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Interim Report **IR-03-013**

The Pontryagin Maximum Principle for Infinite-Horizon Optimal Controls

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April 2003

Abstract

This paper (motivated by recent works on optimization of long-term economic growth) suggests some further developments in the theory of first-order necessary optimality conditions for problems of optimal control with infinite time horizons. We describe an approximation technique involving auxiliary finite-horizon optimal control problems and use it to prove new versions of the Pontryagin maximum principle. A special attention is paid to behavior of the adjoint variables and the Hamiltonian. Typical cases, in which standard transversality conditions hold at infinity, are described. Several significant earlier results are generalized.

Key words: Optimal Control, Infinite Horizon, the Pontryagin Maximum Principle, Transversality Conditions, Optimal Economic Growth
Mathematics Subject Classification (2000): 37N40, 49KXX, 91B62

Contents

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1 Introduction

We deal with an infinite-horizon optimal control problem referred to as problem (P), in which an integral goal functional is maximized across the set of controls and trajectories of a nonlinear finite-dimensional dynamical system operating over an unbounded interval of time. Problems of this type emerge in mathematical economics; they are closely related to the concept of economic sustainability (see, e.g., [39]) and arise in numerous studies on optimization of economic growth (see [1], [2], [16], [17], [23], [32], [37]). A progress in this field of economics was initiated by Ramsey in the 1920s [34], and fundamental contributions were made by Koopmans [27] and Solow [40] in the 1960s.

Throughout this paper, $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ denote, respectively, the norm and scalar product in a finite-dimensional Euclidean space, * stands for the matrix transposition, and "a.a." replaces "almost all with respect to the Lebeague measure".

Using standard notations of control theory, we represent the optimal control problem (P) as follows.

Problem (P):

$$\dot{x}(t) = f(x(t), u(t)); \tag{1.1}$$

$$u(t) \in U;$$

$$x(0) = x_0; \tag{1.2}$$

$$\text{maximize } J(x,u) = \int_0^\infty e^{-\rho t} g(x(t), u(t))dt. \tag{1.3}$$

Here t is time varying in $[0,\infty)$; (1.1) is the equation of a dynamical control system; $x(t) = (x^1(t),\dots,x^n(t))^*$ and $u(t) = (u^1(t),\dots,u^m(t))^*$ are the current values of system's states and controls treated as column vectors in the n- and m-dimensional Euclidean spaces R^n and R^m, respectively; U is a nonempty convex compactum in R^m, which constrains the values of the controls; x_0 is a given initial state; and $\rho \geq 0$ is a given parameter. The functions $f : G \times U \mapsto R^n$ and $g : G \times U \mapsto R^1$, are differentiable; here G is an open set in R^n such that $x_0 \in G$. The matrix $\partial f/\partial x = (\partial f^i/\partial x^j)_{i,j=1,\dots,n}$ (here f^i is the ith coordinate map for f) and the gradient $\partial g/\partial x = (\partial g/\partial x^1,\dots,\partial g/\partial x^n)^*$ are assumed to be continuous on $G \times U$.

As usual a *control* (in system (1.1)) is identified with an arbitrary (Lebeague) measurable function $u : [0,\infty) \mapsto U$. A *trajectory* (of system (1.1)) corresponding to a control u

*This work was partially supported by the Russian Foundation for Basic Research (projects # 99-01-01051 and # 03-01-00737) and by the Fujitsu Research Institute (IIASA-FRI contract # 01-109).

is a Charatheodory solution x to (1.1), which satisfies the initial condition (1.2). The continuous differentiability of f implies that if a trajectory corresponding to a certain control exists, then it is unique. We assume that for any control u the trajectory x corresponding to u exists on $[0,\infty)$ and takes values in G. Any pair (u,x) where u is a control and x is the trajectory corresponding to u will further be called an *admissible pair* (for system (1.1)). An admissible pair (u_*,x_*) that maximizes the integral (1.3) across the set of all admissible pairs (u,x) is said to be *optimal* (in problem (P)); its components u_* and x_* are called an *optimal control* (in problem (P)) and an *optimal trajectory* (in problem (P)), respectively.

Our basic assumptions are the following.

(A1) There exists a $C_0 \geq 0$ such that

$$\langle x, f(x,u)\rangle \leq C_0(1+\|x\|^2) \quad \text{for all} \quad x\in G \quad \text{and all} \quad u\in U.$$

(A2) For each $x\in G$ the function $u\mapsto f(x,u)$ is affine, i.e.,

$$f(x,u) = f_0(x) + \sum_{i=1}^{m} f_i(x)u^i \quad \text{for all} \quad x\in G \quad \text{and all} \quad u\in U$$

where $f_i : G \mapsto R^n$, $i=0,1,\ldots,m$, are continuously differentiable.

(A3) For each $x\in G$ the function $u\mapsto g(x,u)$ is concave.

(A4) There exist positive-valued functions μ and ω on $[0,\infty)$ such that $\mu(t)\to 0$, $\omega(t)\to 0$ as $t\to\infty$ and for any admissible pair (u,x)

$$e^{-\rho t}\max{}_{u\in U}|g(x(t),u)| \leq \mu(t) \quad \text{for all} \quad t>0; \tag{1.4}$$

$$\int_T^\infty e^{-\rho t}|g(x(t),u(t))|dt \leq \omega(T) \quad \text{for all} \quad T>0. \tag{1.5}$$

Remark 1.1 As shown in [14] (Theorem 3.6), assumptions (A1) – (A4) guarantee the existence of an admissible pair optimal in problem (P).

Remark 1.2 Assumption (A1) is conventionally used in existence theorems in theory of optimal control (see [20], [22]). Assumptions (A2) and (A3) imply that problem (P) is "linear-convex" in control; the "linear-convex" structure is important for the implementation of approximation techniques. Assumption (A4) (see (1.5)) implies that the integral (1.3) converges absolutely for any admissible pair (u,x), which excludes any ambiguity in interpreting problem (P). Finally, we note that assumptions (A1) – (A4) are satisfied for typical problems arising in economic applications.

In this paper we analyze conditions necessary for the optimality of an admissible pair in problem (P).

In theory of optimal control standard necessary conditions of optimality are given by the Pontryagin maximum principle [33]. Well-known are classical versions of the Pontryagin maximum principle, holding for problems of optimal control with finite time horizons.

For infinite-horizon optimal control problems without discounting factor ($\rho=0$) the Pontryagin maximum principle was stated in [33] under the constraint $\lim_{t\to\infty} x(t) = x_1$ where x_1 is a prescribed terminal state. However, the latter constraint is not critical for the proof given in [33], which, therefore, provides a version of the Pontryagin maximum principle for problem (P) in case of $\rho=0$. For infinite-horizon optimal control problems involving the discounting factor ($\rho > 0$), a rigorous proof of a general statement on the Pontryagin maximum principle was given in [24]. In application to problem (P), the formulations of [33] and [24] are, however, incomplete, since they establish only "core" relations of the Pontryagin maximum principle and do not suggest any analogue of the transversality conditions, which constitute an immanent component of the Pontryagin maximum principle for classical finite-horizon optimal control problems with nonconstrained terminal states. The issue of transversality conditions for problem (P) is in the focus of our study.

Note that such characteristic features of problem (P) as the lack of constraints on behavior of optimal trajectories in a neighborhood of infinity, and the involvement of a nontrivial discounting factor in the goal functional (if $\rho > 0$) prevent the efficient use of the standard needle variations technique [33] for proving analogues of the transversality conditions.

For problem (P), the "core" relations of the Pontryagin maximum principle are as usual formulated in terms of the *Hamilton-Pontryagin function* $\mathcal{H} : R^n \times [0,\infty) \times U \times R^n \times R^1 \mapsto R^1$ and the *maximized Hamilton-Pontryagin function*, or *Hamiltonian* $H : R^n \times [0,\infty) \times R^n \times R^1 \mapsto R^1$ defined by

$$\mathcal{H}(x,t,u,\psi,\psi^0) = \langle f(x,u),\psi\rangle + \psi^0 e^{-\rho t} g(x,u)$$

and

$$H(x,t,\psi,\psi^0) = \sup_{u\in U} \mathcal{H}(x,t,u,\psi,\psi^0).$$

The formulation involves an admissible pair (u_*, x_*) and a pair (ψ,ψ^0) of *adjoint variables associated with* (u_*, x_*) (in problem (P)); here ψ is a (Caratheodory) solution to the *adjoint equation*

$$\dot{\psi}(t) = -\left[\frac{\partial f(x_*(t),u_*(t))}{\partial x}\right]^* \psi(t) - \psi^0 e^{-\rho t}\frac{\partial g(x_*(t),u_*(t))}{\partial x} \tag{1.6}$$

on $[0,\infty)$ and ψ^0 is a nonnegative real; (ψ,ψ^0) is said to be *nontrivial* if

$$\|\psi(0)\| + \psi^0 > 0. \tag{1.7}$$

We give the formulation in the following form. We shall say that an admissible pair (u_*, x_*) satisfies the *core Pontryagin maximum principle* (in problem (P)) *together* with a pair (ψ,ψ^0) of adjoint variables associated with (u_*, x_*) if (ψ,ψ^0) is nontrivial and the following *maximum condition* holds:

$$\mathcal{H}(x_*(t),t,u_*(t),\psi(t),\psi^0) = H(x_*(t),t,\psi(t),\psi^0) \quad \text{for a.a.} \quad t \geq 0. \tag{1.8}$$

Of special interest is the case where problem (P) is not abnormal, i.e., the Lagrange multiplier ψ^0 in the core Pontryagin maximum principle does not vanish. In this case we say that the normal-form core Pontryagin maximum principle holds. More accurately, we shall say that an admissible pair (u_*, x_*) satisfies the *normal-form* core Pontryagin maximum principle *together* with a pair (ψ,ψ^0) of adjoint variables associated with (u_*, x_*) if (u_*, x_*) satisfies the core Pontryagin maximum principle together with (ψ,ψ^0) and, moreover, $\psi^0 > 0$. In this case we do not lose in generality if we set $\psi^0 = 1$ (indeed, multiplying both ψ and ψ^0 by $1/\psi^0$, we get the new pair of adjoint variables, $(\overline{\psi},\overline{\psi}^0) = (\psi/\psi^0, 1)$, associated with (u_*, x_*) and such that (u_*, x_*) satisfies the normal-form core Pontryagin maximum principle together with $(\overline{\psi},\overline{\psi}^0)$).

Therefore, we simplify the previous definition as follows. Define the *normal-form Hamilton-Pontryagin function* $\tilde{\mathcal{H}} : R^n \times [0, \infty) \times U \times R^n \mapsto R^1$ and *normal-form Hamiltonian* $\tilde{H} : R^n \times [0, \infty) \times R^n \mapsto R^1$ by

$$\tilde{\mathcal{H}}(x, t, u, \psi) = \mathcal{H}(x, t, u, \psi, 1) = \langle f(x, u), \psi \rangle + e^{-\rho t} g(x, u)$$

and

$$\tilde{H}(x, t, \psi) = H(x, t, \psi, 1) = \sup_{u \in U} \tilde{\mathcal{H}}(x, t, u, \psi).$$

Given an admissible pair (u_*, x_*), introduce the *normal-form adjoint equation*

$$\dot{\psi}(t) = - \left[\frac{\partial f(x_*(t), u_*(t))}{\partial x} \right]^* \psi(t) - e^{-\rho t} \frac{\partial g(x_*(t), u_*(t))}{\partial x} \tag{1.9}$$

(representing the adjoint equation (1.6) where $\psi^0 = 1$). Any (Caratheodory) solution ψ to (1.9) on $[0, \infty)$ will be called an *adjoint variable* associated with (u_*, x_*). We shall say that an admissible pair (u_*, x_*) satisfies the *normal-form* core Pontryagin maximum principle *together* with an adjoint variable ψ associated with (u_*, x_*) if the following *normal-form maximum condition* holds:

$$\tilde{\mathcal{H}}(x_*(t), t, u_*(t), \psi(t)) = \tilde{H}(x_*(t), t, \psi(t)) \quad \text{for a.a.} \quad t \geq 0. \tag{1.10}$$

In the context of problem (P), [24] states the following (see also [19]):

Theorem 1.1 *If an admissible pair (u_*, x_*) is optimal in problem (P), then (u_*, x_*) satisfies relations* (1.6)–(1.8) *of the core Pontryagin maximum principle together with some pair (ψ, ψ^0) of adjoint variables associated with (u_*, x_*).*

Qualitatively, this formulation is weaker than the corresponding statement known for finite-horizon optimal control problems with nonconstrained terminal states. Indeed, consider a following finite-horizon counterpart of problem (P),

Problem (P^*):

$$\dot{x}(t) = f(x(t), u(t));$$

$$u(t) \in U;$$

$$x(0) = x_0;$$

$$\text{maximize } J^*(x, u) = \int_0^T e^{-\rho t} g(x(t), u(t)) dt;$$

here $T > 0$ is a fixed positive real. The classical theory [33] says that if an admissible pair (u_*, x_*) is optimal in problem (P^*), then (u_*, x_*) satisfies the core Pontryagin maximum principle together with some pair (ψ, ψ^0) of adjoint variables associated with (u_*, x_*), and, moreover, (ψ, ψ^0) satisfies the *transversality conditions*

$$\psi^0 = 1, \quad \psi(T) = 0 \tag{1.11}$$

(in shorter words, for (u_*, x_*) and (ψ, ψ^0) the normal-form core Pontryagin maximum principle is satisfied; we use the extended formulation just to make it closer to Theorem 1.1). In Theorem 1.1 any analogue of the transversaility conditions (1.11) is missing.

