha}z + \frac{\ln(\frac{\mu u}{\rho_T})}{1-\alpha}, \\ \tau = \frac{1-\beta}{\alpha}z + \frac{\ln(\frac{\rho_Z}{\mu(1-u)})}{\alpha}. \end{array}\right. \tag{1.14}$$

For any solution $(z,\tau)$ of (1.14) we have

$$\left(\frac{\beta+\gamma}{1-\alpha} - \frac{1-\beta}{\alpha}\right)z = \frac{\ln(\frac{\rho_Z}{\mu(1-u)})}{\alpha} - \frac{\ln(\frac{\mu u}{\rho_T})}{1-\alpha},$$

or

$$(\alpha\beta + \alpha\gamma - 1 + \alpha + \beta - \alpha\beta)z = (1-\alpha)\ln\left(\frac{\rho_Z}{\mu(1-u)}\right) - \alpha\ln\left(\frac{\mu u}{\rho_T}\right),$$

implying

$$z = \frac{\ln(\frac{\rho_Z}{\mu(1-u)})^{1-\alpha} - \ln(\frac{\mu u}{\rho_T})^{\alpha}}{\alpha+\alpha\gamma+\beta-1} = \frac{\ln((\frac{\rho_Z}{\mu(1-u)})^{1-\alpha}(\frac{\rho_T}{\mu u})^{\alpha})}{\alpha+\alpha\gamma+\beta-1} = \ln\left(\frac{(\frac{\rho_T}{u})^{\alpha}(\frac{\rho_Z}{1-u})^{1-\alpha}}{\mu}\right)^{\frac{1}{\alpha+\alpha\gamma+\beta-1}}.$$

Substituting $z$ into the second equation in (1.14), we find:

$$\begin{aligned}
\tau &= \frac{1-\beta}{\alpha}\ln\left(\frac{(\frac{\rho_T}{u})^{\alpha}(\frac{\rho_Z}{1-u})^{1-\alpha}}{\mu}\right)^{\frac{1}{\alpha+\alpha\gamma+\beta-1}} + \frac{\ln(\frac{\rho_Z}{\mu(1-u)})}{\alpha} \\
&= \ln\left(\frac{(\frac{\rho_T}{u})^{\frac{1-\beta}{\alpha+\alpha\gamma+\beta-1}}(\frac{\rho_Z}{1-u})^{\frac{1-\alpha-\beta+\alpha\beta}{\alpha(\alpha+\alpha\gamma+\beta-1)}}(\frac{\rho_Z}{\mu(1-u)})^{\frac{1}{\alpha}}}{\mu^{\frac{1-\beta}{\alpha(\alpha+\alpha\gamma+\beta-1)}}}\right) \\
&= \ln\left(\frac{(\frac{\rho_T}{u})^{\frac{1-\beta}{\alpha+\alpha\gamma+\beta-1}}(\frac{\rho_Z}{1-u})^{\frac{1-\alpha-\beta+\alpha\beta+\alpha+\alpha\gamma+\beta-1}{\alpha(\alpha+\alpha\gamma+\beta-1)}}}{\mu^{\frac{1-\beta}{\alpha(\alpha+\alpha\gamma+\beta-1)}+\frac{1}{\alpha}}}\right) \\
&= \ln\left(\frac{(\frac{\rho_T}{u})^{\frac{1-\beta}{\alpha+\alpha\gamma+\beta-1}}(\frac{\rho_Z}{1-u})^{\frac{\alpha\beta+\alpha\gamma}{\alpha(\alpha+\alpha\gamma+\beta-1)}}}{\mu^{\frac{1-\beta+\alpha+\alpha\gamma+\beta-1}{\alpha(\alpha+\alpha\gamma+\beta-1)}}}\right) \\
&= \ln\left(\frac{(\frac{\rho_T}{u})^{1-\beta}(\frac{\rho_Z}{1-u})^{\beta+\gamma}}{\mu^{1+\gamma}}\right)^{\frac{1}{\alpha+\alpha\gamma+\beta-1}}.
\end{aligned}$$

Therefore, the rest point $(z,\tau) = (z^*(u), \tau^*(u))$ of system (1.9) is unique and given by

$$(z^*(u), \tau^*(u)) = \left(\ln\left(\frac{(\frac{\rho_T}{u})^{\alpha}(\frac{\rho_Z}{1-u})^{1-\alpha}}{\mu}\right)^{\frac{1}{\alpha+\alpha\gamma+\beta-1}}, \ln\left(\frac{(\frac{\rho_T}{u})^{1-\beta}(\frac{\rho_Z}{1-u})^{\beta+\gamma}}{\mu^{1+\gamma}}\right)^{\frac{1}{\alpha+\alpha\gamma+\beta-1}}\right). \tag{1.15}$$

Hence, the rest point of the original system (1.1) is also unique and it is represented through the inverse transformation:

$$T^*(u) = e^{\tau^*(u)}, \quad Z^*(u) = e^{z^*(u)}.$$

Statement (i) is proved.

Equations (1.10) and (1.12) for the straight lines $G_\tau(u)$ and $G_z(u)$ show that in case (1.6) the slope of $G_z(u)$ is greater than that of $G_\tau(u)$ and in case (1.6) the former is smaller than the latter. Looking at system (1.9) we see that $\dot{\tau} > 0$ below $G_\tau(u)$ and $\dot{\tau} < 0$ above $G_\tau(u)$ on the $(z,\tau)$ plane; symmetrically, $\dot{z} > 0$ above $G_z(u)$ and $\dot{z} < 0$ below $G_z(u)$ on the $(z,\tau)$ plane. These obsrvations prove statements (ii) and (iii).

For every $u \in (0,1)$ we use notation $(z^*(u), \tau^*(u))$ for the image of the rest point $(Z^*(u), T^*(u))$ under the transformation (1.8); recall that $(z^*(u), \tau^*(u))$ is given by (1.15).

**Proposition 1.2** For any $u \in (0,1)$ the transformation

$$\theta = \tau - \tau^*(u), \quad \zeta = z - z^*(u) \tag{1.16}$$

brings system (1.9) to the form

$$\left\{ \begin{array}{l} \dot{\zeta} = \rho_Z e^{\alpha\theta + (\beta-1)\zeta} - \rho_Z, \\ \dot{\theta} = \rho_T e^{(\alpha-1)\theta + (\beta+\gamma)\zeta} - \rho_T \end{array} \right. \tag{1.17}$$

invariant to $u$.

