OPTIMAL ALLOCATION OF ARTIFICIAL IN-STREAM AERATION[†]

S. RINALDI[††] and R. SONCINI-SESSA[††]

APRIL 1977

[†]This work has been supported by the Centro Teoria dei Sistemi, C.N.R., Via Ponzio 34/5, Milano, Italy and by the International Institute for Applied Systems Analysis (IIASA), Laxenburg, Austria.

[††]Centro Teoria dei Sistemi, C.N.R., Milano, Italy.

International Institute for Applied Systems Analysis
2361 Laxenburg, Austria

PREFACE

This report is one of a series describing IIASA research into approaches for comparing alternative models that could be applied to the establishment of control policies to meet water quality standards. In addition to model evaluation, this project has focused on problems of optimization and conflict resolution in large river basins.

ABSTRACT

This paper presents some simple properties of the problem of optimal allocation and design of a system of mechanical surface aerators. These properties are proved to be valid for an extremely wide class of river quality models and it is shown how they can be usefully employed to simplify the problem and to improve the efficiency of some dynamic programming algorithms. Finally a method is suggested for dealing with the allocation problem in a river basin composed of a main stream and its tributaries.

Optimal Allocation of Artificial In-Stream Aeration

INTRODUCTION

In the development of water quality management programs one usually assumes that the biological oxygen demand (BOD) load coming from recorded effluents constitutes the major fraction of the total BOD load entering the river system. Thus it would seem to be sufficient to select the right degree of purification of the recorded effluents in order to achieve any desired level of water quality in the river. Unfortunately, it has been remarked (Whipple et al., 1970) that even in well administered areas the recorded effluents represent hardly more than one half of the total load. When this is the case, improvement of waste water treatment efficiency may be insufficient. Moreover, it may happen that high treatment levels are required to prevent the occurrence of too low dissolved oxygen (DO) levels only during short periods of adverse waste water assimilation characteristics, while less costly treatment plants are sufficient to obtain the desired water quality for the rest of the year. In either case, artificial in-stream aeration turns out to be more effective than advanced waste water treatment (Ortolano, 1972; Whipple and Yu, 1971). It would be a relatively complex optimization problem to allocate the available budget between these two alternatives. However, we discuss only the simpler case of mechanical aerators. Nevertheless, the same kind of algorithm can probably be used for solving the more complex problem.

This paper presents some simple properties of the problem of optimal allocation and design of a system of mechanical surface aerators able to attain a given DO standard during periods of low flow and high temperature. These properties are proved to be valid for an extremely wide class of river quality models including all currently used ones. It is shown how these properties can be usefully employed to simplify the allocation problem by breaking it down into a set of simpler problems, and to improve the efficiency of an already proposed solution algorithm based on dynamic programming (Chang and Yeh, 1973; Fioramonti et al., 1973; Koivo and Phillips, 1975). Finally a method is suggested for dealing with the allocation problem for a whole basin composed of a main stream and its tributaries.

The procedure presented in this paper has been successfully applied by the authors to solve the allocation and design problem of aeration devices in a specific portion of the Rhine river basin in West Germany.

STATEMENT OF THE PROBLEM

The optimal control problem consists of determining the number of units to be used N, their location ℓ_i $(i = 1,\dots,N)$, and their power Π_i, which is a function of the DO increment u_i, in such a way that the DO standard is not violated at any point in a given river stretch L $(L = \{\ell: 0 \leq \ell \leq L\})$ and that the total aeration cost is minimized. Once the steady state design conditions (flow rate $q(\ell)$, temperature T, BOD load, etc.) have been fixed, a river quality model can be selected to describe the system. Such a model is generally of the form

$$\frac{dz(\ell)}{d\ell} = f(z(\ell),v(\ell)) \tag{1a}$$

$$\frac{dc(\ell)}{d\ell} = k(\ell)(c_s - c(\ell)) + g(z(\ell),w(\ell)) + u(\ell) \tag{1b}$$