Information provided by the transversality conditions (1.11) is substantial. As noted in [33], the core Pontryagin maximum principle represented by the system equation (1.1) (for $(u, x) = (u_*, x_*)$), the adjoint equation (1.6) and the maximum condition (1.8), together with the transversality conditions (1.11), form a complete system of equations, in which the number of equations equals the number of the unknowns in them. Situations where this system of equations has a unique solution (u_*, x_*, ψ) are quite common; in such situations problem (P^*) is resolved uniquely. Conversely, the system of equations of the core Pontryagin maximum principle not complemented by the transvesality conditions has, generically, infinitely many solutions. In other words, for problem (P^*) the core Pontryagin maximum principle is essentially less informative unless it is complemented by the transversality conditions.

The situation is different for problems with constrained terminal states. Consider the following

Problem $(\overline{P}^*)$:

$$\dot{x}(t) = f(x(t), u(t));$$

$$u(t) \in U;$$

$$x(0) = x_0;$$

$$x(T) = x_1; \tag{1.12}$$

$$\text{maximize } J^*(x, u) = \int_0^T e^{-\rho t} g(x(t), u(t))dt;$$

here x_1 is a prescribed terminal state in a fixed terminal time $T > 0$. For problem $(\overline{P}^*)$ the classical necessary optimality conditions [33] include the core Pontryagin maximum principle and do not involve any additional (transversality) conditions. However, due to the additional terminal constraint (1.12), the core Pontryagin maximum principle is as informative for problem $(\overline{P}^*)$ as the core Pontryagin maximum principle together with the transversality conditions for problem (P^*).

Thus, the core Pontryagin maximum principle is "complete" for problem $(\overline{P}^*)$ (with constrained terminal states) and is "incomplete" for problem (P^*) (with non-constrained terminal states) unless it is complemented by the transversality conditions.

As mentioned above, the core Pontryagin maximum principle representing a necessary condition of optimality for problem (P) was stated in [33] under the assumption that the goal functional does not involve the discounting factor $e^{-\rho t}$ (or $\rho = 0$), which made the problem fully stationary. In case of a nondegenerate discounting factor ($\rho > 0$), the needle variations technique used in [33] is not applicable to problem (P) directly. However, in this case the core Pontryagin maximum principle can be stated using simple manipulations with the core Pontryagin maximum principle for approximating finite-horizon problems $(\overline{P}^*)$ with large horizons T. Indeed, every admissible pair (u_*, x_*) optimal in problem (P) is, clearly, optimal in problem $(\overline{P}^*)$ where $x_1 = x_*(T)$. Hence, in problem $(\overline{P}^*)$ (u_*, x_*) satisfies the core Pontryagin maximum principle together with some pair of adjoint variables. Letting $T \to \infty$ and taking the limit, we find that in problem (P) (u_*, x_*) also satisfies the core Pontryagin maximum principle together with some pair of adjoint variables (see [24] and [19] for details).

The lack of analogues of the transversality conditions in the formulations of the Pontryagin maximum principle is a generic feature of infinite-horizon optimal control problems with nonconstrained terminal states. In case of no discounting ($\rho = 0$), illustrating counter-examples were given in [24] and [36], and for problems with discounting ($\rho > 0$) in [13] and [30]. In Section 2 we construct a set of further counter-examples for problem (P) (which, generally, differs from the settings analyzed in [13] and [30]).

There were numerous attempts to find specific situations, in which the infinite-horizon Pontryagin maximum principle holds together with transversality conditions at infinity (see [13], [15], [18], [21], [26], [30], [35], [38]). The major results were established under rather severe assumptions of linearity or full convexity, which made it difficult to apply them to particular meaningful problems (see, e.g., [28] discussing application of the Pontryagin maximum principle to a particular infinite-horizon optimal control problem).

In this paper we develop necessary optimality conditions for problem (P), which complement the core Pontryagin maximum principle by non-trivial conditions characterizing behavior of the adjoint variables and Hamiltonian; under some reasonable assumptions these conditions take the form of a natural extension of the finite-horizon transversality conditions (1.11). In our analysis we follow the approximation approach suggested in [10] – [12]. We approximate problem (P) by a sequence of finite-horizon optimal control problems (P_k) whose horizons go to infinity. As we noted earlier, the use of finite-horizon approximating problems $(\overline{P}^*)$ with constrained terminal states leads to the core Pontryagin maximum principle (Theorem 1.1) but is unable to provide any analogues of the transversality conditions. Unlike problems $(\overline{P}^*)$, problems (P_k) impose no constrains on the terminal states, in this sense, they inherit the structure of problem (P); on the other hand, problems (P_k) are not plain "restrictions" of problem (P) to finite intervals like problem (P^*): the goal functionals in problems (P_k) include special penalty terms associated with a certain control optimal in problem (P). These key features of our technique allow us to find limit forms of the classical transversality conditions for problems (P_k) and formulate conditions that complement the core Pontryagin maximum principle and hold with a necessity for every admissible pair optimal in problem (P).

Earlier, similar approximation approach was used to derive necessary optimality conditions for various nonclassical optimal control problems (see, e.g., [3] – [6], [8], [31]; and also survey [7]). Basing on relevant approximation techniques and the methodology presented here, one can extend the results of this paper to more complex infinite-horizon problems of optimal control such as problems with nonsmooth terminal constraints, problems with state constraints, problems for systems described by differential inclusions, etc. In this paper, our primary goal is to show how the regularized approximation approach allows us to resolve the major singularity emerging due to the infinitness of the time horizon. Therefore, we restrict our consideration to the relatively simple nonlinear infinite-horizon problem (P), which is smooth, "linear-convex" in control and free from any constraints on the system's states. The results presented here generalize [10]–[12].

Finally, we note that the suggested regularized approximation methodology, appropriately modified, can be used directly in analysis of particular nonstandard optimal control problems with infinite time horizons (see, e.g., [9]).

2 Transversality conditions: counter-examples

Considering problem (P) as the "limit" of finite-horizon problems (P^*) whose horizons T tend to infinity, one can expect the following "limit" transversality conditions for problem (P):

$$\psi^0 = 1, \quad \lim_{t\to\infty} \psi(t) = 0; \tag{2.13}$$

here (ψ, ψ^0) is a pair of adjoint variables satisfying the core Pontryagin maximum principle together with an admissible pair (u_*, x_*) optimal in problem (P). Relations

$$\psi^0 = 1, \quad \lim_{t\to\infty} \langle \psi(t), x_*(t)\rangle = 0 \tag{2.14}$$

represent alternative transversality conditions for problem (P), which are frequently used in economic applications (see, e.g., [16]). The interpretation of (2.14) as transversality conditions for problem (P) is also motivated by Arrow's statement on sufficient conditions of optimality (see [1], [2] and [35]), which (under some additional assumptions) asserts that if (2.14) holds for an admissible pair (u_*, x_*) and a pair (ψ, ψ^0) of adjoint variables, jointly satisfying the core Pontryagin maximum principle, then (u_*, x_*) is optimal in problem (P) provided the superposition $H(x, t, \psi(t), \psi^0)$ is concave in x.

Generally, for infinite-horizon optimal control problems neither of the "natural" transversality conditions (2.13) and (2.14) holds; illustrating counter-examples were given in [24] and [36] for problems without discounting ($\rho = 0$). Here, we provide a set of further counter-examples for problem (P), in the case when discounting ρ is positive.

Example 1 is a slight modification of an example given in [30]; it shows that problem (P) can be *abnormal*, i.e., in the core Pontryagin maximum principle the Lagrange multiplier ψ^0 may necessarily vanish (which contradicts both (2.13) and (2.14)).

Example 1. Consider the optimal control problem

$$\dot{x}(t) = (2x(t) + u(t))\phi(x(t));$$

$$u(t) \in U = [-1, 0];$$

$$x(0) = 0;$$

$$\text{maximize } J(x, u) = \int_0^\infty e^{-t}(2x(t) + u(t))dt.$$

Here ϕ is smooth, nonnegative, bounded and such that $\phi(x) = 1$ if $|x| \leq 1$ and $\phi(x) = 0$ if $|x| \geq 2$.

Viewing the above problem as problem (P) and setting $G = R^1$, we easily find that assumptions (A1) – (A4) are satisfied. It is easily seen that (u_*, x_*) where $u_*(t) = 0$ and $x_*(t) = 0$ for all $t \geq 0$ is the unique optimal admissible pair[1]. Indeed, any control u taking negative values in any set whose Lebeague measure is positive produces a negative value of the goal functional, wehereas $J(u_*, x_*) = 0$. The Hamilton-Pontryagin function is given by

$$\mathcal{H}(x, t, u, \psi, \psi^0) = \psi(2x + u)\phi(x) + \psi^0 e^{-t}(2x + u) = (\psi\phi(x) + \psi^0 e^{-t})(2x + u).$$

Let (ψ, ψ^0) be an arbitrary pair of adjoint variables such that (u_*, x_*) satisfies the core Pontryagin maximum principle together with (ψ, ψ^0). The adjoint equation (1.6) has the form

$$\dot{\psi}(t) = -2(\psi(t) + \psi^0 e^{-t}),$$

and the maximum condition (1.8) implies

$$\psi(t) + \psi^0 e^{-t} \geq 0 \quad \text{for all} \quad t \geq 0. \tag{2.15}$$

Solving the adjoint equation, we get

$$\psi(t) = -2\psi^0 e^{-t} + (\psi(0) + 2\psi^0)e^{-2t}.$$

Thus, if $\psi^0 > 0$, then for all $t > 0$ large enough

$$\psi(t) + \psi^0 e^{-t} = -\psi^0 e^{-t} + (\psi(0) + 2\psi^0)e^{-2t} < 0$$

which contradicts (2.15). Consequently, $\psi^0 = 0$ with a necessity.

[1] Here and in Examples 2, 3 and 4 the uniqueness of u_* implies that every optimal control is equivalent to u_* with respect to the Lebeague measure on $[0, \infty)$.

The next example shows that for problem (P) the limit relation in (2.13) may be violated, whereas the alternative transversality conditions (2.14) may hold.

Example 2. Consider the optimal control problem

$$\dot{x}(t) = u(t) - x(t);$$

$$u(t) \in U = [0,1];$$

$$x(0) = 1/2;$$

$$\text{maximize } J(x,u) = \int_0^\infty e^{-t} \ln \frac{1}{x(t)} dt.$$

We set $G = (0,\infty)$ and treat the above problem as problem (P). Assumptions (A1) – (A4) are, obviously, satisfied. For an arbitrary trajectory x we have $e^{-t}/2 \le x(t) < 1$ for all $t \ge 0$. Hence, (u_*, x_*) where $u_*(t) = 0$ and $x_*(t) = e^{-t}/2$ for all $t \ge 0$ is the unique optimal admissible pair. The Hamilton-Pontryagin function is given by

$$\mathcal{H}(x,t,u,\psi,\psi^0) = (u-x)\psi - \psi^0 e^{-t} \ln x.$$

Let (ψ,ψ^0) be an arbitrary pair of adjoint variables such that (u_*, x_*) satisfies the core Pontryagin maximum principle together with (ψ,ψ^0). The adjoint equation (1.6) has the form

$$\dot{\psi}(t) = \psi(t) + \psi^0 e^{-t} \frac{1}{x_*(t)} = \psi + 2\psi^0,$$

and the maximum condition (1.8) implies

$$\psi(t) \le 0 \quad \text{for all} \quad t \ge 0. \tag{2.16}$$

Assume $\psi^0 = 0$. Then $\psi(0) < 0$ and $\psi(t) = e^t \psi(0) \to -\infty$ as $t \to \infty$, i.e., the limit relation in (2.13) does not hold. Let $\psi^0 > 0$. With no loss of generality (or multiplying both ψ and ψ^0 by $1/\psi^0$), we assume $\psi^0 = 1$. Then $\psi(t) = (\psi(0)+2)e^t - 2$. By (2.16) only two cases are admissible: (a) $\psi(0) = -2$ and (b) $\psi(0) < -2$. In case (a) $\psi(t) \equiv -2$, and in case (b) $\psi(t) \to -\infty$ as $t \to \infty$. In both situations the limit relation in (2.13) is violated. Note that $\psi(t) \equiv -2$ $(t \ge 0)$ and $\psi^0 = 1$ satisfy the alternative transversality conditions (2.14).

The next example is complementary to Example 2; it shows that for problem (P) the limit relation in (2.14) may be violated, whereas (2.13) may hold.

Example 3. Consider the following optimal control problem:

$$\dot{x}(t) = 1 + u(t); \tag{2.17}$$

$$u(t) \in U = \left[-\frac{1}{2}, 0\right];$$

$$x(0) = 0;$$

$$\text{maximize } J(x,u) = \int_0^\infty e^{-t}(1+\gamma(x(t)))(1+u(t))dt. \tag{2.18}$$

Here γ is a nonnegative continuously differentiable real function such that

$$I = \int_0^\infty e^{-t}\gamma(t)dt < \infty. \tag{2.19}$$

We set $G = R^1$ and view the above problem as problem (P). Clearly, assumptions (A1) – (A3) are satisfied. Below, we specify the form of γ and show that assumption (A4) is satisfied too.