**Proof.** In the $(\theta, \zeta)$ variables, system (1.9) takes the form

$$\left\{ \begin{array}{l} \dot{\zeta} = \mu(1-u)e^{\alpha(\theta+\theta^*)+(\beta-1)(\zeta+\zeta^*)} - \rho_Z, \\ \dot{\theta} = \mu u e^{(\alpha-1)(\theta+\theta^*)+(\beta+\gamma)(\zeta+\zeta^*)} - \rho_T \end{array} \right.$$

which is sequentially transformed into

$$\left\{ \begin{array}{l} \dot{\theta} = \mu u \left( \frac{(\frac{\rho_T}{u})^{1-\beta}(\frac{\rho_Z}{1-u})^{\beta+\gamma}}{\mu^{1+\gamma}} \right)^{\frac{\alpha-1}{\alpha+\alpha\gamma+\beta-1}} \left( \frac{(\frac{\rho_T}{u})^{\alpha}(\frac{\rho_Z}{1-u})^{1-\alpha}}{\mu} \right)^{\frac{\beta+\gamma}{\alpha+\alpha\gamma+\beta-1}} e^{(\alpha-1)\theta+(\beta+\gamma)\zeta} - \rho_T, \\ \dot{\zeta} = \mu(1-u) \left( \frac{(\frac{\rho_T}{u})^{1-\beta}(\frac{\rho_Z}{1-u})^{\beta+\gamma}}{\mu^{1+\gamma}} \right)^{\frac{\alpha}{\alpha+\alpha\gamma+\beta-1}} \left( \frac{(\frac{\rho_T}{u})^{\alpha}(\frac{\rho_Z}{1-u})^{1-\alpha}}{\mu} \right)^{\frac{\beta-1}{\alpha+\alpha\gamma+\beta-1}} e^{\alpha\theta+(\beta-1)\zeta} - \rho_Z, \end{array} \right.$$

$$\left\{ \begin{array}{l} \dot{\theta} = \mu^{\frac{(\alpha+\alpha\gamma+\beta-1)-(1+\gamma)(\alpha-1)-(\beta+\gamma)}{\alpha+\alpha\gamma+\beta-1}} \rho_T^{\frac{(1-\beta)(\alpha-1)+\alpha(\beta+\gamma)}{\alpha+\alpha\gamma+\beta-1}} \rho_Z^{\frac{(\beta+\gamma)(\alpha-1)+(1-\alpha)(\beta+\gamma)}{\alpha+\alpha\gamma+\beta-1}} \times \\ \qquad \times u^{\frac{(\alpha+\alpha\gamma+\beta-1)-(1-\beta)(\alpha-1)-\alpha(\beta+\gamma)}{\alpha+\alpha\gamma+\beta-1}} (1-u)^{\frac{-(\beta+\gamma)(\alpha-1)-(1-\alpha)(\beta+\gamma)}{\alpha+\alpha\gamma+\beta-1}} e^{(\alpha-1)\theta+(\beta+\gamma)\zeta} - \rho_T, \\ \dot{\zeta} = \mu^{\frac{(\alpha+\alpha\gamma+\beta-1)-(1+\gamma)\alpha-(\beta-1)}{\alpha+\alpha\gamma+\beta-1}} \rho_T^{\frac{(1-\beta)\alpha+\alpha(\beta-1)}{\alpha+\alpha\gamma+\beta-1}} \rho_Z^{\frac{(\beta+\gamma)\alpha+(1-\alpha)(\beta-1)}{\alpha+\alpha\gamma+\beta-1}} \times \\ \qquad \times u^{\frac{-(1-\beta)\alpha-\alpha(\beta-1)}{\alpha+\alpha\gamma+\beta-1}} (1-u)^{\frac{(\alpha+\alpha\gamma+\beta-1)-(\beta+\gamma)\alpha-(1-\alpha)(\beta-1)}{\alpha+\alpha\gamma+\beta-1}} e^{\alpha\theta+(\beta-1)\zeta} - \rho_Z \end{array} \right.$$

and finally into (1.17).

In what follows, we call system (1.17) the *invariant system*.

On the state plane of the invariant system the images of the curves $G_T(u)$ (1.5) and $G_Z(u)$ (1.4) under the transformations (1.8) and (1.16) (or the images of the straight lines $G_\tau(u)$ (1.10) and $G_z(u)$ (1.12) under transformation (1.16)) are, respectively, the straight line $G_\theta(u)$ given by

$$\theta = \frac{\beta+\gamma}{1-\alpha}\zeta \tag{1.18}$$

and straight line $G_\zeta(u)$ given by

$$\theta = \frac{1-\beta}{\alpha}\zeta. \tag{1.19}$$

We denote by $G_\theta^+(u)$ and by $G_\theta^-(u)$ the parts of the straight line $G_\theta(u)$ which lie in the non-negative and non-positive orthants, respectively; symmetrically, we denote by $G_\zeta^+(u)$ and by $G_\zeta^-(u)$ the parts of the straight line $G_\zeta(u)$ which lie in the non-negative and non-positive orthants, respectively. The images of the north-east, south-east, south-west and north-west angles $G_{ZT}^{++}(u)$, $G_{ZT}^{+-}(u)$, $G_{ZT}^{--}(u)$, $G_{ZT}^{-+}(u)$ under the transformations (1.8) and (1.16) will be denoted as $G_{\zeta\theta}^{++}(u)$, $G_{\zeta\theta}^{+-}(u)$, $G_{\zeta\theta}^{--}(u)$, $G_{\zeta\theta}^{-+}(u)$, respectively, and called the *invariant* north-east, south-east, south-west and north-west angles, respectively.

The next remark follows straightforwardly from the given definitions.

**Remark 1.1** 1. If case 1 (1.6), stagnation, takes place, then
(i) the slope of $G_\zeta$ on the $(\zeta, \theta)$ plain is greater than the slope of $G_\theta$,
(ii) the invariant north-east angle $G_{\zeta\theta}^{++}$ is bordered by the half-lines $G_\zeta^+$ and $G_\theta^+$,
(iii) the invariant south-west angle $G_{\zeta\theta}^{--}$ is bordered by the half-lines $G_\zeta^-$ and $G_\theta^-$,
(iv) the invariant north-west angle $G_{\zeta\theta}^{+-}$ is bordered by the half-lines $G_\zeta^+$ and $G_\theta^-$,
(v) the invariant south-east angle $G_{\zeta\theta}^{-+}$ is bordered by the half-lines $G_\zeta^-$ and $G_\theta^+$,
(vi) the vector field of the invariant system (1.17) has the form shown in Fig. 1.3.

2. If case 2 (1.7), progress, takes place, then
(i) the slope of $G_\zeta$ on the $(\zeta, \theta)$ plain is smaller than the slope of $G_\theta$,
(ii) the invariant north-east angle $G_{\zeta\theta}^{++}$ is bordered by the half-lines $G_\zeta^+$ and $G_\theta^+$,
(iii) the invariant south-west angle $G_{\zeta\theta}^{--}$ is bordered by the half-lines $G_\zeta^-$ and $G_\theta^-$,
(iv) the invariant north-west angle $G_{\zeta\theta}^{+-}$ is bordered by the half-lines $G_\zeta^-$ and $G_\theta^+$,
(v) the invariant south-east angle $G_{\zeta\theta}^{-+}$ is bordered by the half-lines $G_\zeta^+$ and $G_\theta^-$,
(vi) the vector field of the invariant system (1.17) has the form shown in Fig. 1.4.