where c is the mean cross-sectional DO concentration, c_s is the DO saturation value at temperature T, and $z(\ell)$ is a suitable nth order vector describing the various stages in the degradation of the organic pollutants. For example, at one extreme (the Streeter-Phelps model) $z(\ell)$ is simply the BOD, while at the other (the complex ecological model) $z(\ell)$ is the concentration of different types of pollutant and the biomasses of various stages of the food chain. The vectors $v(\ell)$ and $w(\ell)$ take into account all the sources and sinks of the components of z and of the dissolved oxygen respectively, while

$$u(\ell) = \sum_{i=1}^{N} u_i \delta(\ell - \ell_i)$$

is the artificial in-stream aeration. (δ is the impulse function and u_i is the difference between the DO concentration downstream and upstream of the point ℓ_i induced by the ith aerator.) Finally, f, g, k, v and w are continuous functions. In Equations (1a) and (1b) it is assumed that the effects of in-stream aeration can be described as a set of point sources of DO and that the presence of the aerator does not influence either the natural aeration process or the self-purification one.

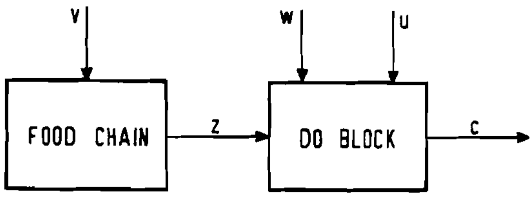

Figure 1. Block diagram of model.

The block diagram of the model is shown in Figure 1. Note that there is no feedback from the second block to the first. Thus the function $z(\ell)$ is completely independent of the input u and can be computed once and for all, so that the model employed in solving the optimization problem is always one-dimensional (Equation 1b), even if extremely sophisticated, multi-dimensional ecological models are selected to describe the river.

The objective function to be minimized is the sum of the costs of all the aerators. The cost C_i of any unit is in general proportional to the power π_i of the unit and is higher for higher values of the induced DO increment u_i, but it is also an increasing function of the DO concentration c_i present upstream of the aerator turbulent area ($c_i = c_i(\ell_i^-)$), and it depends linearly upon the flow rate $q_i = q(\ell_i)$. (In the following, for the sake of simplicity of notation ℓ_i will often be used instead of ℓ_i^-; the right meaning should always be clear from the context.) Moreover, the cost $C_i = C(u_i, c_i, q_i)$ of the ith unit reflects the practical impossibility of attaining supersaturation so that C_i goes to infinity when the induced increment u_i approaches the deficit $(c_s - c_i)$.

Thus the problem of determining the best aeration system is as follows:

Select N, $\{u_i\}_{i=1}^{N}$, *and* $\{\ell_i\}_{i=1}^{N}$ *so that*

$$J = \sum_{i=1}^{N} C(u_i, c_i, q_i) = \min \tag{2a}$$

and

$$c(\ell) \geq \underline{c} \qquad \forall \ell \in L \tag{2b}$$

where $c(\ell)$ *is the solution of Equation (1b) with given initial condition* $c(0) = c_{in} \geq \underline{c}$.

In some cases the problem may be more complex, since additional constraints (e.g. an upper limit on the number of units to be used--see Fioramonti et al., 1973) or a stream standard varying over space might be imposed, but the algorithm presented in this paper can be modified to account for extra constraints.

The optimization problem described by Equation (2) (from now on called Problem 2) is not in standard form, since it is not characterized by a finite number of constraints (see Equation (2b)). In order to transform the problem into a standard mathematical programming model one can simply discretize constraint (2b) over space, as proposed by many authors (Chang and Yeh, 1973; Koivo and Phillips, 1975; Liebman and Lynn, 1966; Revelle et al., 1968).

Moreover it will be shown that the problem can be simplified if the cost function C of an aerator does not exhibit economies of scale. The aeration cost C is said to exhibit economies of scale if the cost of one aerator, which improves the DO load from c to $c + u$, is lower than the sum of the costs of two aerators in series producing the same effect (i.e. two aerators of which the first improves the DO level from c to $c + u'$ and the second from $c + u'$ to $c + u$.