The admissible pair (u_*, x_*) where $u_*(t) = 0$ and $x_*(t) = t$ for all $t \geq 0$ is optimal. Indeed, let (u, x) be an arbitrary admissible pair. Observing (2.17), we find that $\dot{x}(t) > 0$ for almost all $t \geq 0$. Taking $\tau(t) = x(t)$ for a new integration variable in (2.18), we get $d\tau = (1 + u(t))dt$ and

$$t(\tau) = \int_0^\tau \frac{1}{1 + u(t(s))} ds \quad \text{for all} \quad \tau \geq 0.$$

As far as

$$\int_0^\tau \frac{1}{1 + u(t(s))} ds \geq \tau,$$

we get

$$\begin{aligned} J(x, u) &= \int_0^\infty e^{-t}(1 + \gamma(x(t)))(1 + u(t))dt \\ &= \int_0^\infty e^{-\int_0^\tau \frac{1}{1+u(t(s))} ds}(1 + \gamma(\tau))d\tau \\ &\leq \int_0^\infty e^{-\tau}(1 + \gamma(\tau))d\tau \\ &= J(u_*, x_*). \end{aligned}$$

Hence, (u_*, x_*) is an optimal admissible pair. It is easy to see that there are no other optimal admissible pairs. The Hamilton-Pontryagin function has the form

$$\mathcal{H}(x, t, u, \psi, \psi^0) = (1 + u)\psi + \psi^0 e^{-t}(1 + \gamma(x))(1 + u).$$

Let (ψ, ψ^0) be an arbitrary pair of adjoint variables such that (u_*, x_*) satisfies the core Pontryagin maximum principle together with (ψ, ψ^0). The adjoint equation (1.6) has the form

$$\dot{\psi}(t) = -\psi^0 \dot{\gamma}(t) e^{-t}.$$

If $\psi^0 = 0$, then the maximum condition (1.8) implies $\psi(t) \equiv \psi(0) > 0$; hence, $\psi(t)x_*(t) = \psi(0)t \to \infty$ as $t \to \infty$, and the limit relation in (2.14) is violated.

Suppose $\psi^0 > 0$, or, equivalently, $\psi^0 = 1$. The adjoint equation (1.6) takes the form

$$\dot{\psi}(t) = -\dot{\gamma}(t)e^{-t}$$

and we have

$$\psi(t) = \psi(0) - \int_0^t \dot{\gamma}(s)e^{-s} ds.$$

The limit relation in (2.14) has the form $\lim_{t\to\infty} t\psi(t) = 0$. Let us show that one can define γ so that the latter relation is violated i.e., for any $\psi(0) \in R^1$

$$p(t) \not\to 0 \quad \text{as} \quad t \to \infty \tag{2.20}$$

where

$$p(t) = t\psi(t).$$

We represent $p(t)$ as follows:

$$\begin{aligned} p(t) &= t\psi(0) - t\int_0^t \dot{\gamma}(s)e^{-s} ds \\ &= t\psi(0) - t\left[\gamma(s)e^{-s}|_0^t + \int_0^t \gamma(s)e^{-s} ds\right] \\ &= t\psi(0) - t\gamma(t)e^{-t} + t\gamma(0) - tI(t) \end{aligned}$$

where

$$I(t) = \int_0^t \gamma(s)e^{-s}ds.$$

Introducing

$$\nu(t) = \gamma(t)e^{-t}, \tag{2.21}$$

rewrite:

$$I(t) = \int_0^t \nu(s)ds, \tag{2.22}$$

$$p(t) = t\psi(0) - t\nu(t) + t\nu(0) - tI(t). \tag{2.23}$$

Note that

$$\lim_{t\to\infty} I(t) = I \tag{2.24}$$

due to (2.19).

Now let us specify the form of ν. For each natural k we fix a positive $\varepsilon_k < 1/2$ and denote by Δ_k the ε_k-neighborhood of k. Clearly, $\Delta_k \cup \Delta_j = \emptyset$ for $k \neq j$. We set

$$\nu(k) = \frac{1}{k} \quad \text{for} \quad k = 1, 2, \ldots;$$

$$\nu(t) = 0 \quad \text{for} \quad t \notin \cup_{k=1}^{\infty}\Delta_k;$$

$$\nu(t) \in \left[0, \frac{1}{k}\right] \quad \text{for} \quad t \in \Delta_k \quad (k = 1, 2, \ldots).$$

Moreover, we require that

$$\sum_{k=j}^{\infty} \int_{\Delta_k} \nu(t)dt \leq \frac{1}{j^2}. \tag{2.25}$$

This can be achieved, for example, by letting

$$\frac{2\varepsilon_k}{k} \leq \frac{a_k}{k^2}$$

where $\sum_{k=1}^{\infty} a_k = 1$, $a_k > 0$. Indeed, in this case

$$\sum_{k=j}^{\infty} \int_{\Delta_k} \nu(t)dt \leq \sum_{k=j}^{\infty} \frac{2\varepsilon_k}{k} \leq \sum_{k=j}^{\infty} \frac{a_k}{k^2} \leq \frac{1}{j^2}\sum_{k=j}^{\infty} a_k \leq \frac{1}{j^2},$$

i.e., (2.25) holds. Note that for $j = 1$ the left hand side in (2.25) equals I (see (2.19)); thus, (2.25) implies that assumption (2.19) holds.

Another fact following from (2.25) is that

$$\lim_{t\to\infty} t(I - I(t)) = 0. \tag{2.26}$$

Indeed, by (2.22)

$$I(j + \varepsilon_j) = \sum_{k=1}^{j} \int_{\Delta_k} \nu(t)dt,$$

hence, due to (2.25),

$$I - I(j + \varepsilon_j) = \sum_{k=j+1}^{\infty} \int_{\Delta_k} \nu(t)dt \leq \frac{1}{(j+1)^2}.$$

For $t \in [j + \varepsilon_j, j + 1 + \varepsilon_{j+1}]$

$$I(j + \varepsilon_j) \leq I(t) \leq I,$$

therefore, for $t \geq 1$

$$0 \leq I - I(t) \leq \frac{1}{(j+1)^2} \leq \frac{1}{(t-\varepsilon_{j+1})^2} \leq \frac{1}{(t-1/2)^2},$$

which yields (2.26). The given definition of ν (see (2.21)) is, clearly, equivalent to defining γ by

$$\begin{aligned} \gamma(k) &= \frac{e^k}{k} \quad \text{for} \quad k = 1, 2, \ldots; \\ \gamma(t) &= 0 \quad \text{for} \quad t \notin \cup_{k=1}^{\infty} \Delta_k; \\ \gamma(t) &\in \left[0, \frac{e^k}{k}\right] \quad \text{for} \quad t \in \Delta_k \quad (k = 1, 2, \ldots) \end{aligned} \tag{2.27}$$

and requiring (2.25). Let us show that assumption (A4) is satisfied. Let (u, x) be an arbitrary admissible pair. By (2.17) $t/2 \leq x(t) \leq t$ for all $t \geq 0$. Hence, by the definition of ν

$$\nu(x(t)) \leq \left(\frac{t}{2} - 1\right)^{-1} = \frac{2}{(t-2)} \quad \text{for all} \quad t > 2.$$

Hence, due to (2.21),

$$\begin{aligned} 0 &\leq e^{-\rho t} \max{}_{u \in U}[(1 + \gamma(x(t))(1+u)] = e^{-\rho t}(1 + \gamma(x(t))) \\ &\leq \mu(t) = e^{-\rho t} + \frac{2}{(t-2)} \to 0 \quad \text{as} \quad t \to \infty. \end{aligned}$$

Thus, condition (1.4) holds. Furthermore, introducing the integration variable $\tau(t) = x(t)$ and taking into account (2.21), we get

$$\begin{aligned} \int_T^{\infty} e^{-t}(1 + \gamma(x(t)))(1 + u(t))dt &= \int_{x(T)}^{\infty} e^{-\int_0^{\tau} \frac{1}{1+u(t(s))} ds}(1 + \gamma(\tau))d\tau \\ &\leq \int_{x(T)}^{\infty} e^{-\tau}(1 + \gamma(\tau))d\tau \\ &\leq \omega(T) = \int_{\frac{T}{2}}^{\infty} e^{-t}(1 + \gamma(t))dt \to 0 \quad \text{as} \quad T \to \infty. \end{aligned}$$

Hence, condition (1.5) holds. We stated the validity of assumption (A4).

By the definition of γ, for $t \in \Delta_k$, $k = 1, 2, \ldots$ we have

$$0 \leq t\nu(t) \leq \frac{k + \varepsilon_k}{k} \leq 1 + \frac{1}{k}.$$

Hence,

$$0 \leq t\nu(t) \leq 2 \quad \text{for all} \quad t \geq 0, \tag{2.28}$$

i.e., the function $t\nu(t)$ is bounded. Furthermore, $k\nu(k) = 1$, and due to (2.27) for any sequence $t_k \to \infty$ such that $t_k \in [k, k+1] \setminus (\Delta_k \cup \Delta_{k+1})$ we have $t_k \nu(t_k) = 0$. Therefore, $\lim_{t\to\infty} t\nu(t)$ does not exist.

Using $\nu(0) = 0$, we specify (2.23) as

$$p(t) = t\psi(0) - t\nu(t) - tI(t). \tag{2.29}$$

If $\psi(0) > I$, then, in view of (2.24), $\lim_{t\to\infty} t(\psi(0)+I(t)) = \infty$, which implies $\lim_{t\to\infty} p(t) = \infty$, since $t\nu(t)$ is bounded. Similarly, we find that if $\psi(0) < I$, then $\lim_{t\to\infty} p(t) = -\infty$. Let, finally, $\psi(0) = I$. Then

$$\lim_{t\to\infty} t(\psi(0) - I(t)) = \lim_{t\to\infty} t(I - I(t)) = 0$$

as follows from (2.26). Thus, in the right hand side of (2.29) the sum of the first and third terms has the zero limit at infinity, whereas the second term, $t\nu(t)$, has no limit at infinity, as we noticed earlier. Consequently, $p(t)$, the left hand side in (2.29), has no limit at infinity. We showed that (2.20) holds for every $\psi(0) \in R^1$.

Thus, the limit relation in the transversality conditions (2.14) is violated. Note that setting $\psi^0 = 1$ and $\psi(0) = I$, we make the adjoint variable ψ satisfy the transversality conditions (2.13). Indeed, in this case $\psi(t) = p(t)/t = \psi(0) - I - \nu(t)$ for all $t > 0$, and the conditions $\psi(0) = I$ and (2.28) imply that $\psi(t) \to 0$ as $t \to \infty$.

Examples 1, 2 and 3 show that assumptions (A1) – (A4) are insufficient for the validity of the core Pontryagin maximum principle together with the transversality conditions (2.13) or (2.14) as necessary conditions of optimality in problem (P). Below, we find mild additional assumptions that guarantee that necessary conditions of optimality in problem (P) include the core Pontryagin maximum principle and transversality conditions (2.13) or (2.14).

3 Basic constructions

In this section we define a sequence of finite-horizon optimal control problems (P_k) with horizons $T_k \to \infty$; we treat problems (P_k) as approximations to the infinite-horizon problem (P). Unlike the "natural" approximation problem $(\overline{P}^*)$ (see Section 1), problems (P_k) are explicitly associated with a fixed control u_* optimal in problem (P). Following the approximation scheme of [10] – [12], we complement the goal functional in problem (P_k) by a penalty term $-\alpha_k\Omega_k$ where α_k is a positive parameter tending to 0 as $k \to \infty$ and Ω_k is a functional of a control u in system (1.1):

$$\Omega_k(u) = \int_0^{T_k} e^{-(\rho+1)t}\|u(t) - z_k(t)\|^2 dt \tag{3.30}$$

where z_k is an appropriate smooth approximation to u_*. The convergences $\alpha_k \to 0$, $T_k \to \infty$ and $z_k \to u_*$ imply that problems (P_k) approximate problem (P) more and more "accurately" as $k \to \infty$. Our basic approximation lemma (Lemma 3.1) states that any sequence of controls optimal in problems (P_k) L^2-converges to u_* on every bounded interval (in this context, one can notice a certain parallelism with the Tikhonov regularization method widely used in theory of ill-posed problems [41]).

Let us describe the data defining problems (P_k). Given a control u_* optimal in problem (P), we fix a sequence of continuously differentiable functions $z_k : [0, \infty) \to R^m$ and a sequence of positive σ_k such that

$$\sup_{t\in[0,\infty)} \|z_k(t)\| \le \max{}_{u\in U}\|u\| + 1; \tag{3.31}$$

$$\int_0^\infty e^{-(\rho+1)t}\|z_k(t) - u_*(t)\|^2 dt \le \frac{1}{k}; \tag{3.32}$$

$$\sup_{t\in[0,\infty)} \|\dot{z}_k(t)\| \le \sigma_k < \infty; \tag{3.33}$$

$$\sigma_k \to \infty \quad \text{as} \quad k \to \infty$$

(obviously, such sequences exist). Next, we take a monotonically increasing sequence of positive T_k such that $T_k \to \infty$ as $k \to \infty$ and

$$\omega(T_k) \le \frac{1}{k(1+\sigma_k)} \quad \text{for all} \quad k = 1, 2, \ldots; \tag{3.34}$$

recall that ω is defined in (A4). For every $k = 1, 2, \ldots$ we define problem (P_k) as follows.