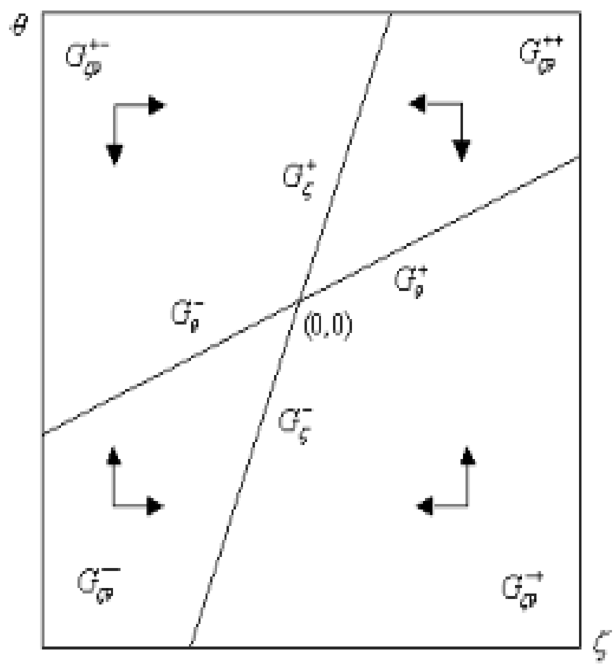


Fig. 1.3.
The vector field of the invariant system in case 1, stagnation.

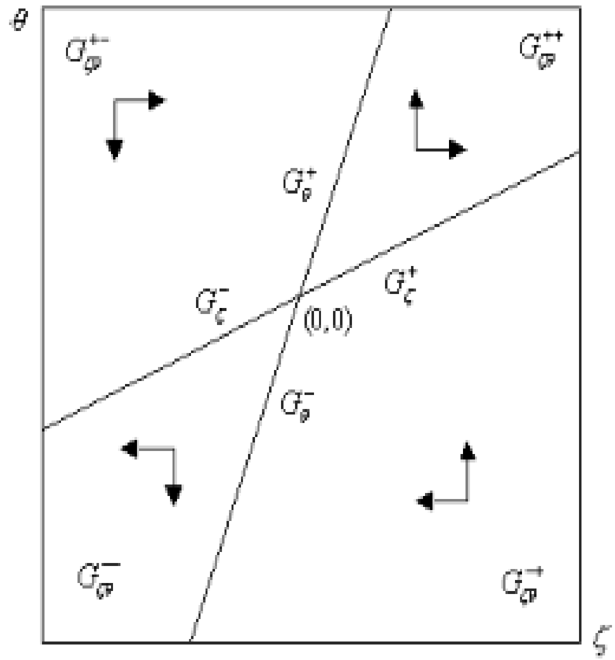

Fig. 1.4.
The vector field of the invariant system in case 2, progress.

## 2 Basic definitions

Kryazhimskii, et. al., 2002, definies the basic bahaviors of the techno-labor system (1.1) as follows.

It is said that

(i) the techno-labor system (1.1) with the initial state $(Z_0, T_0)$ exhibits *homeostasis* under control $u$ if for the solution $t \mapsto (Z(t), T(t))$ of the Cauchy problem (1.1), (1.3) the functions $t \mapsto Z(t)$ and $t \mapsto T(t)$ are strictly increasing on interval $[0, \infty)$;

(ii) if, in addition, both $Z(t)$ and $T(t)$ tend to $\infty$ as $t$ tends to $\infty$, we shall say that the techno-labor system (1.1) with the initial state $(Z_0, T_0)$ exhibits *progressive homeostasis* under control $u$;

(iii) finally, if both $Z(t)$ and $T(t)$ tend to finite limits as $t$ tends to $\infty$, we shall say that the techno-labor system (1.1) with the initial state $(Z_0, T_0)$ exhibits *regressive homeostasis* under control $u$.

It is said that

(i) the techno-labor system (1.1) with the initial state $(Z_0, T_0)$ exhibits *pre-homeostasis* under control $u$ if for the solution $t \mapsto (Z(t), T(t))$ of the Cauchy problem (1.1), (1.3) there exists a $t_0 \geq 0$ such that the functions $t \mapsto Z(t)$ and $t \mapsto T(t)$ are strictly increasing on interval $[t_0, \infty)$;

(ii) if, in addition, both $Z(t)$ and $T(t)$ tend to $\infty$ as $t$ tends to $\infty$, we shall say that the techno-labor system (1.1) with the initial state $(Z_0, T_0)$ exhibits *progressive pre-homeostasis* under control $u$;

(iii) finally, if both $Z(t)$ and $T(t)$ tend to finite limits as $t$ tends to $\infty$, we shall say that the techno-labor system (1.1) with the initial state $(Z_0, T_0)$ exhibits *regressive pre-homeostasis* under control $u$. It is said that

(i) the techno-labor system (1.1) with the initial state $(Z_0, T_0)$ exhibits *collapse* under control $u$ if for the solution $t \mapsto (Z(t), T(t))$ of the Cauchy problem (1.1), (1.3) the functions $t \mapsto Z(t)$ and $t \mapsto T(t)$ are strictly decreasing on interval $[0, \infty)$;

(ii) if, in addition, both $Z(t)$ and $T(t)$ tend to positive limits as $t$ tends to $\infty$, we shall say that the techno-labor system (1.1) with the initial state $(Z_0, T_0)$ exhibits *limited collapse* under control $u$;

(iii) finally, if both $Z(t)$ and $T(t)$ tend to 0 as $t$ tends to $\infty$, we shall say that the techno-labor system (1.1) with the initial state $(Z_0, T_0)$ exhibits *total collapse* under control $u$. It is said that

(i) the techno-labor system (1.1) with the initial state $(Z_0, T_0)$ exhibits *pre-collapse* under control $u$ if for the solution $t \mapsto (Z(t), T(t))$ of the Cauchy problem (1.1), (1.3) there exists a $t_0 \geq 0$ such that the functions $t \mapsto Z(t)$ and $t \mapsto T(t)$ are strictly decreasing on interval $[t_0, \infty)$;

(ii) if, in addition, both $Z(t)$ and $T(t)$ tend to positive limits as $t$ tends to $\infty$, we shall say that the techno-labor system (1.1) with the initial state $(Z_0, T_0)$ exhibits *limited pre-collapse* under control $u$;

(iii) finally, if both $Z(t)$ and $T(t)$ tend to 0 as $t$ tends to $\infty$, we shall say that the techno-labor system (1.1) with the initial state $(Z_0, T_0)$ exhibits *total pre-collapse* under control $u$.

It is said that

(i) the techno-labor system (1.1) with the initial state $(Z_0, T_0)$ exhibits *growth in welfare and decline in technologies* under control $u$ if for the solution $t \mapsto (Z(t), T(t))$ of the Cauchy problem (1.1), (1.3) the function $t \mapsto Z(t)$ is strictly increasing and the function $t \mapsto T(t)$ strictly decreasing on interval $[0, \infty)$;

(ii) the techno-labor system (1.1) with the initial state $(Z_0, T_0)$ exhibits *growth in technologies and decline in welfare* under control $u$ if for the solution $t \mapsto (Z(t), T(t))$ of the Cauchy problem (1.1), (1.3) the function $t \mapsto Z(t)$ is strictly decreasing and the function $t \mapsto T(t)$ strictly increasing on interval $[0, \infty)$.

The next definitions are given for a fixed control $u$.