For example Susag et al. (1966) give for the cost of a mechanical aerator:

$$C_i = C(u_i,c_i,q_i) = \rho q_i \ell n\left[\frac{c_s - c_i}{c_s - (c_i + u_i)}\right]$$

where ρ is a suitable constant. This function does not exhibit economies of scale, since

$$C(u,c,q) = C(u',c,q) + C(u-u',c+u',q) \quad .$$

SOME PROPERTIES OF THE OPTIMAL SOLUTION

Some properties of the optimal solution of Problem 2 will now be given for the particular case in which the flow rate $q(\ell)$ is constant along the river. The problem will be simplified and, in the absence of economies of scale, broken down into a set of subproblems.

For all the points ℓ between two aerators, Equation (1b) is a linear differential equation with $u(\ell) = 0$. If $c(\cdot)$ and $c'(\cdot)$ are two solutions with initial value c_0 and $c_0 + \Delta c_0$ ($\Delta c_0 > 0$), one obtains

$$\frac{d}{d\ell}(c'(\ell) - c(\ell)) = - k(\ell)(c'(\ell) - c(\ell)) < 0 \quad ,$$

which means that $c'(\ell) - c(\ell)$ is a decreasing function of ℓ. On this basis it is possible to demonstrate the following property:

Property A
The optimal solution must have all the aerator devices located at points ℓ_i *where* $c_i = \underline{c}$.

In fact if one aerator is not in this position, it is possible to lower its cost, without violating the standard, by shifting it downstream as shown in Figure 2. (Recall that the cost function increases with u_i and c_i.) This property can easily be understood by remembering that the natural re-aeration process is more efficient for lower oxygen levels. Therefore the least costly solution will be one where DO reaches its lower possible value $\underline{c}$.

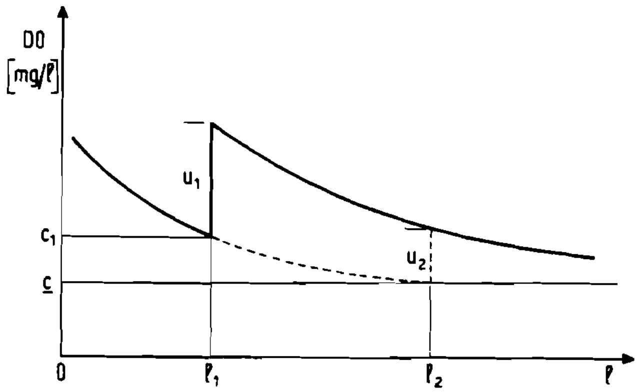

Figure 2. If an aerator is placed in a point where $c > \underline{c}$ (position ℓ_1), its cost can be reduced by shifting it downstream to the first place where $c = \underline{c}$ (position ℓ_2), since $u_2 < u_1$ and $\underline{c} < c_1$.

As a consequence, the optimal solution will be characterized by the absence of aerators upstream of the point $\underline{\ell}_1$ where the natural oxygen profile (i.e. the solution c^1 of Equation (1b), with $u = 0$ and $c^1(0) = c_{in}$) reaches the standard for the first time.

Determine now, if it exists, the point $\bar{\ell}_1$, such that

$$\bar{\ell}_1 = \min \ell \tag{3a}$$

subject to

$$g(z(\ell),w(\ell)) = -k(\ell)(c_s = \underline{c}) \tag{3b}$$

$$\frac{d}{d\ell}\, g(z(\ell),w(\ell)) > -(c_s - \underline{c})\, \frac{d}{d\ell}\, k(\ell) \tag{3c}$$