Problem (P_k):

$$\dot{x}(t) = f(x(t), u(t));$$
$$u(t) \in U;$$
$$x(0) = x_0;$$
$$\text{maximize } J_k(x,u) = \int_0^{T_k} e^{-\rho t} g(x(t), u(t))dt - \frac{1}{1+\sigma_k} \int_0^{T_k} e^{-(\rho+1)t} \|u(t) - z_k(t)\|^2 dt \quad (3.35)$$

(the last integral in (3.35) represents the penalty term $-\alpha_k \Omega_k(u)$ with $\Omega_k(u)$ given by (3.30) and $\alpha_k = 1/(1+\sigma_k)$). As usual, any admissible pair (u_k, x_k) maximizing (3.35) across all admissible pairs (u, x) is said to be optimal in problem (P_k); its components u_k and x_k are called an optimal control in problem (P_k) and an optimal trajectory in problem (P_k), respectively. By Theorem 9.3.i of [20] for every $k = 1, 2, \ldots$ there exists an admissible pair (u_k, x_k) optimal in problem (P_k). We assume that this optimal pair (u_k, x_k) is extended to the whole infinite time interval $[0, \infty)$ by an arbitrary admissible way.

The above defined sequence of problems, $\{(P_k)\}$ $(k = 1, 2, \ldots)$, will be said to be *associated* with the control u_*.

We are ready to formulate our basic approximation lemma.

Lemma 3.1 *Let assumptions* (A1) – (A4) *be satisfied; u_* be a control optimal in problem (P); $\{(P_k)\}$ be the sequence of problems associated with u_*; and for every $k = 1, 2, \ldots$ u_k be a control optimal in problem (P_k). Then for every $T > 0$ it holds that $u_k \to u_*$ in $L^2([0,T], R^m)$ as $k \to \infty$.*

Proof. Take a $T > 0$. Below $\|\cdot\|_{L^2}$ stands for the norm in $L^2([0,T], R^m)$. Let k_1 be such that $T_{k_1} \geq T$. For every $k \geq k_1$ we have

$$\begin{aligned} J_k(x_k, u_k) &= \int_0^{T_k} e^{-\rho t} \left[g(x_k(t), u_k(t)) - e^{-t} \frac{\|u_k(t) - z_k(t)\|^2}{1+\sigma_k} \right] dt \\ &\leq \int_0^{T_k} e^{-\rho t} g(x_k(t), u_k(t))dt - \frac{e^{-(\rho+1)T}}{1+\sigma_k} \int_0^T \|u_k(t) - z_k(t)\|^2 dt \end{aligned}$$

where x_k is the trajectory corresponding to u_k. Hence, introducing the trajectory x_* corresponding to u_* and taking into account the optimality of u_k in problem (P_k), optimality of u_* in problem (P) and conditions (1.5), (3.32) and (3.34), we find that for all sufficiently large k

$$\begin{aligned} \frac{e^{-(\rho+1)T}}{1+\sigma_k} \|u_k - z_k\|_{L^2}^2 &= \frac{e^{-(\rho+1)T}}{1+\sigma_k} \int_0^T \|u_k(t) - z_k(t)\|^2 dt \\ &\leq \int_0^{T_k} e^{-\rho t} g(x_k(t), u_k(t))dt - J_k(x_*, u_*) \\ &\leq \int_0^{T_k} e^{-\rho t} g(x_k(t), u_k(t))dt - J(x_*, u_*) + \\ &\quad\; \omega(T_k) + \int_0^\infty \frac{e^{-(\rho+1)t}}{1+\sigma_k} \|u_*(t) - z_k(t)\|^2 dt \\ &\leq \int_0^{T_k} e^{-\rho t} g(x_k(t), u_k(t))dt - J(x_*, u_*) + \frac{2}{k(1+\sigma_k)} \\ &\leq J(x_k, u_k) - J(x_*, u_*) + \frac{3}{k(1+\sigma_k)} \\ &\leq \frac{3}{k(1+\sigma_k)}. \end{aligned}$$

Hence,

$$\|u_k - z_k\|_{L^2}^2 \le \frac{3e^{(\rho+1)T}}{k}.$$

Then in view of (3.32)

$$\begin{aligned}
\|u_k - u_*\|_{L^2} &= \left(\int_0^T \|u_k(t) - u_*(t)\|^2 dt\right)^{1/2} \\
&\le \left(\int_0^T \|u_*(t) - z_k(t)\|^2 dt\right)^{1/2} + \left(\int_0^T \|u_k(t) - z_k(t)\|^2 dt\right)^{\frac{1}{2}} \\
&\le \left(\frac{e^{(\rho+1)T}}{k}\right)^{1/2} + \left(\frac{3e^{(\rho+1)T}}{k}\right)^{1/2} \\
&= (1 + 3^{1/2}) \left(\frac{e^{(\rho+1)T}}{k}\right)^{1/2}.
\end{aligned}$$

Therefore, for any $\epsilon > 0$ there exists a $k_2 \ge k_1$ such that $\|u_k - u_*\|_{L^2} \le \epsilon$ for all $k \ge k_2$. The lemma is proved.

Remark 3.3 In the above proof we used estimates (3.32) and (3.34), and did not use (3.31) and (3.33); these estimates will be utilized in the proof of Lemma 3.2.

Now, basing on Lemma 3.1, we derive a limit form of the classical Pontryagin maximum principle for problems (P_k), which leads us to the core Pontryagin maximum principle for problem (P). It is important that the adjoint variables involved in the latter core "infinite-horizon" Pontryagin maximum principle are designed as limits of the adjoint variables emerging in the "finite-horizon" Pontryagin maximum principle for problems (P_k); in this seance, the limit "infinite-horizon" adjoint variables carry some "limit" information on the transversality conditions in problems (P_k).

We use the following formulation of the Pontryagin maximum principle [33] for problems (P_k). Let an admissible pair (u_k, x_k) be optimal in problem (P_k) for some k. Then there exists a pair (ψ_k, ψ_k^0) of adjoint variables associated with (u_k, x_k) such that (u_k, x_k) satisfies relations (1.6)–(1.8) of the core Pontryagin maximum principle (in problem (P_k)) together with (ψ_k, ψ_k^0) and, moreover, $\psi_k^0 > 0$ and the transversality condition

$$\psi_k(T_k) = 0 \tag{3.36}$$

holds; recall that ψ_k is a (Caratheordory) solution on $[0, T_k]$ to the adjoint equation associated with (u_k, x_k) in problem (P_k), i.e.,

$$\dot{\psi}_k(t) = -\left[\frac{\partial f(x_k(t), u_k(t))}{\partial x}\right]^* \psi_k(t) - \psi^0 e^{-\rho t} \frac{\partial g(x_k(t), u_k(t))}{\partial x} \quad \text{for a.a.} \quad t \in [0, T_k], \tag{3.37}$$

and the core Pontryagin maximum principle satisfied by (u_k, x_k) together with (ψ_k, ψ_k^0) implies that the following maximum condition holds:

$$\mathcal{H}_k(x_k(t), t, u_k(t), \psi_k(t), \psi_k^0) = H_k(x_k(t), t, \psi_k(t), \psi_k^0) \quad \text{for a.a.} \quad t \in [0, T_k]; \tag{3.38}$$

here $\mathcal{H}_k$ and H_k given by

$$\mathcal{H}_k(x, t, u, \psi, \psi^0) = \langle f(x, u), \psi \rangle + \psi^0 e^{-\rho t} g(x, u) - \psi^0 e^{-(\rho+1)t} \frac{\|u - z_k(t)\|^2}{1 + \sigma_k}; \tag{3.39}$$

$$H_k(x,t,\psi,\psi^0) = \sup_{u\in U} \mathcal{H}_k(x,t,u,\psi,\psi^0)$$

are, respectively, the Hamilton-Pontryagin function and the Hamiltonian in problem (P_k); note that in [33] it is shown that (3.37) and (3.38) imply

$$\frac{d}{dt}H_k(x_k(t),t,\psi_k(t),\psi_k^0) = \frac{\partial \mathcal{H}_k}{\partial t}(x_k(t),t,u_k(t),\psi_k(t),\psi_k^0) \quad \text{for a.a.} \quad t\in[0,T_k]. \qquad (3.40)$$

Lemma 3.2 *Let assumptions* (A1) – (A4) *be satisfied;* (u_*, x_*) *be an admissible pair optimal in problem* (P)*;* $\{(P_k)\}$ *be the sequence of problems associated with* u_**; for every* $k = 1,2,\dots$ (u_k,x_k) *be an admissible pair optimal in problem* (P_k)*; for every* $k=1,2,\dots$ (ψ_k,ψ_k^0) *be a pair of adjoint variables associated with* (u_k,x_k) *in problem* (P_k) *such that* (u_k,x_k) *satisfies relations* (3.37), (3.38) *of the core Pontryagin maximum principle in problem* (P_k) *together with* (ψ_k,ψ_k^0)*; and for every* $k = 1,2,\dots$ *one have* $\psi_k^0 > 0$ *and the transversality condition* (3.36) *hold. Let, finally, the sequences* $\{\psi_k(0)\}$ *and* $\{\psi_k^0\}$ *be bounded and*

$$\|\psi_k(0)\| + \psi_k^0 \geq a \quad (k=1,2,\dots) \qquad (3.41)$$

for some $a>0$*. Then there exists a subseguence of* $\{(u_k,x_k,\psi_k,\psi_k^0)\}$*, further denoted again as* $\{(u_k,x_k,\psi_k,\psi_k^0)\}$*, such that*

(i) *for every* $T>0$

$$u_k(t)\to u_*(t) \quad \text{for a.a.} \quad t\in[0,T] \quad \text{as} \quad k\to\infty; \qquad (3.42)$$

$$x_k \to x_* \quad \text{uniformly on} \quad [0,T] \quad \text{as} \quad k\to\infty; \qquad (3.43)$$

(ii)

$$\psi_k^0 \to \psi^0 \quad \text{as} \quad k\to\infty \qquad (3.44)$$

and for every $T>0$

$$\psi_k \to \psi \quad \text{uniformly on} \quad [0,T] \quad \text{as} \quad k\to\infty, \qquad (3.45)$$

where (ψ,ψ^0) *is a nontrivial pair of adjoint variables associated with* (u_*,x_*) *in problem* (P)*;*

(iii) (u_*,x_*) *satisfies relations* (1.6)–(1.8) *of the core Pontryagin maximum principle in problem* (P) *together with* (ψ,ψ^0)*;*

(iv) *the stationarity condition holds:*

$$H(x_*(t),t,\psi(t),\psi^0) = \psi^0\rho\int_t^\infty e^{-\rho s}g(x_*(s),u_*(s))ds \quad \text{for all} \quad t\geq 0. \qquad (3.46)$$

Remark 3.4 Convergence (3.45) is defined correctly, since for all k large enough the interval $[0,T_k]$ on which ψ_k is defined contains T.

Proof of Lemma 3.2. Lemma 3.1 and the Ascoli theorem (see, e.g., [20]) imply that, selecting if needed a subsequence, we get (3.42) and (3.43) for every $T>0$. By assumption the sequence $\{\psi_k^0\}$ is bounded; therefore, selecting if needed a subsequence, we obtain (3.44) for some $\psi^0\geq 0$.

Now our goal is to select a subsequence of $\{(u_k,x_k,\psi_k)\}$ such that for every $T>0$ (3.45) holds and (ψ,ψ^0) is a nontrivial pair of adjoint variables associated with (u_*,x_*) (we do not change notations after the selection of a subsequence).

Consider the sequence $\{\psi_k\}$ restricted to $[0,T_1]$. Observing (3.37), taking into account the boundedness of the sequence $\{\psi_k(0)\}$ (see the assumptions of this lemma), using the

Gronwall lemma (see, e.g., [25]) and selecting if needed a subsequence denoted further as $\{\psi_k^1\}$, we get that $\psi_k^1 \to \psi^1$ uniformly on $[0,T_1]$ and $\dot\psi_k^1 \to \dot\psi^1$ weakly in $L^1[0,T_1]$ as $k\to\infty$ for some absolutely continuous $\psi^1 : [0,T_1] \to R^n$; here and in what follows $L^1[0,T] = L^1([0,T],R^n)$ $(T>0)$.

Now consider the sequence $\{\psi_k^1\}$ restricted to $[0,T_2]$. Taking if necessary a subsequence $\{\psi_k^2\}$ of $\{\psi_k^1\}$, we get that $\psi_k^2 \to \psi^2$ uniformly on $[0,T_2]$ and $\dot\psi_k^2 \to \dot\psi^2$ weakly in $L^1[0,T_2]$ as $k\to\infty$ for some absolutely continuous $\psi^2 : [0,T_2]\to R^n$ whose restriction to $[0,T_1]$ coincides with ψ^1.