We denote by $H^{++}(u)$ the set of all $(Z_0, T_0) \in O^+$ such that the techno-labor system (1.1) with the initial state $(Z_0, T_0)$ exhibits homeostasis under control $u$ and by $H(u)$ the set of all $(Z_0, T_0) \in O^+$ such that the techno-labor system (1.1) with the initial state $(Z_0, T_0)$ exhibits pre-homeostasis under control $u$. We call $H^{++}(u)$ the *zone of homeostasis under control* $u$ and $H(u)$ the *zone of pre-homeostasis under control* $u$.

We denote by $C^{--}(u)$ the set of all $(Z_0, T_0) \in O^+$ such that the techno-labor system (1.1) with the initial state $(Z_0, T_0)$ exhibits collapse under control $u$ and by $C(u)$ the set of all $(Z_0, T_0)$ in $O^+$ such that the techno-labor system (1.1) with the initial state $(Z_0, T_0)$ exhibits pre-collapse under control $u$. We call $C^{--}(u)$ the *zone of collapse under control* $u$ and $C(u)$ the *zone of pre-collapse under control* $u$.

## 3 Case 1: stagnation

The next proposition provides an entire characterization of the behaviors of the techno-labor system (1.1) in case 1, stagnaion. A graphical illustration is given in Fig. 3.1.

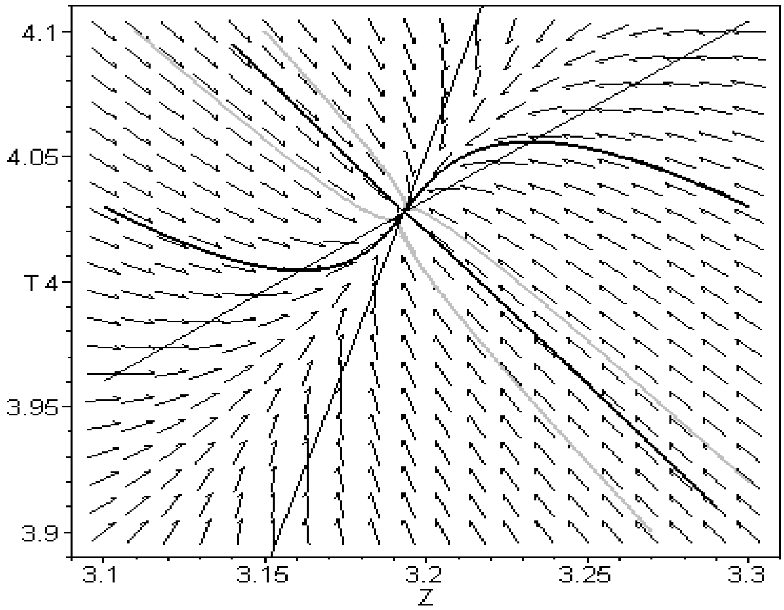


Fig. 3.1.
Trajectories of the techno-labor system in case 1, stagnation.
The separation curves in the north-west and south-east
angles are shown in grey.

**Proposition 3.1** (Kryazhimskii, et. al., 2002, Proposition 4.1). Let case 1, stagnation, take place, i.e., (1.6) hold. Let $u \in (0, 1)$ be an arbitrary control. Then

(i) the rest point $(Z^*(u), T^*(u))$ is the unique attractor for the techno-labor system (1.1) under control $u$; more accurately, for any initial state $(Z_0, T_0)$, the solution $t \mapsto (Z(t), T(t))$ of the Cauchy problem (1.1), (1.3) satisfies $\lim_{t\to\infty} Z(t) = Z^*(u)$ and $\lim_{t\to\infty} T(t) = T^*(u)$;

(ii) the zone of homeostasis under control $u$, $H^{++}(u)$, is the south-west angle $G^{--}_{ZT}(u)$; moreover, the zone of regressive homeostasis under control $u$ coincides with $H^{++}(u)$;

(iii) the zone of collapse under control $u$, $C^{--}(u)$, is the north-east angle $G^{++}_{ZT}(u)$; moreover, the zone of limited collapse under control $u$ coincides with $C^{--}(u)$;

(iv) there exists the unique solution $t \mapsto (Z^{+-}_{-}(t), T^{+-}_{-}(t))$ of system (1.1), which is defined on $(-\infty, \infty)$, takes values, in the north-west angle, $G^{+-}_{ZT}(u)$, and is minimal in the following sense: for every $(Z_0, T_0)$ located to the south-west of the trajectory, $\Lambda^{+-}_{-}(u)$, of the solution $t \mapsto (Z^{+-}_{-}(t), T^{+-}_{-}(t))$, the solution $t \mapsto (Z(t), T(t))$ of system (1.1), with the initial state $(Z_0, T_0)$ crosses the boundary of the north-west angle, $G^{+-}_{ZT}(u)$;

(v) there exists the unique solution $t \mapsto (Z^{+-}_{+}(t), T^{+-}_{+}(t))$ of system (1.1), which is defined on $(-\infty, \infty)$, takes values in the north-west angle, $G^{+-}_{ZT}(u)$, and is maximal in the following sense: for every $(Z_0, T_0)$ located to the north-east of the trajectory, $\Lambda^{+-}_{+}(u)$, of the solution $t \mapsto (Z^{+-}_{+}(t), T^{+-}_{+}(t))$, the solution $t \mapsto (Z(t), T(t))$ of system (1.1), with the initial state $(Z_0, T_0)$ crosses the boundary of the north-west angle, $G^{+-}_{ZT}(u)$;

(vi) there exists the unique solution $t \mapsto (Z^{-+}_{-}(t), T^{-+}_{-}(t))$ of system (1.1), which is defined on $(-\infty, \infty)$, takes values in the south-east angle, $G^{-+}_{ZT}(u)$, and is minimal in the following sense: for every $(Z_0, T_0)$ located to the south-west of the trajectory, $\Lambda^{-+}_{-}(u)$, of the solution $t \mapsto (Z^{-+}_{-}(t), T^{-+}_{-}(t))$, the solution $t \mapsto (Z(t), T(t))$ of system (1.1), with the initial state $(Z_0, T_0)$ crosses the boundary of the south-east angle, $G_{ZT}^{-+}(u)$;

(vii) there exists the unique solution $t \mapsto (Z_+^{-+}(t), T_+^{-+}(t))$ of system (1.1), which is defined on $(-\infty, \infty)$, takes values in the south-east angle, $G_{ZT}^{-+}(u)$, and is maximal in the following sense: for every $(Z_0, T_0)$ located to the north-east of the trajectory, $\Lambda_+^{-+}(u)$, of the solution $t \mapsto (Z_+^{-+}(t), T_+^{-+}(t))$, the solution $t \mapsto (Z(t), T(t))$ of system (1.1), with the initial state $(Z_0, T_0)$ crosses the boundary of the south-east angle, $G_{ZT}^{-+}(u)$;

(viii) $H(u)$, the zone of pre-homeostasis under control $u$, is the union of the domain $\hat{H}^{+-}(u)$ located in the north-west angle, $G_{ZT}^{+-}(u)$, to the south-west of trajectory $\Lambda_-^{+-}(u)$, and the domain $\hat{H}^{-+}(u)$ located in the south-east angle $G_{ZT}^{-+}(u)$ to the south-west of trajectory $\Lambda_-^{-+}(u)$; moreover, the zone of regressive pre-homeostasis under control $u$ coincides with $H(u)$;

(ix) $C(u)$, the zone of pre-collapse under control $u$, is the union of the domain $\hat{C}^{+-}(u)$ located in the north-west angle, $G_{ZT}^{+-}(u)$, to the north-east of trajectory $\Lambda_+^{+-}(u)$, and the domain $\hat{C}^{-+}(u)$ located in the south-east angle $G_{ZT}^{-+}(u)$ to the north-east of trajectory $\Lambda_+^{-+}(u)$; moreover, the zone of limited pre-collapse under control $u$ coincides with $C(u)$;

(x) for every $(Z_0, T_0)$ located in the north-west angle, $G_{ZT}^{+-}(u)$, between the trajectories $\Lambda_-^{+-}(u)$ and $\Lambda_+^{+-}(u)$ the techno-labor system (1.1) with the initial state $(Z_0, T_0)$ exhibits growth in welfare and decline in technologies under control $u$;

(xi) for every $(Z_0, T_0)$ located in the south-east angle, $G_{ZT}^{-+}(u)$, between the trajectories $\Lambda_-^{-+}(u)$ and $\Lambda_+^{-+}(u)$ the techno-labor system (1.1) with the initial state $(Z_0, T_0)$ exhibits growth in technologies and decline in welfare under control $u$.

**Proof.** We use the transformations (1.8) and (1.16) and represent the techno-labor system (1.1) as the invariant system (1.17) thus shifting the stationary point $(Z^*(u), T^*(u))$ to the origin.

Let us prove (i).

We notice that the powers of the exponent in (1.17) go to 0 as $\theta, \zeta \to 0$ and linearize system (1.17) in a neighborhood of the origin using the relation $\lim_{x\to 0}(e^x - 1)/x = 1$. The linarized system has the form

$$\begin{aligned}\dot{\theta} &= \rho_T((\alpha-1)\theta + (\beta+\gamma)\zeta),\\ \dot{\zeta} &= \rho_Z(\alpha\theta + (\beta-1)\zeta).\end{aligned}$$

Its characteristic equation,

$$\begin{vmatrix} \rho_T(\alpha-1)-\lambda & \rho_T(\beta+\gamma) \\ \rho_Z\alpha & \rho_Z(\beta-1)-\lambda \end{vmatrix} = 0,$$

is sequenially transformed into

$$\rho_T\rho_Z(\alpha\beta - \alpha - \beta + 1) - \rho_Z(\beta-1)\lambda - \rho_T(\alpha-1)\lambda + \lambda^2 - \rho_T\rho_Z(\alpha\beta + \alpha\gamma) = 0$$

and

$$\lambda^2 + \lambda(\rho_T(1-\alpha) + \rho_Z(1-\beta)) - \rho_T\rho_Z(\alpha + \alpha\gamma + \beta - 1) = 0.$$

The roots of the characteristic equation, $\lambda_1$ and $\lambda_2$, are given by

$$\lambda_{1,2} = \frac{1}{2}\left(\rho_T(\alpha-1) + \rho_Z(\beta-1) \pm \sqrt{(\rho_T(\alpha-1) + \rho_Z(\beta-1))^2 + 4\rho_T\rho_Z(\alpha + \alpha\gamma + \beta - 1)}\right). \tag{3.1}$$

For the determinant we have:

$$\begin{aligned}&(\rho_T(\alpha-1)+\rho_Z(\beta-1))^2+4\rho_T\rho_Z(\alpha+\alpha\gamma+\beta-1)=\\&\rho_T^2(\alpha-1)^2+\rho_Z^2(\beta-1)^2+2\rho_T\rho_Z(\alpha\beta-\alpha-\beta+1)+4\rho_T\rho_Z(\alpha+\alpha\gamma+\beta-1)=\\&\rho_T^2(\alpha-1)^2+\rho_Z^2(\beta-1)^2+2\rho_T\rho_Z(\alpha\beta+\alpha+\beta+2\alpha\gamma-1)=\\&\rho_T^2(\alpha-1)^2+\rho_Z^2(\beta-1)^2-2\rho_T\rho_Z(\alpha\beta-\alpha-\beta+1)+4\rho_T\rho_Z(\alpha\beta+\alpha\gamma)=\\&(\rho_T(\alpha-1)-\rho_Z(\beta-1))^2+4\rho_T\rho_Z\alpha(\beta+\gamma)>0.\end{aligned}\tag{3.2}$$

Hence, $\lambda_1$ and $\lambda_2$ are real. Moreover, (1.6) implies that $\lambda_1$ and $\lambda_2$ are negative. Therefore, the rest point $(0,0)$ is a knot, implying that it is the unique attraction point for system (1.17). This proves statement (i).

Let us prove (ii).

The right-hand sides of both equations in (1.17) are positive if and only if $(\theta,\zeta)$ belongs to the interior of the invariant south-west angle, $G^{--}_{\zeta\theta}$ (see Fig. 1.3). Therefore, the transformed zone of homeostasis under control $u$, $\bar{H}^{++}(u)$, lies necessarily in $G^{--}_{\zeta\theta}$. In order to state that $\bar{H}^{++}=G^{--}_{\zeta\theta}$ (which implies that $H^{++}(u)=G^{--}_{ZT}(u)$) it suffices to show that the invariant system (1.17) survives in $G^{--}_{\zeta\theta}$ (see Aubin, 1991), i.e., its solution $t\mapsto(\zeta(t),\theta(t))$ satisfies $(\zeta(t),\theta(t))\in G^{--}_{\zeta\theta}$ for all $t\ge 0$ provided $(\zeta(0),\theta(0))\in G^{--}_{\zeta\theta}$. The latter property holds if the vector field of (1.17) points inside $G^{--}_{\zeta\theta}$ at every point $(\zeta,\theta)$ on the boundary of $G^{--}_{\zeta\theta}$. The boundary of $G^{--}_{\zeta\theta}$ is the union of the half-lines lines $G^-_\theta$ and $G^-_\zeta$ (the former is located above the latter, see Remark 1.1, 1, (i), and Fig. 1.3). Let $(\zeta,\theta)$ lie on the "upper" border $G^-_\theta$. At this point, the right hand side of the invariant system (1.17) is represented as

$$f=\left(\rho_Z e^{\alpha(\frac{\beta+\gamma}{1-\alpha}\zeta)+(\beta-1)\zeta}-\rho_Z,\rho_T e^{(\alpha-1)(\frac{\beta+\gamma}{1-\alpha}\zeta)+(\beta+\gamma)\zeta}-\rho_T\right)=\left(\rho_Z e^{\frac{\alpha+\alpha\gamma+\beta-1}{1-\alpha}\zeta}-\rho_Z,0\right).\tag{3.3}$$

The inequalities $\zeta<0$ and (1.6) imply that $f$ points to the right (on the $(\zeta,\theta)$ plane), i.e., inside $G^{--}_{\zeta\theta}$. Let $(\zeta,\theta)$ lie on the "lower" border $G^-_\zeta$. At this point, the right-hand side of (1.17) is represented as

$$f=\left(\rho_Z e^{\alpha(\frac{1-\beta}{\alpha}\zeta)+(\beta-1)\zeta}-\rho_Z,\rho_T e^{(\alpha-1)(\frac{1-\beta}{\alpha}\zeta)+(\beta+\gamma)\zeta}-\rho_T\right)=\left(0,\rho_T e^{\frac{\alpha+\alpha\gamma+\beta-1}{\alpha}\zeta}-\rho_T\right).\tag{3.4}$$

The inequalities $\zeta<0$ and (1.6) imply that $f$ points upwards, i.e., inside $G^{--}_{\zeta\theta}$. Thus, system (1.17) survives in $G^{--}_{\zeta\theta}$, which proves that $\hat{H}^{++}=G^{--}_{\zeta\theta}$ implying $H^{++}(u)=G^{--}_{ZT}(u)$. Finally, the fact that all the solutions of system (1.17) converge to the origin (see statement (i)) yields that $H^{++}=G^{--}_{ZT}$ is the zone of regressive homeostasis. Statement (ii) is proved.