Repeating this procedure sequentially for $[0,T_i]$ with $i=3,4,\ldots$, we find that there exist absolutely continuous $\psi^i : [0,T_i]\to R^n$ $(i=1,2,\ldots)$ and $\psi_k^i : [0,T_i] \to R^n$ $(i,k = 1,2,\ldots)$ such that for every $i=1,2,\ldots$ the restriction of ψ^{i+1} to $[0,T_i]$ is ψ^i, the restriction of the sequence $\{\psi_k^{i+1}\}$ to $[0,T_i]$ is a subsequence of $\{\psi_k^i\}$ and, moreover, $\psi_k^i\to\psi$ uniformly on $[0,T_i]$ and $\dot\psi_k^i \to \dot\psi^i$ weakly in $L^1[0,T_i]$ as $k\to\infty$.

Define $\psi : [0,\infty)\mapsto R^n$ so that the restriction of ψ to $[0,T_i]$ is ψ^i for every $i=1,2,\ldots$. Clearly, ψ is absolutely continuous. Furthermore, without changing notations, for every $i=1,2,\ldots$ and every $k=1,2,\ldots$ we extend ψ_k^i to $[0,\infty)$ so that the extended function is absolutely continuous and, moreover, the family $\dot\psi_k^i$ $(i,k=1,2,\ldots)$ is bounded in $L^1[0,T]$ for every $T>0$. Since $T_i\to\infty$ as $i\to\infty$, for every $T>0$ we get that ψ_k^k converges to ψ uniformly on $[0,T]$ and $\dot\psi_k^k\to\dot\psi$ weakly in $L^1[0,T]$ as $k\to\infty$. Simplifying notations, we, again, write ψ_k instead of ψ_k^k and note that for ψ_k (3.37) holds $(k=1,2,\ldots)$. Thus, for every $T>0$ we have (3.45) and also get that $\dot\psi_k\to\dot\psi$ weakly in $L^1[0,T]$ as $k\to\infty$. These convergences together with equalities (3.37) and convergences (3.42) and (3.43) (holding for every $T>0$) yield that ψ solves the adjoint equation (1.6). Thus, (ψ,ψ^0) is a pair of adjoint variables associated with (u_*,x_*) in problem (P). The nontriviality of (ψ,ψ^0) (see (1.7)) is ensured by (3.41).

For every $k=1,2,\ldots$ consider the maximum condition (3.38) and specify it as

$$\langle f(x_k(t),u_k(t)),\psi_k(t)\rangle + \psi_k^0 e^{-\rho t} g(x_k(t),u_k(t)) - \psi_k^0 e^{-(\rho+1)t}\frac{\|u_k(t)-z_k(t)\|^2}{1+\sigma_k} =$$

$$\max{}_{u\in U}\left[\langle f(x_k(t),u),\psi_k(t)\rangle + \psi_k^0 e^{-\rho t} g(x_k(t),u) - \psi_k^0 e^{-(\rho+1)t}\frac{\|u-z_k(t)\|^2}{1+\sigma_k}\right] \qquad (3.47)$$

$$\text{for a.a.} \quad t\in[0,T_k].$$

Taking into account that $T_k\to\infty$ and $\sigma_k\to\infty$ as $k\to\infty$ and using convergences (3.44), (3.45), (3.42) and (3.43) (holding for every $T>0$), we obtain

$$\langle f(x_*(t),u_*(t)),\psi(t)\rangle + \psi^0 e^{-\rho t} g(x_*(t),u_*(t)) = \max{}_{u\in U}\left[\langle f(x_*(t),u),\psi(t)\rangle + \psi^0 e^{-\rho t} g(x_*(t),u)\right]$$

$$\text{for a.a.} \quad t\ge 0$$

as the limit of (3.47); this is equivalent to the maximum condition (1.8). Thus, (u_*,x_*) satisfies the core Pontryagin maximum principle together with the pair (ψ,ψ^0) of adjoint variables associated with (u_*,x_*).

Now we specify (3.40) using the form of $\mathcal{H}_k$ (see (3.39)). We get

$$\begin{aligned}\frac{d}{dt}H_k(x_k(t),t,\psi_k(t),\psi_k^0) &= \frac{\partial\mathcal{H}_k}{\partial t}(x_k(t),t,u_k(t),\psi_k(t),\psi_k^0)\\ &= -\psi_k^0\rho e^{-\rho t}\left[g(x_k(t),u_k(t)) + (\rho+1)e^{-(\rho+1)t}\frac{\|u_k(t)-z_k(t)\|^2}{1+\sigma_k}\right] + \\ &\quad 2\psi_k^0 e^{-(\rho+1)t}\frac{\langle u_k(t)-z_k(t),\dot z_k(t)\rangle}{1+\sigma_k} \quad \text{for a.a.} \quad t\in[0,T_k]. \qquad (3.48)\end{aligned}$$

Take an arbitrary $t > 0$ and an arbitrary k such that $T_k > t$ and integrate (3.48) over $[t, T_k]$ taking into account the boundary condition (3.36). We arrive at

$$\begin{aligned} H_k(x_k(t), t, \psi_k(t), \psi_k^0) &= \psi_k^0 e^{-\rho T_k} \max{}_{u\in U} \left[g(x_k(T_k), u) - e^{-\rho T_k} \frac{\|u - z_k(T_k)\|^2}{1+\sigma_k}) \right] - \\ & \psi_k^0 \rho \int_t^{T_k} e^{-\rho s} g(x_k(s), u_k(s)) ds + \\ & \psi_k^0 (\rho + 1) \int_t^{T_k} e^{-(\rho+1)s} \frac{\|u_k(s) - z_k(s)\|^2}{1+\sigma_k} ds + \\ & 2\psi_k^0 \int_t^{T_k} e^{-(\rho+1)s} \frac{\langle u_k(s) - z_k(s), \dot z_k(s) \rangle}{1+\sigma_k} ds. \end{aligned}$$

Now we take the limit using convergences (3.44), (3.45), (3.42) and (3.43) (holding for every $T > 0$), and also estimates (3.31) – (3.33). We end up with (3.46). The lemma is proved.

Remark 3.5 Relation (3.46) stated in Lemma 3.2 is a reflection of the fact that the limit "infinite-horizon" pair of adjoint variables, (ψ, ψ^0), carries some "limit" information on the transversality conditions in the finite-horizon approximating problems (P_k). Indeed, (3.46) cannot be derived from the core Pontryagin maximum principle in problem (P); we proved it using the transversality conditions (3.36).

Relation (3.46) implies the *asymptotic stationarity condition* introduced in [30]:

$$\lim_{t\to\infty} H(x_*(t), t, \psi(t), \psi^0) = 0. \tag{3.49}$$

Indeed, (3.49) follows strightforwardly from (3.46) and assumption (A4) (see (1.5)). However, assuming that (u_*, x_*) satisfies the core Pontryagin maximum principle in problem (P) together with (ψ, ψ^0) (see Lemma 3.2, (iii)), one can easily state that (3.46) and (3.49) are equivalent. Indeed, let (u_*, x_*) satisfy the core Pontryagin maximum principle in problem (P) together with (ψ, ψ^0) and (3.49) hold. Taking into account that ψ solves the adjoint equation (1.6) and using the maximum condition (1.8), we get

$$\begin{aligned} \frac{d}{dt} H(x_*(t), t, \psi(t), \psi^0) &= \frac{\partial \mathcal{H}}{\partial t}(x_*(t), t, u_*(t), \psi(t), \psi^0) \\ &= -\psi^0 \rho e^{-\rho t} g(x_*(t), u_*(t)) \end{aligned}$$

for a.a. $t \geq 0$. The integration over an arbitrary interval $[t, T]$ where $T > t \geq 0$ yields

$$H(x_*(t), t, \psi(t), \psi^0) = H(x_*(T), T, \psi(T), \psi^0) + \psi^0 \rho \int_t^T e^{-\rho s} g(x_*(s), u_*(s)) ds.$$

Letting $T \to \infty$ and using (A4) and (3.49), we obtain (3.46) for any $t \geq 0$.

The corollary given below specifies Lemma 3.2 for the case where the Pontryagin maximum principle for problems (P_k) is taken in the normal form (implying that the corresponding Lagrange multipliers ψ_k^0 equal 1 [33]). We use the following formulation of the normal-form Pontryagin maximum principle for problems (P_k). Let an admissible pair (u_k, x_k) be optimal in problem (P_k) for some k. Then there exists an adjoint variable ψ_k associated with (u_k, x_k) such that (u_k, x_k) satisfies the normal-form core Pontryagin maximum principle (in problem (P_k)) together with ψ_k and the transversality condition (3.36) holds; here ψ_k is a (Caratheordory) solution on $[0, T_k]$ of the normal-form adjoint equation associated with (u_k, x_k) in problem (P_k), i.e.,

$$\dot\psi_k(t) = -\left[\frac{\partial f(x_k(t), u_k(t))}{\partial x}\right]^* \psi_k(t) - e^{-\rho t} \frac{\partial g(x_k(t), u_k(t))}{\partial x} \quad \text{for a.a.} \quad t \in [0, T_k], \tag{3.50}$$

and the normal-form core Pontryagin maximum principle satisfied by (u_k, x_k) together with ψ_k implies that the following normal-form maximum condition holds:

$$\tilde{\mathcal{H}}_k(x_k(t), t, u_k(t), \psi(t)) = \tilde{H}_k(x_k(t), t, \psi_k(t)) \quad \text{for a.a.} \quad t \geq 0; \tag{3.51}$$

here $\tilde{\mathcal{H}}_k$ and $\tilde{H}_k$ given by

$$\tilde{\mathcal{H}}_k(x, t, u, \psi) = \langle f(x, u), \psi \rangle + e^{-\rho t} g(x, u) - e^{-(\rho+1)t} \frac{\|u - z_k(t)\|^2}{1 + \sigma_k};$$

$$\tilde{H}_k(x, t, \psi) = \sup_{u \in \tilde{U}} \tilde{\mathcal{H}}_k(x, t, \tilde{u}, \psi)$$

are, respectively, the normal-form Hamilton-Pontryagin function and normal-form Hamiltonian in problem (P_k); note that (3.40) takes the form

$$\frac{d}{dt} \tilde{H}_k(x_k(t), t, \psi_k(t)) = \frac{\partial \tilde{\mathcal{H}}_k}{\partial t}(x_k(t), t, u_k(t), \psi_k(t)) \quad \text{for a.a.} \quad t \in [0, T_k].$$

The next corollary follows from Lemma 3.2 straightforwardly.

Corollary 3.1 *Let assumptions* (A1) – (A4) *be satisfied;* (u_*, x_*) *be an admissible pair optimal in problem* (P)*;* $\{(P_k)\}$ *be the sequence of problems associated with* u_**; for every* $k = 1, 2, \ldots$ (u_k, x_k) *be an admissible pair optimal in problem* (P_k)*; and for every* $k = 1, 2, \ldots$ ψ_k *be an adjoint variable associated with* (u_k, x_k) *in problem* (P_k) *such that* (u_k, x_k) *satisfies relations* (3.50), (3.51) *of the normal-form core Pontryagin maximum principle in problem* (P_k) *together with* ψ_k *and the transversality condition* (3.36) *holds. Let, finally, the sequence* $\{\psi_k(0)\}$ *be bounded. Then there exists a subsequence of* $\{(u_k, x_k, \psi_k)\}$*, further denoted again as* $\{(u_k, x_k, \psi_k)\}$*, such that*

(i) *for every* $T > 0$ (3.42) *and* (3.43) *hold;*

(ii) *for every* $T > 0$ (3.45) *holds where* ψ *is an adjoint variable associated with* (u_*, x_*) *in problem* (P)*;*

(iii) (u_*, x_*) *satisfies relations* (1.9), (1.10) *of the normal-form core Pontryagin maximum principle in problem* (P) *together with* ψ*;*

(iv) *the normal-form stationarity condition holds:*

$$\tilde{H}(x_*(t), t, \psi(t)) = \rho \int_t^\infty e^{-\rho s} g(x_*(s), u_*(s)) ds \quad \text{for all} \quad t \geq 0. \tag{3.52}$$

4 Maximum principle and stationarity condition

In this section and in Sections 5 and 6 specific necessary conditions of optimality in problem (P) are derived. Our basic instruments are Lemma 3.2 and Corollary 3.1 providing limit relations in the Pontryagin maximum principle for the approximating finite-horizon problems (P_k) associated with a given control u_* optimal in problem (P).

The next theorem which is in fact an immediate corollary of Lemma 3.2 is an adaptation of a result of [30] to problem (P).

Theorem 4.2 *Let assumptions* (A1) – (A4) *be satisfied and* (u_*, x_*) *be an admissible pair optimal in problem* (P)*. Then there exists a pair* (ψ, ψ^0) *of adjoint variables associated with* (u_*, x_*) *such that*

(i) (u_*, x_*) *satisfies relations* (1.6)–(1.8) *of the core Pontryagin maximum principle together with* (ψ, ψ^0)*, and*

(ii) (u_*, x_*) *and* (ψ, ψ^0) *satisfy the stationarity condition* (3.46)*.*

Remark 4.6 Theorem 4.2 (which is, evidently, stronger than Theorem 1.1) suggests the most complete formulation of the Pontryagin maximum principle for problem (P) under assumptions (A1) – (A4). Formally, the necessary optimality conditions given by Theorem 4.2 are equivalent to those stated in [30], in application to setting (P). One can, though, anticipate that beyond this setting (for example, for problems of infinite-horizon optimal control of systems with non-smooth right-hand sides or systems described by differential inclusions) (3.46) complementing the core Pontryagin maximum principle can be substantially stronger than the asymptotic stationarity condition (3.49) stated in [30].