Statement (iii) is proved identically.

Let us prove (iv).

Again we argue in terms of the invariant system (1.17). Consider the angle area $G^{+-}_{\zeta\theta}$ whose "lower" boudary is the half-line $G^-_\theta$ and "upper" boundary is the half-line $G^+_\zeta$ (see Remark 1.1, 1, (iv), and Fig. 1.3). Take a $(\zeta_0,\theta_0)\in G^-_\theta$ and a $(\zeta_1,\theta_1)\in G^+_\zeta$. For every $\lambda\in[0,1]$ define the solution $t\mapsto(\zeta_\lambda(t),\theta_\lambda(t))$ on $[0,\infty)$ of (1.17) by

$$\zeta_\lambda(0)=\zeta_0+\lambda(\zeta_1-\zeta_0),\quad \theta_\lambda(0)=\theta_0+\lambda(\theta_1-\theta_0).$$

Obviously, $(\zeta_\lambda(0),\theta_\lambda(0))\in G^{+-}_{\zeta\theta}$ for all $\lambda\in(0,1)$. Let $L$ be the set of all $\bar\lambda\in[0,1]$ such that for every $\lambda\in[0,\bar\lambda]$ there is a $t_*\ge 0$ for which $(\zeta_\lambda(t_*),\theta_\lambda(t_*))\in G_\theta$ and $(\zeta_\lambda(t),\theta_\lambda(t))\in$

$G^{+-}_{\zeta\theta}$ for all $t \in [0, t_*)$. Obviously, $0 \in L$. We set $\lambda_* = \sup L$. Consider the solution $t \mapsto (\zeta_*(t), \theta_*(t)) = (\zeta_{\lambda_*}(t), \theta_{\lambda_*}(t))$.

Let us prove that $(\zeta_*(t), \theta_*(t)) \in G^{+-}_{\zeta\theta}$ for all $t \geq 0$. Suppose this is untrue. Then $(\zeta_*(t_*), \theta_*(t_*))$ belongs to the boundary of $G^{+-}_{\zeta\theta}$ for some $t_* \geq 0$. Suppose $(\zeta_*(t_*), \theta_*(t_*))$ belongs to $G^-_\theta$, the "lower" boundary of $G^{+-}_{\zeta\theta}$. With no loss of generality we assume that $t_*$ is the minimal point in time with this property, i.e., $(\zeta_*(t), \theta_*(t)) \in G^{+-}_{\zeta\theta}$ for all $t \in [0, t_*)$. As shown in the proof of statement (ii), at point $(\zeta, \theta) = (\zeta_*(t), \theta_*(t))$ the right-hand side of system (1.17), $f$ (3.3), points to the right. Therefore, $(\zeta_{\lambda_*}(t_* + \delta), \theta_{\lambda_*}(t_* + \delta)) = (\zeta_*(t_* + \delta), \theta_*(t_* + \delta))$ lies in the negative orthant below the straight line $G_\theta$ for all $\delta \in (0, \delta_*]$ with some $\delta_* > 0$. By the continuity of the solution of (1.17) in the initial state we conclude that for every $\lambda \in [0, 1]$ sufficiently close to $\lambda_*$, $(\zeta_\lambda(t_* + \delta), \theta_\lambda(t_* + \delta))$ lies in the negative orthant for all $\delta \in (0, \delta_*]$ and $(\zeta_\lambda(t_* + \delta_*), \theta_\lambda(t_* + \delta_*))$ lies below $G_\theta$ on the $(\zeta, \theta)$ plane. Consequently, all $\lambda \in [0, 1]$ sufficiently close to $\lambda_*$ belong to $L$ which contradicts the equality $\lambda_* = \sup L$ if $\lambda_* < 1$. Thus, $\lambda_* = 1$. Then $(\zeta_*(0), \theta_*(0)) = (\zeta_1, \theta_1)$ lies in the intersection of the straight line $G_\zeta$ and the positive orthant. One can easily show that at point $(\zeta_1, \theta_1)$ the right-hand side of the invariant system (1.17), points downwards. Therefore, $(\zeta_{\lambda_*}(t_* + \delta), \theta_{\lambda_*}(t_* + \delta)) = (\zeta_*(t_* + \delta), \theta_*(t_* + \delta))$ lies in the positive orthant below the half-line $G^+_\zeta$ for all $\delta \in (0, \delta_*]$ with some $\delta_* > 0$. By the continuity of the solution of (1.17) with respect to the initial state we conclude that for every $\lambda \in [0, 1]$ sufficiently close to $\lambda_*$, $(\zeta_\lambda(t_* + \delta), \theta_\lambda(t_* + \delta))$ lies in the positive orthant for all $\delta \in (0, \delta_*]$ and $(\zeta_\lambda(t_* + \delta_*), \theta_\lambda(t_* + \delta_*))$ lies below $G^+_\zeta$ on the $(\zeta, \theta)$ plane. Therefore, for all $\lambda \in [0, 1]$ sufficiently close to $\lambda_*$ we have $\lambda \notin L$ which contradicts the equality $\lambda_* = \sup L$. Thus, $\lambda_* = 1$ is not possible. This shows that for all $t \geq 0$ $(\zeta_*(t), \theta_*(t))$ is not on the "lower" boundary of $G^{+-}_{\zeta\theta}$. A similar argument leads to the symmetric conclusion that for all $t \geq 0$ $(\zeta_*(t), \theta_*(t))$ does not belong to $G^+_\zeta$, the "upper" boundary of $G^{+-}_{\zeta\theta}$. This proves that $\zeta_*(t), \theta_*(t)) \in G^{+-}_{\zeta\theta}$ for all $t \geq 0$.

Now we extend the solution $t \mapsto (\zeta_*(t), \theta_*(t))$ to $(-\infty, \infty)$. In $G^{+-}_{\zeta\theta}$ the vector field of system (1.17) points south-east; therefore, $(\zeta_*(t), \theta_*(t)) G^{+-}_{\zeta\theta}$ for all $t \in (\infty, 0]$ and, consequently, for all $t \in (\infty, \infty)$. Let $t \mapsto (Z^{+-}_-(t), T^{+-}_-(t)$ be the image of $t \mapsto (\zeta_*(t), \theta_*(t))$ under the transformations inverse to (1.8) and (1.16). Obviously $t \mapsto (Z^{+-}_-(t), T^{+-}_-(t)$ is a solution of the techno-labor system (1.1) which takes values in the north-west angle $G^{+-}_{ZT}$.

Let us show that $t \mapsto (Z^{+-}_-(t), T^{+-}_-(t)$ is minimal in the sense explained in (iv). Take arbitrary $(\zeta, \theta) \in G^{+-}_{\zeta\theta}$ located below the trajectory $l_* = \{(\zeta_*(t), \theta_*(t)) : t \in (-\infty, \infty)\}$ and consider the solution $t \mapsto (\zeta(t), \theta(t))$ on $(-\infty, \infty)$ of (1.17) such that $(\zeta(0), \theta(0)) = (\zeta, \theta)$. It is sufficient to show that this solution crosses $G^-_\theta$, the "lower" boundary of $G^{+-}_{\zeta\theta}$. Suppose this is untrue. Then $(\zeta(t), \theta(t)) \in G^{+-}_{\zeta\theta}$ for all real $t$. By (i) $(\zeta(t), \theta(t)) \to (0, 0)$ as $t \to \infty$. Therefore, there is a $t_*$ such that $(\zeta(t_*), \theta(t_*))$ lies in the triagnle formed by the the "lower" and "upper" boundaries of $G^{+-}_{\zeta\theta}$ and the segment $l$ connecting $(\zeta_0, \theta_0)$ and $(\zeta_1, \theta_1)$ (recall that $(\zeta_0, \theta_0)$ and $(\zeta_1, \theta_1)$ lie on the "lower" and "upper" boundaries of $G^{+-}_{\zeta\theta}$, respectively). In $G^{+-}_{\zeta\theta}$ the vector field of system (1.17) points south-east; therefore, there is a $t_0 \leq t_*$ such that $(\zeta(t_0), \theta(t_0))$ lies on the segment $l$ with the end points $(\zeta_0, \theta_0)$ and $(\zeta_1, \theta_1)$. Then $(\zeta(t_0), \theta(t_0)) = (\zeta_\lambda), \theta_\lambda)$ for some $\lambda \in [0, 1]$. Trajectory $l_*$ crosses segment $l$ at $(\zeta_{\lambda_*}), \theta_{\lambda_*})$ by definition. Since $(\zeta(t_0), \theta(t_0)) = (\zeta_\lambda), \theta_\lambda)$ lies below $l_*$, it lies below $(\zeta_{\lambda_*}, \theta_{\lambda_*}) \in l_*$, implying $\lambda < \lambda_* = \sup L$. Hence, $\lambda \in L$. By the definition of $L$ the solution $t \mapsto (\zeta_\lambda(t), \theta_\lambda(t)) = (\zeta(t), \theta(t))$ crosses the "lower" boundary of $G^{+-}_{\zeta\theta}$ which contradicts the assumption. This completes the proof of statement (iv).

Statements (v) – (vii) are proved using similar arguments.

Statements (viii) – (xi) follow straightforwadrly from (i) and (iv) – (vii) and from the definitions of the zone of regressive pre-homeostasis, zone of limited pre-collapse, zone of growth in welfare and decline in technologies and zone of growth in technologies and decline in welfare.

The proposition is proved.

## 4 Case 2: progress

The next proposition provides an entire characterization of the behaviors of the techno-labor system (1.1) in case 2, stagnaion. A graphical illustration is given in Fig. 4.1.

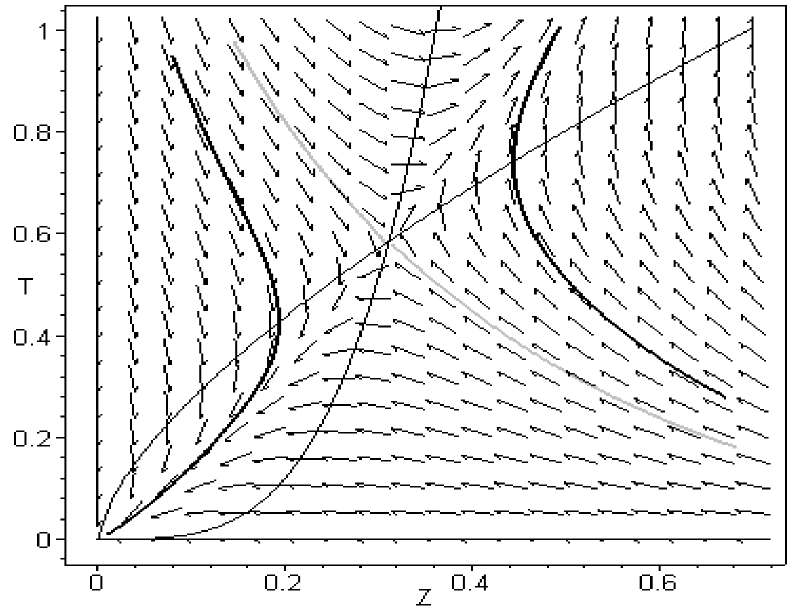


Fig. 4.1.
Trajectories of the techno-labor system in case 2, progress.
The separation curves in the north-west and south-east
angles are shown in grey.

**Proposition 4.1** (Kryazhimskii, et. al., 2002, Proposition 4.2). Let case 2 (progress) take place, i.e., (1.7) hold. Let $u$ be an arbitrary control. Then

(i) the rest point $(Z^*(u), T^*(u))$ of the techno-labor system (1.1) under control $u$ is unstable;

(ii) the zone of homeostasis under control $u$, $H^{++}(u)$, is the north-east angle $G^{++}_{ZT}(u)$; moreover, the zone of progressive homeostasis under control $u$ coincides with $H^{++}(u)$;

(iii) the zone of collapse under control $u$, $C^{--}(u)$, is the south-west angle $G^{--}_{ZT}(u)$; moreover, the zone of total collapse under control $u$ coincides with $C^{--}(u)$;

(iv) there exists the unique solution $t \mapsto (Z^{-+}(t), T^{-+})$ of system (1.1), which is defined on $(-\infty, \infty)$ and takes values in the north-west angle, $G^{+-}_{ZT}(u)$; moreover, the trajectory $\Lambda^{+-}(u)$ of this solution splits $G^{+-}_{ZT}(u)$, in two open areas, $\hat{H}^{+-}(u)$ and $\hat{C}^{+-}(u)$, adjoining the north-east angle $G^{++}_{ZT}(u)$ and south-west angle $G^{--}_{ZT}(u)$ respectively;

(v) symmetrically, there exists the unique solution $t \mapsto (Z^{-+}(t), T^{-+})$ of system (1.1), which is defined on $(-\infty, \infty)$ and takes values in the south-east angle, $G^{-+}_{ZT}(u)$; moreover, the trajectory $\Lambda^{-+}(u)$ of this solution splits $G^{-+}_{ZT}(u)$, in two open areas, $\hat{H}^{-+}(u)$ and $\hat{C}^{-+}(u)$, adjoining the north-east angle $G^{++}_{ZT}(u)$ and south-west angle $G^{--}_{ZT}(u)$ respectively;

(vi) $H(u)$, the zone of pre-homeostasis under control $u$, is the union of $\hat{H}^{+-}(u)$ and $\hat{H}^{-+}(u)$; moreover, the zone of progressive pre-homeostasis under control $u$ coincides with $H(u)$;

(vii) $C(u)$, the zone of pre-collapse under control $u$, is the union of $\hat{C}^{+-}(u)$ and $\hat{C}^{-+}(u)$; moreover, the zone of total pre-collapse under control $u$ coincides with $C(u)$.