Remark 4.7 Under the assumptions of Theorem 4.2 we have

$$\lim_{t\to\infty} \max{}_{u\in U}\langle f(x_*(t),u),\psi(t)\rangle = 0; \tag{4.53}$$

the latter follows from (3.46) and assumption (A4).

Remark 4.8 Recall that Example 1 (modifying an example given in [30]) shows that problem (P) can be abnormal, i.e., under the assumptions of Theorem 4.2 the nontriviality condition (1.7) can hold with $\psi^0 = 0$. In Section 5 we find additional assumptions excluding abnormality of problem (P).

Proof of Theorem 4.2. Let $\{(P_k)\}$ be the sequence of problems associated with u_* and for every $k = 1, 2, \ldots$ (u_k, x_k) be an admissible pair optimal in problem (P_k). In accordance with the classical formulation of the Pontryagin maximum principle, for every $k = 1, 2, \ldots$ there exists a pair (ψ_k, ψ_k^0) of adjoint variables associated with (u_k, x_k) in problem (P_k) such that (u_k, x_k) satisfies the core Pontryagin maximum principle (in problem (P_k)) together with (ψ_k, ψ_k^0) and for every $k = 1, 2, \ldots$ $\psi_k^0 > 0$ and the transversality condition (3.36) holds.

Since $\psi_k^0 > 0$, the value $c_k = \|\psi_k(0)\| + \psi_k^0$ is positive. We keep the notations ψ_k and ψ_k^0 for the normalized elements ψ_k/c_k and ψ_k^0/c_k, thus, achieving $\|\psi_k(0)\| + \psi_k^0 = 1$ and, clearly, preserving the transversality condition (3.36) and the fact that (u_k, x_k) satisfies the core Pontryagin maximum principle (in problem (P_k)) together with (ψ_k, ψ_k^0) $(k = 1, 2, \ldots)$. Now the sequences $\{\psi_k(0)\}$ and $\{\psi_k^0\}$ are bounded and (3.41) holds with $a = 1$. Thus, the sequence $\{(u_k, x_k, \psi_k, \psi_k^0)\}$ satisfies all the assumptions of Lemma 3.2. By Lemma 3.2 there exists a subseguence of $\{(u_k, x_k, \psi_k, \psi_k^0)\}$, further denoted again as $\{(u_k, x_k, \psi_k, \psi_k^0)\}$, such that for the pairs (ψ_k, ψ_k^0) of adjoint variables convergences (3.44) and (3.45) hold with an arbitrary $T > 0$; the limit element (ψ, ψ^0) is a nontrivial pair of adjoint variables associated with (u_*, x_*) in problem (P); (u_*, x_*) satisfies the core Pontryagin maximum principle in problem (P) together with (ψ, ψ^0); and, finally, (u_*, x_*) and (ψ, ψ^0) satisfy the stationarity condition (3.46). The theorem is proved.

As noted in Remark 3.5, the stationarity condition (3.46) stated in Theorem 4.2 does not follow from the core Pontryagin maximum principle in problem (P). In other words, (3.46) complements the core Pontryagin maximum principle substantially. Example 4 given below illustrates this fact. It shows that the usage of the core Pontryagin maximum principle may not lead to the specification of an optimal control, whereas the latter can be selected if one applies the core Pontryagin maximum principle together with (3.46). It is remarkable that in Example 4 the "natural" transversality conditions (2.13) are violated (Example 4 deals with the situation of Example 2), i.e., the "additional" information (3.46) is by no means identical to (2.13).

Example 4. Let us come back to the problem analyzed in Example 2:

$$\dot{x}(t) = u(t) - x(t);$$

$$u(t) \in U = [0, 1];$$

$$x(0) = 1/2;$$

$$\text{maximize } J(x, u) = \int_0^\infty e^{-t} \ln \frac{1}{x(t)} dt = -\int_0^\infty e^{-t} \ln x(t) dt,$$

with $G = (0, \infty)$. As noted in Example 2, assmptions (A1) – (A4) are satisfied.

Let an admissible pair (u_*, x_*) satisfy the core Pontryagin maximum principle together with some pair (ψ, ψ^0) of adjoint variables associated with (u_*, x_*). The Hamilton-Pontryagin function is given by

$$\mathcal{H}(x, t, u, \psi, \psi^0) = (u - x)\psi - \psi^0 e^{-t} \ln x;$$

and the adjoint equation (1.6) has the form

$$\dot{\psi}(t) = \psi(t) + \psi^0 e^{-t} \frac{1}{x_*(t)}. \tag{4.54}$$

By assumption ψ solves (4.54). The maximum condition (1.8) yields

$$\begin{array}{llll} u_*(t) = 1 & \text{for a.a.} & t & \text{such that} \quad \psi(t) > 0; \\ u_*(t) = 0 & \text{for a.a.} & t & \text{such that} \quad \psi(t) < 0. \end{array} \tag{4.55}$$

Resolving (4.54), we get

$$\psi(t) = \psi(\xi) e^{t - \xi} + \psi^0 \int_\xi^t \frac{e^{t-2s}}{x_*(s)} ds \tag{4.56}$$

for all $\xi \geq 0$ and all $t \geq \xi$.

Suppose $\psi^0 > 0$. Consider three cases:

$$\psi(0) \geq 0; \tag{4.57}$$

$$-2\psi^0 < \psi(0) < 0; \tag{4.58}$$

$$\psi(0) \leq -2\psi^0. \tag{4.59}$$

In case (4.57) by (4.56) where $\xi = 0$ we have $\psi(t) > 0$ for all $t > 0$, hence, by (4.55) $u_*(t) = 1$ for a.a. $t \geq 0$.

Consider case (4.58). Clearly, $\zeta = \sup\{t > 0 : \psi(s) < 0 \text{ for all } s \in [0, t]\} > 0$. By (4.55) $u_*(t) = 0$ for a.a. $t \in [0, \zeta)$ and hence $x_*(t) = \frac{1}{2} e^{-t}$ for all $t \in [0, \zeta)$. Due to (4.56) we have $\zeta < \infty$. Then by (4.56) $\psi(t) > 0$ for all $t > \zeta$ and hence, by (4.55) $u_*(t) = 1$ for a.a. $t \geq \zeta$.

Finally, in case (4.59) by (4.56) where $\xi = 0$ we get $\psi(t) < 0$ for all $t > 0$; hence, by (4.55) $u_*(t) = 0$ for a.a. $t \geq 0$.

Now suppose $\psi^0 = 0$. By the nontriviality condition (1.7) $\psi(0) \neq 0$ and by (4.56) with $\xi = 0$ we have either $\psi(t) > 0$ for all $t > 0$, implying $u_*(t) = 1$ for a.a. $t \geq 0$, or $\psi(t) < 0$ for all $t > 0$, implying $u_*(t) = 0$ for a.a. $t \geq 0$.

Thus, we showed that if an admissible pair (u_*, x_*) satisfies the core Pontryagin maximum principle together with a pair (ψ, ψ^0) of adjoint variables associated with (u_*, x_*), then we have either $u_*(t) = 0$ for a.a. $t \geq 0$, or $u_*(t) = 1$ for a.a. $t \geq 0$, or

$$u_*(t) = \begin{cases} 0 & \text{for a.a.} \quad t \in [0, \zeta); \\ 1 & \text{for a.a.} \quad t \geq \zeta \end{cases} \tag{4.60}$$

for some $\zeta \geq 0$. Moreover, both situations are admissible. Indeed, as shown in Example 2, the admissible pair (u_*, x_*) where $u_*(t) = 0$ for a.a. $t \geq 0$ is the unique optimal one; therefore, (u_*, x_*) satisfies the core Pontryagin maximum principle together with some pair (ψ, ψ^0) of adjoint variables associated with (u_*, x_*). Our analysis of cases (4.57), (4.58) shows that the non-optimal admissible pair (u_*, x_*) of the form (4.60) also satisfies the core Pontryagin maximum principle together with a corresponding pair (ψ, ψ^0) of adgoint variables.

Thus, the core Pontryagin maximum principle (not complemented by (3.46)) is unable to reject all non-optimal controls of form (4.60).

Let us show that we reject all non-optimal controls of form (4.60) if we take into account (3.46). Let (u_*, x_*) be some admissible pair such that u_* is given by (4.60) and let (u_*, x_*) satisfy the core Pontryagin maximum principle together with some pair (ψ, ψ^0) of adjoint variables associated with (u_*, x_*). Due to (4.60), $\dot{x}_*(t) = 1 - x_*(t)$ for a.a. $t \geq \zeta$; it is also clear that $0 < x_*(\zeta) < 1$. Therefore,

$$x_*(t) = ce^{-t+\zeta} + 1 \quad \text{for all} \quad t \geq \zeta, \quad c < 0. \tag{4.61}$$

By (4.55) necessarily $\psi(t) \geq 0$ for all $t \geq \zeta$; in particular, $\psi(\zeta) \geq 0$. The latter inequality together with the nontriviality condition (1.7) and representation (4.56) yield that ψ is strictly increasing on $[\zeta, \infty)$. Again using (4.56), we state that

$$\psi(\nu) > 0, \quad \psi(t) \geq \psi(\nu)e^{t-\nu} \quad \text{for all} \quad t > \nu. \tag{4.62}$$

where $\nu > \zeta$ is some fixed value.

Now suppose (3.46) holds. Using the maximum condition (1.8), we rewrite (3.46) as

$$\mathcal{H}(x_*(t), t, u_*(t), \psi(t), \psi^0) = -\psi^0 \int_t^\infty e^{-s} \ln x_*(s) ds \quad \text{for a.a.} \quad t \geq 0.$$

More specifically, we have

$$(1 - x_*(t))\psi(t) - \psi^0 e^{-t} \ln x_*(t) = -\psi^0 \int_t^\infty e^{-s} \ln x_*(s) ds \quad \text{for all} \quad t \geq \nu. \tag{4.63}$$

Consider the left-hand side in (4.63). In view of (4.61) $1 - x_*(t) = -ce^{-t+\zeta} > 0$ for $t \geq \nu$, which together with (4.62) yield the following lower estimate for the left-hand side in (4.63):

$$\begin{aligned} (1 - x_*(t))\psi(t) - \psi^0 e^{-t} \ln x_*(t) &\geq -ce^{-t+\zeta}\psi(\nu)e^{t-\nu} - \psi^0 e^{-t} \ln x_*(t) \\ &= b - \psi^0 e^{-t} \ln x_*(t) \end{aligned} \tag{4.64}$$

where $b = -ce^{\zeta-\nu}\psi(\nu) > 0$. By (4.61) $0 < x_*(\zeta) \leq x_*(t) < 1$ for all $t \geq \zeta$; hence,

$$|\ln x_*(t)| \leq |\ln x_*(\zeta)| \quad \text{for all} \quad t \geq \zeta. \tag{4.65}$$

The latter implies that $e^{-t} \ln x_*(t) \to 0$ as $t \to \infty$. Now (4.64) yields the next lower estimate for the lower limit of the left-hand side in (4.63):

$$\liminf_{t\to\infty} [(1 - x_*(t))\psi(t) - \psi^0 e^{-t} \ln x_*(t)] \geq b > 0. \tag{4.66}$$

For the right-hand side in (4.63), due to (4.65), we have

$$\lim_{t\to\infty} [-\psi^0 \int_t^\infty e^{-s} \ln x_*(s) ds] = 0,$$

which together with (4.66) imply that (4.63) is not possible.

Thus, the core Pontryagin maximum principle in combination with (3.46) is satisfied by the single admissible pair (u_*, x_*) (where $u_*(t) = 0$ for a.a. $t \geq 0$) together with some pair (ψ, ψ^0) of adjoint variables associated with (u_*, x_*). The latter admissible pair (u_*, x_*) is the unique optimal one.

5 Normal-form maximum principle and transversality conditions

As noted in Remark 4.8, Theorem 4.2 holding under assumptions (A1) – (A4) does not exclude abnormality of problem (P); in other words, Theorem 4.2 admits that the Pontryagin maximum principle can hold with $\psi^0 = 0$ only. In this section, we suggest an assumption that excludes abnormality of problem (P), i.e., ensures that for problem (P) the normal-form Pontryagin maximum principle (see Section 1) provides a necessary condition of optimality. Moreover, our basic result formulated in Theorem 5.3 states that all the coordinates of the adjoint variable ψ in the Pontryagin maximum principle are necessarily positive-valued. Basing on Theorem 5.3, we formulate conditions ensuring that the core Pontryagin maximum principle is complemented by the transversality conditions discussed in Section 2. The proof of Theorem 5.3 is based on Corollary 3.1.

In what follows, the notation $z > 0$ (respectively, $z \geq 0$) for a vector $z \in R^n$ designates that all coordinates of z are positive (respectively, nonnegative). Similarly, the notation $Z > 0$ (respectively, $Z \geq 0$) for a matrix Z designates that all elements of Z are positive (respectively, nonnegative).

The assumption complementing assumptions (A1) – (A4) is the following.