**Proof.** We use the transformations (1.8) and (1.16) and reduce the techno-labor system (1.1) to the invariant system (1.17) whose unique stationary point is the origin.

Let us prove (i).

As in the proof of statement (i) of Proposition 3.1 we linarize system (1.17) in a neighborhood of the origin and find that the roots of of characteristic equation, $\lambda_1$ and $\lambda_2$, are given by (3.1) where the determiant is positive (see (3.2)). Hence, $\lambda_1$ and $\lambda_2$ are real. Inequatlity (1.7) implies that $\lambda_1$ and $\lambda_2$ have different signes. Therefore, the rest point $(0, 0)$ is unstable. Statement (i) is proved.

Let us prove (ii).

The right-hand sides of both equations in (1.17) are positive if and only if $(\theta, \zeta)$ belongs to the interior of the invariant north-east angle, $G^{++}_{\zeta\theta}$ (see Fig. 1.4). Therefore, the transformed zone of homeostasis under control $u$, $\bar{H}^{++}(u)$, lies necessarily in $G^{++}_{\zeta\theta}$. In order to state that $\bar{H}^{++} = G^{--}_{\zeta\theta}$ (which implies that $H^{++}(u) = G^{--}_{ZT}(u)$) it suffices to show that the invariant system (1.17) survives in $G^{++}_{\zeta\theta}$. Thi is so if the vector field of (1.17) points inside $G^{++}_{\zeta\theta}$ at every point $(\zeta, \theta)$ on the boundary of $G^{++}_{\zeta\theta}$. The boundary of $G^{++}_{\zeta\theta}$ is the union of the half-lines lines $G^{+}_{\theta}$ and $G^{+}_{\zeta}$ (the former is located above the latter, see Remark 1.1, 2, (i), and Fig. 1.4). Let $(\zeta, \theta)$ lie on the "upper" border $G^{+}_{\theta}$. At this point, the right hand side of the invariant system (1.17) is represented as (3.3) The inequalities $\zeta > 0$ and (1.7) imply that $f$ points to the right (on the $(\zeta, \theta)$ plane), i.e., inside $G^{++}_{\zeta\theta}$. Let $(\zeta, \theta)$ lie on the "lower" border $G^{+}_{\zeta}$. At this point, the right-hand side of (1.17) is represented as (3.4). The inequalities $\zeta > 0$ and (1.7) imply that $f$ points upwards, i.e., inside $G^{++}_{\zeta\theta}$. Thus, system (1.17) survives in $G^{++}_{\zeta\theta}$, which proves that $\hat{H}^{++} = G^{++}_{\zeta\theta}$ implying $H^{++}(u) = G^{--}_{ZT}(u)$. Every solution $t \mapsto (\zeta(t), \theta(t))$ such that $(\zeta(0), \theta(0)) \in G^{++}_{\zeta\theta}$ remains in $G^{++}_{\zeta\theta}$ and does not converge to the origin, implying that $\zeta(t) \to \infty$ and $\theta(t) \to \infty$ as $t \to \infty$. This proves that $H^{++} = G^{++}_{ZT}$ is the zone of progressive homeostasis. Statement (ii) is proved.

Statement (iii) is proved identically.

Let us prove (iv).

Arguing like in the proof of statement (iv) of Proposition 3.1, we show that there is a solution $t \mapsto (\zeta_*(t), \theta_*(t))$ on $(-\infty, \infty)$ of the invariant system (1.17) such that $(\zeta_*(t), \theta_*(t)) \in G^{+-}_{\zeta\theta}$ for all $t \geq 0$. The fact that in $G^{+-}_{\zeta\theta}$ the right-hand side of (1.17) points south-east (see Fig. 3.4) implies that $(\zeta_*(t), \theta_*(t)) \to (0, 0)$ as $t \to \infty$. Let us show that there is no other solution of (1.17) possessing these properties; this will immediately complete the proof of statement (iv). The trajectory of every solution of (1.17) which takes values in $G^{+-}_{\zeta\theta}$ is the graph of a solution of the differential equation

$$\frac{d\theta}{d\zeta} = h(\zeta, \theta) \tag{4.1}$$

where

$$h(\zeta,\theta) = \frac{\rho_T}{\rho_Z}\frac{e^{(\alpha-1)\theta+(\beta+\gamma)\zeta}-1}{e^{\alpha\theta+(\beta-1)\zeta}-1}$$

(we get (4.1) if we divide the second equation in (1.17) by the first). The trajectory of the solution $t \mapsto (\zeta_*(t), \theta_*(t))$ is the graph of a solution $\zeta \mapsto \theta_*(\zeta)$ of (4.1) which is defined on $(-\infty, 0)$ and satisfies the conditions $(\zeta, \theta_*(\zeta)) \in G^{+-}_{\zeta\theta}$ and $\theta_*(\zeta) \to 0$ as $\zeta \to 0$. It is suffivient to state that (4.1) has the unique solution possessing these properties. The latter fact takes place if we show that for every $\zeta < 0$ the function $\theta \mapsto h(\zeta, \theta)$ is strictly increasing in a neighborhood of $\theta_*(\zeta)$. Let us check this. Take a $\zeta < 0$ and set $\theta = \theta_*(\zeta)$. We have

$$\frac{\partial h(\zeta,\theta)}{\partial\theta} = \frac{\rho_T}{\rho_Z}\frac{(\alpha-1)e^{(\alpha-1)\theta+(\beta+\gamma)\zeta}(e^{\alpha\theta+(\beta-1)\zeta}-1) - \alpha e^{\alpha\theta+(\beta-1)\zeta}(e^{(\alpha-1)\theta+(\beta+\gamma)\zeta}-1)}{(e^{\alpha\theta+(\beta-1)\zeta}-1)^2}$$

Point $(\zeta, \theta) \in G^{+-}_{\zeta\theta}$ lies above the straight line $G_\theta(u)$ (1.18), i.e.,

$$(\alpha-1)\theta + (\beta+\gamma)\zeta < 0.$$

By (1.7)

$$\frac{\beta-1}{\alpha} > \frac{\beta+\gamma}{\alpha-1}.$$

Then using $\zeta < 0$, we get

$$\alpha\theta + (\beta-1)\zeta = \alpha\left(\theta + \frac{\beta-1}{\alpha}\zeta\right) < \alpha\left(\theta + \frac{\beta+\gamma}{\alpha-1}\zeta\right) < 0.$$

Hence,

$$e^{(\alpha-1)\theta+(\beta+\gamma)\zeta}-1 < 0, \quad e^{\alpha\theta+(\beta-1)\zeta}-1 < 0.$$

Consequently,

$$\frac{\partial h(\zeta,\theta)}{\partial\theta} > 0.$$

This completes the proof of statement (iv).

Statement (v) is proved similarly.

Statements (vi) and (vii) follow straightforwardly from (iv) and (v) and the definitions of the zones of progressive pre-homeostasis and zone of total pre-collapse (under control $u$).

The proposition is proved.

## References


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