(A5) For every admissible pair (u, x) one has

$$\frac{\partial g(x(t), u(t))}{\partial x} > 0 \quad \text{for a.a.} \quad t \geq 0$$

and

$$\frac{\partial f(x(t), u(t))}{\partial x} \geq 0 \quad \text{for a.a.} \quad t \geq 0.$$

Remark 5.9 In typical models of regulated economic growth the coordinates of the state vector x represent positive-valued production factors. Normally it is assumed that the utility flow and the rate of growth of every production factor increase as all the production factors grow. In terms of problem (P), this implies that the integrand $g(x, u)$ in the goal functional (1.3) together with every coordinate of the right-hand side $f(x, u)$ of the system equation (1.1) are monotonically increasing in every coordinate of x. These monotonicity properties (specified so that $g(x, u)$ is strictly increasing in every coordinate of x) imply that assumption (A5) is satisfied. Note that the utility flow and the rates of growth of the production factors are normally positive, implying $g(x, u) > 0$ and $f(x, u) > 0$. The latter assumptions, as well as the assumption $x > 0$ mentioned earlier appear in different combinations in the formulations of the results of this section.

The next theorem strengthens Theorem 4.2 under assumption (A5) and some positivity assumptions for f (recall that the formulation of the normal-form Pontryagin maximum principle for problem (P) is given in Section 1, where also the normal-form Hamilton-Pontryagin function $\tilde{\mathcal{H}}$ and the normal-form Hamiltonian $\tilde{H}$ in problem (P) are defined).

Theorem 5.3 *Let assumptions* (A1) – (A5) *be satisfied, there exist a* $u_0 \in U$ *such that* $f(x_0, u_0) > 0$ *and for every admissible pair* (u, x) *it hold that* $f(x(t), u(t)) \geq 0$ *for a.a.* $t \geq 0$. *Let* (u_*, x_*) *be an admissible pair optimal in problem* (P). *Then there exists an adjoint variable* ψ *associated with* (u_*, x_*) *such that*

(i) (u_*, x_*) *satisfies relations* (1.9), (1.10) *of the normal-form core Pontryagin maximum principle together with* ψ;

(ii) (u_*, x_*) *and* ψ *satisfy the normal-form stationarity condition* (3.52);

(iii)

$$\psi(t) > 0 \quad \text{for all} \quad t \geq 0. \tag{5.67}$$

Remark 5.10 Condition (3.52) is the specification of condition (3.46) in Theorem 4.2 for the case where $\psi^0 = 1$. Condition (5.67) is non-standard for the Pontryagin maximum principle; it usually arises in problems of optimal economic growth and plays an important role in our analysis of the transversality conditions for problem (P) (see Corollaries 5.2 and 5.3).

Proof of Theorem 5.3. Let $\{(P_k)\}$ be the sequence of problems associated with u_* and for every $k = 1, 2, \ldots$ (u_k, x_k) be an admissible pair optimal in problem (P_k). In accordance with the classical formulation of the normal-form Pontryagin maximum principle, for every $k = 1, 2, \ldots$ there exists an adjoint variable ψ_k associated with (u_k, x_k) in problem (P_k) such that (u_k, x_k) satisfies the normal-form core Pontryagin maximum principle (in problem (P_k)) together with ψ_k and for every $k = 1, 2, \ldots$ the transversality condition (3.36) holds.

Observing assumption (A5), the adjoint equation resolved by ψ_k (see (3.50)) and transversality condition (3.36) for ψ_k, we easily find that $\psi_k(t) > 0$ for all t sufficiently close to T_k. Let us show that

$$\psi_k(t) > 0 \quad \text{for all} \quad t \in [0, T_k). \tag{5.68}$$

Suppose the contrary. Then for some k there exists a $\tau \in [0, T_k)$ such that at least one coordinate of the vector $\psi_k(\tau)$ vanishes. Let ξ be the maximum of all such $\tau \in [0, T_k)$ and $i \in \{1, 2, \ldots, n\}$ be such that $\psi_k^i(\xi) = 0$. Then

$$\psi_k(t) > 0 \quad \text{for all} \quad t \in (\xi, T_k) \tag{5.69}$$

and

$$\psi_k^i(t) = -\int_\xi^t \left\langle \frac{\partial f^i(x_k(s), u_k(s))}{\partial x}, \psi_k(s) \right\rangle ds - \int_\xi^t e^{-\rho s} \frac{\partial g^i(x_k(s), u_k(s))}{\partial x} ds \tag{5.70}$$

$$\text{for all} \quad t \in [\xi, T_k].$$

The latter equation and assumption (A5) imply that $\psi_k^i(t) \leq 0$ for all $t \in (t_*, T_k)$, which contradicts (5.69). The contradiction proves (5.68).

Let us show that the sequence $\{\psi_k(0)\}$ is bounded. The equation for ψ_k (see (3.50)) and maximum condition (3.51) yield

$$\begin{aligned}
\frac{d}{dt}\tilde{H}_k(x_k(t), t, \psi_k(t)) &= \frac{\partial \tilde{\mathcal{H}}_k}{\partial t}(x_k(t), t, \tilde{u}_k(t), \psi_k(t)) \\
&= -\rho e^{-\rho t} g(x_k(t), u_k(t)) + (\rho+1) e^{-(\rho+1)t} \frac{\|u_k(t) - z_k(t)\|^2}{1 + \sigma_k} + \\
&\quad 2e^{-(\rho+1)t} \frac{\langle u_k(t) - z_k(t), \dot{z}_k(t) \rangle}{1 + \sigma_k} \quad \text{for a.a.} \quad t \geq 0.
\end{aligned}$$

Integrating over $[0, T_k]$ and using the transversality condition (3.36), we arrive at

$$\tilde{H}_k(x_0, 0, \psi_k(0)) \;=\; e^{-\rho T_k} \max{}_{u \in U} \left[g(x_k(T_k), u) - e^{-T_k} \frac{\|u - z_k(T_k)\|^2}{1 + \sigma_k} \right] +$$

$$\rho \int_0^{T_k} e^{-\rho t} g(x_k(t), u_k(t)) dt -$$
$$(\rho+1) \int_0^{T_k} e^{-(\rho+1)t} \frac{\|u_k(t) - z_k(t)\|^2}{1+\sigma_k} dt -$$
$$2 \int_0^{T_k} e^{-(\rho+1)t} \frac{\langle u_k(t) - z_k(t), \dot{z}_k(t)\rangle}{1+\sigma_k} dt.$$

This together with (3.31) – (3.33) imply that $\tilde{H}_k(x_0, 0, \psi_k(0)) \leq M$ for some $M > 0$ and all $k = 1, 2, \ldots$. Hence, by virtue of

$$\langle f(x_0, u_0), \psi_k(0)\rangle + g(x_0, u_0) - \frac{\|u_0 - z_k(0)\|^2}{1+\sigma_k} \leq \tilde{H}_k(x_0, 0, \psi_k(0)),$$

we have

$$\langle f(x_0, u_0), \psi_k(0)\rangle \leq M + |g(x_0, u_0)| + (2|U| + 1)^2$$

where $|U| = \max_{u\in U}\|u\|$. The latter estimate, assumption $f(x_0, u_0) > 0$ and (5.68) yield that the sequence $\{\psi_k(0)\}$ is bounded.

Therefore, the sequence $\{(u_k, x_k, \psi_k)\}$ satisfies all the assumptions of Corollary 3.1. By Corollary 3.1 there exists a subseguence of $\{(u_k, x_k, \psi_k)\}$, further denoted again as $\{(u_k, x_k, \psi_k)\}$, such that for every $T > 0$ one has convergence (3.45) for the adjoint variables ψ_k where the limit element ψ is an adjoint variable associated with (u_*, x_*) in problem (P); (u_*, x_*) satisfies the normal-form core Pontryagin maximum principle in problem (P) together with ψ; and, finally, (u_*, x_*) and ψ satisfy the normal-form asymptotic stationarity condition (3.52). Thus, for (u_*, x_*) and ψ statements (i) and (ii) are proved.

From (3.45) and (5.68) it follows that $\psi(t) \geq 0$ for all $t \geq 0$. Now the fact that ψ solves the adjoint equation (1.9) and assumption (A5) imply (5.67), thus, proving (iii). The theorem is proved.

Remark 5.11 Suppose the dimension n of the state space of system (1.1) is 1. Then Theorem 5.3 remains true if one removes the assumption that for every admissible pair (u, x) it holds that $f(x(t), u(t)) \geq 0$ for a.a. $t \geq 0$. Indeed, in the proof of Theorem 5.3 we use the latter assumption to state (5.68) only. If $n = 1$, (5.68) follows straightforwardly from (3.36), (5.70) and the fact that $\partial g(x_k(t), u_k(t))/\partial x > 0$ for a.a. $t \in [0, T_k]$ (see assumption (A5)).

Now, using Theorem 5.3, we formulate conditions coupling the normal-form core Pontryagin maximum principle and the transversality conditions discussed in Section 2.

Corollary 5.2 *Let the assumptions of Theorem* 5.3 *be satisfied and*

$$f(x_*(t), u_*(t)) \geq a_1 \quad \text{for a.a.} \quad t \geq 0 \tag{5.71}$$

where $a_1 > 0$*. Then there exists an adjoint variable* ψ *associated with* (u_*, x_*) *such that statements* (i), (ii) *and* (iii) *of Theorem* 5.3 *hold true and, moreover,* ψ *satisfies the transversality condition*

$$\lim_{t\to\infty} \psi(t) = 0. \tag{5.72}$$

Remark 5.12 The fact that (u_*, x_*) satisfies the normal-form Pontryagin maximum principle (Theorem 5.3, (i)) implies that (5.72) is equivalent to the transversality conditions (2.13) discussed in Section 2.

Proof of Corollary 5.2. By Theorem 5.3 there exists an adjoint variable ψ associated with (u_*, x_*) such that statements (i), (ii) and (iii) of Theorem 5.3 hold true. Let us prove (5.72). By Remark 4.7 we have (4.53). From (4.53) and (5.71) we get

$$\lim_{t\to\infty}\langle a_1, \psi(t)\rangle \leq \lim_{t\to\infty} \max{}_{u\in U}\langle f(x_*(t), u), \psi(t)\rangle = 0;$$

the latter together with (5.67) imply (5.72). The corollary is proved.

Corollary 5.3 *Let the assumptions of Theorem* 5.3 *be satisfied,*

$$x_0 \geq 0, \tag{5.73}$$

$$g(x_*(t), u_*) \geq 0 \quad \text{for a.a.} \quad t \geq 0 \tag{5.74}$$

and

$$\frac{\partial f(x_*(t), u_*(t))}{\partial x} \geq A \quad \text{for a.a.} \quad t \geq 0 \tag{5.75}$$

where $A > 0$. *Then there exists an adjoint variable* ψ *associated with* (u_*, x_*) *such that statements* (i), (ii) *and* (iii) *of Theorem* 5.3 *hold true and, moreover,* ψ *satisfies the transversality condition*

$$\lim_{t\to\infty}\langle x_*(t), \psi(t)\rangle = 0. \tag{5.76}$$

Remark 5.13 The fact that (u_*, x_*) satisfies the normal-form Pontryagin maximum principle (Theorem 5.3, (i)) implies that (5.76) is equivalent to the transversality conditions (2.14) discussed in Section 2.

Proof of Corollary 5.3. By Theorem 5.3 there exists an adjoint variable ψ associated with (u_*, x_*) such that statements (i), (ii) and (iii) of Theorem 5.3 hold true. Let us prove (5.76). The system equation (1.1) and normal-form adjoint equation (1.9) yield

$$\begin{aligned} \frac{d}{dt}\langle x_*(t), \psi(t)\rangle &= \langle f(x_*(t), u_*(t)), \psi(t)\rangle - \\ & \left\langle x_*(t), \left[\frac{\partial f(x_*(t), u_*(t))}{\partial x}\right]^* \psi(t)\right\rangle - \\ & e^{-\rho t}\left\langle x_*(t), \frac{\partial g(x_*(t), u_*(t))}{\partial x}\right\rangle \quad \text{for a.a.} \quad t \geq 0. \end{aligned} \tag{5.77}$$

From (5.73), assumption (A5) and (5.74) follows

$$-e^{-\rho t}\left\langle x_*(t), \frac{\partial g(x_*(t), u_*(t))}{\partial x}\right\rangle \leq 0 \leq e^{-\rho t} g(x_*(t), u_*(t)).$$

Taking this into account and using assumption (A5), the normal-form maximum condition (1.10) and assumption (5.75), we continue (5.77) as follows:

$$\begin{aligned} \frac{d}{dt}\langle x_*(t), \psi(t)\rangle &= \langle f(x_*(t), u_*(t)), \psi(t)\rangle - \\ & \left\langle x_*(t), \left[\frac{\partial f(x_*(t), u_*(t))}{\partial x}\right]^* \psi(t)\right\rangle + e^{-\rho t} g(x_*(t), u_*(t)) \\ &\leq -\langle A x_*(t), \psi(t)\rangle + \tilde{H}(x_*(t), t, \psi(t)) \quad \text{for a.a.} \quad t \geq 0. \end{aligned}$$

Therefore, by (5.75) for some $\theta > 0$ we have

$$\frac{d}{dt}\langle x_*(t), \psi(t)\rangle \leq -\theta\langle x_*(t), \psi(t)\rangle + \alpha(t)$$

where

$$\alpha(t) = \tilde{H}(x_*(t), t, \psi(t)) \to 0 \quad \text{as} \quad t \to \infty$$

(see (3.52)). Then, taking into account (5.73) and (5.67), we get

$$0 \leq \langle x_*(t), \psi(t)\rangle \leq e^{-\theta t}\langle x_0, \psi(0)\rangle + e^{-\theta t}\int_0^t e^{\theta s}\alpha(s)ds. \tag{5.78}$$

Furthermore,

$$\begin{aligned}\dot{\alpha}(t) &= \frac{d}{dt}\tilde{H}(x_*(t), t, \psi(t)) = \frac{\partial}{\partial t}\tilde{\mathcal{H}}(x_*(t), t, u_*(t), \psi(t)) \\ &= -\rho e^{-\rho t} g(x_*(t), u_*(t)) \leq 0 \quad \text{for a.a.} \quad t \geq 0\end{aligned}$$

(here we used (5.74)). Therefore,

$$\int_0^t e^{\theta s}\alpha(s)ds = \frac{1}{\theta}[e^{\theta t}\alpha(t) - \alpha(0)] + \frac{1}{\theta}\int_0^t e^{\theta s}\dot{\alpha}(s)ds \leq \frac{1}{\theta}(e^{\theta t}\alpha(t) - \alpha(0)).$$

Substituting this estimate into (5.78), we get

$$0 \leq \langle x_*(t), \psi(t)\rangle \leq e^{-\theta t}\langle x_0, \psi(0)\rangle + e^{-\theta t}\frac{1}{\theta}[e^{\theta t}\alpha(t) - \alpha(0)] \to 0 \qquad \text{as} \quad t \to \infty.$$

The corollary is proved.

The next theorem is to a certain extent an inversion of Theorem 5.3. It adjoins works treating the Pontryagin maximum principle as a key component in sufficient conditions of optimality. Within the finite-horizon setting, this line of analysis was initiated in [29]. In [1] the approach was extended to infinite-horizon optimal control problems.

Theorem 5.4 *Let assumptions* (A1) – (A5) *be satisfied,* $x_0 \geq 0$ *and for every admissible pair* (u, x) *it hold that* $f(x(t), u(t)) \geq 0$ *and* $g(x(t), u(t)) \geq 0$ *for a.a.* $t \geq 0$*. Let* (u_*, x_*) *be an admissible pair satisfying* (5.75) *with some* $A > 0$*, and there exist an adjoint variable* ψ *associated with* (u_*, x_*) *such that statements* (i), (ii) *and* (iii) *of Theorem* 5.3 *hold true. Let, finally, the set* G *be convex and function* $x \mapsto \tilde{H}(x, t, \psi(t)) : G \mapsto R^1$ *be concave for every* $t \geq 0$*. Then the admissible pair* (u_*, x_*) *is optimal in problem* (P)*.*

We omit the proof, which is similar to the proofs given in [2] and [35].

Combining Corollary 5.3 and Theorem 5.4, we arrive at the following optimality criterion for problem (P).

Corollary 5.4 *Let assumptions* (A1) – (A5) *be satisfied;* $x_0 \geq 0$*; the set* G *be convex; the function* $x \mapsto \tilde{H}(x, t, \psi) : G \mapsto R^1$ *be concave for every* $t \geq 0$ *and for every* $\psi > 0$*; there exist a* $u_0 \in U$ *such that* $f(x_0, u_0) > 0$*; and for every admissible pair* (u, x) *it hold that* $f(x(t), u(t)) \geq 0$ *for a.a.* $t \geq 0$*,* $g(x(t), u(t)) \geq 0$ *for a.a.* $t \geq 0$ *and* $\partial f(x(t), u(t))/\partial x \geq A$ *for a.a.* $t \geq 0$ *with some* $A > 0$*. Then an admissible pair* (u_*, x_*) *is optimal in problem* (P) *if and only if there exists an adjoint variable* ψ *associated with* (u_*, x_*) *such that statements* (i), (ii) *and* (iii) *of Theorem* 5.3 *hold and the transversality condition* (5.76) *is satisfied.*

6 Case of dominating discount

In [13] infinite-horizon necessary optimality conditions involving the normal-form core Pontryagin maximum principle and a characterization of global behavior of the adjoint variable (alternative to the transversality conditions) were stated; in this work the control system was assumed to be linear. In this section we use the approximation scheme developed in Section 3 to prove a nonlinear counterpart of the result of [13].

Following [13], we posit the next growth constraint on g:

(A6) There exist a $\kappa \geq 0$ and an $r \geq 0$ such that

$$\left\| \frac{\partial g(x,u)}{\partial x} \right\| \leq \kappa(1 + \|x\|^r) \quad \text{for all} \quad x \in G \quad \text{and for all} \quad u \in U. \tag{6.79}$$

Given an admissible pair (u, x), we denote by $Y_{(u,x)}$ the normalized fundamental matrix for the linear differential equation

$$\dot{y}(t) = \frac{\partial f(x(t), u(t))}{\partial x} y(t); \tag{6.80}$$

more specifically, $Y_{(x,u)}$ is the $n \times n$ matrix-valued function on $[0, \infty)$ whose columns y_i $(i = 1, \ldots, n)$ are the solutions to (6.80) such that $y_i^j(0) = \delta_{i,j}$ $(i, j = 1, \ldots, n)$ where $\delta_{i,i} = 1$ and $\delta_{i,j} = 0$ for $i \neq j$; for every $t \geq 0$, $\|Y_{(u,x)}(t)\|$ stands for the standard norm of $Y_{(u,x)}(t)$ as a linear operator in R^n. Similarly, given an admissible pair (u, x), we denote by $Z_{(u,x)}$ the normalized fundamental matrix for the linear differential equation

$$\dot{z}(t) = -\left[\frac{\partial f(x(t), u(t))}{\partial x}\right]^* z(t). \tag{6.81}$$

Note that

$$[Z_{(u,x)}(t)]^{-1} = [Y_{(u,x)}(t)]^*. \tag{6.82}$$

Introduce the following growth assumption:

(A7) There exist a $\lambda \in R^1$ a $C_1 \geq 0$, a $C_2 \geq 0$ and a $C_3 \geq 0$ such that for every admissible pair (u, x) one has

$$\|x(t)\| \leq C_1 + C_2 e^{\lambda t} \quad \text{for all} \quad t \geq 0 \tag{6.83}$$

and

$$\|Y_{(u,x)}(t)\| \leq C_3 e^{\lambda t} \quad \text{for all} \quad t \geq 0. \tag{6.84}$$

Remark 6.14 It is easily seen that assumption (A6) implies that there exist a $C_4 \geq 0$ and a $C_5 \geq 0$ such that for every admissible pair (u, x)

$$|g(x(t), u(t)| \leq C_4 + C_5 \|x(t)\|^{r+1} \quad \text{for all} \quad t \geq 0. \tag{6.85}$$

Furthermore, (6.83) and (6.85) imply that

$$e^{-\rho t}|g(x(t), u(t)| \leq C_6 e^{-\rho t} + C_7 e^{-(\rho-(r+1)\lambda)t}$$

holds for every admissible pair (u, x) with $C_6 \geq 0$ and $C_7 \geq 0$ not depending on (u, x). Therefore, assumptions (A6) and (A7) imply (A4) provided $\rho > (r + 1)\lambda$. The latter inequality implying that the discount parameter ρ in the goal functional (1.3) *dominates* the growth parameters r and λ (see (6.79) and (6.83)) is a counterpart of a condition assumed in [13].

The next theorem is a nonlinear extension of Theorem 1 of [13]. The proof is based on Corollary 3.1.

Theorem 6.5 *Let assumptions* (A1) – (A3), (A6) *and* (A7) *be satisfied and* $\rho > (r+1)\lambda$. *Let* (u_*, x_*) *be an admissible pair optimal in problem* (P). *Then there exists an adjoint variable* ψ *associated with* (u_*, x_*) *such that*

(i) (u_*, x_*) *satisfies relations* (1.9), (1.10) *of the normal-form core Pontryagin maximum principle together with* ψ,

(ii) (u_*, x_*) *and* ψ *satisfy the normal-form stationarity condition* (3.52);

(iii) *for every* $t \geq 0$ *the integral*

$$I_*(t) = \int_t^\infty e^{-\rho s}[Z_*(s)]^{-1}\frac{\partial g(x_*(s), u_*(s))}{\partial x}ds \qquad (6.86)$$

where $Z_* = Z_{(u_*,x_*)}$ *converges absolutely and*

$$\psi(t) = Z_*(t)I_*(t). \qquad (6.87)$$

Proof of Theorem 6.5. Let $\{(P_k)\}$ be the sequence of problems associated with u_* and for every $k = 1, 2, \ldots$ (u_k, x_k) be an admissible pair optimal in problem (P_k). In accordance with the classical formulation of the normal-form Pontryagin maximum principle, for every $k = 1, 2, \ldots$ there exists an adjoint variable ψ_k associated with (u_k, x_k) in problem (P_k) such that (u_k, x_k) satisfies the normal-form core Pontryagin maximum principle (in problem (P_k)) together with ψ_k and for every $k = 1, 2, \ldots$ the transversality condition (3.36) holds.

Let us show that the sequence $\{\psi_k(0)\}$ is bounded. Using the standard representation of the solution ψ_k to the linear normal-form adjoint equation (3.50) with the zero boundary condition (3.36) through the fundamental matrix $Z_k = Z_{(u_k,x_k)}$ of the corresponding linear homogeneous equation (see (6.81)), we get

$$\psi_k(0) = \int_0^{T_k} e^{-\rho s}[Z_k(s)]^{-1}\frac{\partial g(x_k(s), u_k(s))}{\partial x}ds.$$

We have (see (6.82))

$$[Z_k(s)]^{-1} = [Y_{(x_k,u_k)}(s)]^*, \quad \|[Y_{(x_k,u_k)}(s)]^*\| = \|Y_{(x_k,u_k)}(s)\| \quad \text{for all} \quad s \geq 0. \qquad (6.88)$$

Therefore,

$$\|\psi_k(0)\| \leq \int_0^{T_k} e^{-\rho s}\|Y_{(x_k,u_k)}(s)\| \left\|\frac{\partial g(x_k(s), u_k(s))}{\partial x}\right\| ds$$

and due to assumptions (A6) and (A7) (see (6.84))

$$\|\psi_k(0)\| \leq \int_0^{T_k} (C_8 e^{-(\rho-\lambda)s} + C_9 e^{-(\rho-(r+1)\lambda)s} ds$$

where $C_8 \geq 0$ and $C_9 \geq 0$ do not depend on k. Now assumption $\rho > (r+1)\lambda$ implies that the sequence $\{\psi_k(0)\}$ is bounded.

Therefore, the sequence $\{(u_k, x_k, \psi_k)\}$ satisfies all the assumptions of Corollary 3.1. By Corollary 3.1 there exists a subseguence of $\{(u_k, x_k, \psi_k)\}$, further denoted again as $\{(u_k, x_k, \psi_k)\}$, such that for every $T > 0$ one has convergences (3.42) and (3.43) for the admissible pairs (u_k, x_k) and convergence (3.45) for the adjoint variables ψ_k where the limit element ψ is an adjoint variable associated with (u_*, x_*) in problem (P); (u_*, x_*)

satisfies the normal-form core Pontryagin maximum principle in problem (P) together with ψ; and, finally, (u_*, x_*) and ψ satisfy the normal-form stationarity condition (3.52). Thus, for (u_*, x_*) and ψ statements (i) and (ii) are proved.

Consider the integral $I_*(t)$ (6.86) for an arbitrary $t \geq 0$. Convergences (3.42) and (3.43) imply

$$Z_k(s) \to Z_*(s) \quad \text{for all} \quad s \geq 0. \tag{6.89}$$

Hence,

$$\begin{aligned} I_*(t) &= \lim_{T\to\infty} \int_t^T e^{-\rho s}[Z_*(s)]^{-1} \frac{\partial g(x_*(s), u_*(s))}{\partial x} ds \\ &= \lim_{T\to\infty} \lim_{k\to\infty} \int_t^T e^{-\rho s}[Z_k(s)]^{-1} \frac{\partial g(x_k(s), u_k(s))}{\partial x} ds. \end{aligned}$$

Furthermore, from (6.88) it follows that for all $s \geq 0$

$$e^{-\rho t} \|[Z_k(s)]^{-1}\| \| \frac{\partial g(x_k(s), u_k(s))}{\partial x} \| \leq C_{10} e^{-(\rho-\lambda)s} + C_{11} e^{-(\rho-(r+1)\lambda)s}$$

with some positive C_{10} and C_{11}. Therefore, $I_*(t)$ converges absolutely. Let us prove (6.87). Integrate the adjoint equation for ψ_k (see (3.50)) over $[t, T_k]$ assuming that k is large enough (i.e., $T_k \geq t$) and taking into account the transversality condition (3.36). We get

$$\psi_k(t) = Z_k(t) \int_t^{T_k} e^{-\rho s} Z_k^{-1}(s) \frac{\partial g(x_k(s), u_k(s))}{\partial x} ds. \tag{6.90}$$

Convergences (3.42) and (3.43) (holding for every $T > 0$) imply that $x_k(s) \to x_*(s)$ for all $s \geq 0$ and $u_k(s) \to u_*(s)$ for a.a. $s \geq 0$. The latter convergences, convergences (6.89) and (3.45) and the absolute convergence of the integral $I_*(t)$ yield that the desired equality (6.87) is the limit of (6.90) with $k \to \infty$. Statement (iii) is proved. The proof is completed.

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