$$\ell \in [\underline{\ell}_1, L] \quad . \tag{3d}$$

Note that $\bar{\ell}_1$ is the first point downstream of $\underline{\ell}_1$ where, for a particular value of $c(\underline{\ell}_1)$, the minimum of the DO sag curve is the tangent to the standard $\underline{c}$. Then, let c^2 be the solution of Equation (1b) for $\ell \geq \bar{\ell}_1$, with $u = 0$ and initial condition $c^2(\bar{\ell}_1)$ $= \underline{c}$. Determine, if it exists, the point $\underline{\ell}_2$, such that

$$\bar{\ell}_2 = \min \ell$$

subject to

$$c^2(\ell) = \underline{c} \qquad\qquad \ell \in [\bar{\ell}_1, L] \quad ,$$

(i.e. determine the first point downstream of $\bar{\ell}_1$ where $c^2(\ell)$ is equal to the standard). Finally determine the point $\bar{\ell}_2$, such that

$$\bar{\ell}_2 = \min \ell$$

subject to constraint (3b), (3c), and

$$\ell \in [\underline{\ell}_2, L]$$

and continue in the same way until all the stretch L is worked out and a finite number (p) of segments

$$[\underline{\ell}_1, \bar{\ell}_1], [\underline{\ell}_2, \bar{\ell}_2], \ldots, [\underline{\ell}_p, \bar{\ell}_p]$$

are obtained (see Figure 3 where p = 2). Then the following property holds:

Property B
The optimal solution of Problem 2 is characterized by the presence of aerators only in the p *segments* $[\underline{\ell}_k, \overline{\ell}_k]$ k = 1,...,p (notice that the segments are open on the right).

The proof of this property is very simple. Let c* be the optimal oxygen profile. Obviously $c^*(\overline{\ell}_{k-1}) \geq \underline{c}$ and this implies

$$c^*(\ell) \geq c^k(\ell) \qquad \forall \ell \in (\overline{\ell}_{k-1}, \underline{\ell}_k) \qquad k = 1,\ldots,p$$

since $c^k(\overline{\ell}_{k-1}) = \underline{c}$. But

$$c^k(\overline{\ell}) > \underline{c} \qquad \forall \ell \in (\overline{\ell}_{k-1}, \underline{\ell}_k) \qquad k = 1,\ldots,p$$

so that we can conclude that

$$c^*(\ell) > \underline{c} \qquad \forall \ell \in (\overline{\ell}_{k-1}, \underline{\ell}_k) \qquad k = 1,\ldots,p \quad .$$

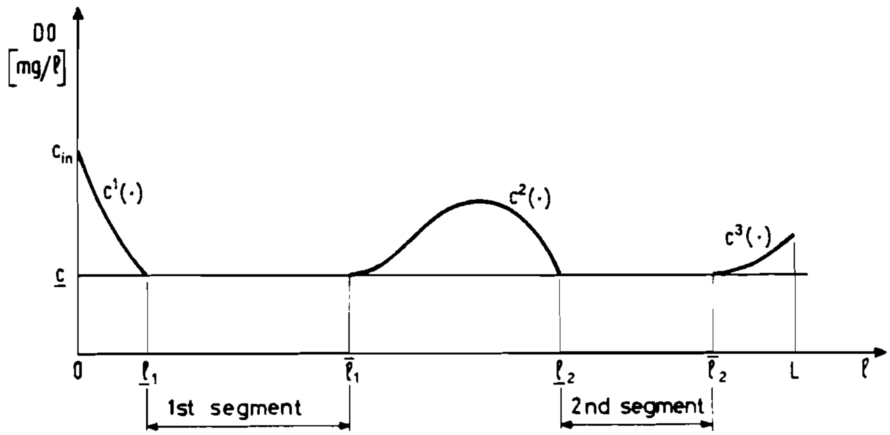

Figure 3. Determination of the segments $[\underline{\ell}_k, \overline{\ell}_k]$.

From property A it follows that no aerators will be present in the interior points of the segment $(\bar{\ell}_{k-1}, \underline{\ell}_k)$. Moreover, no aerator will be placed at the point $\bar{\ell}_{k-1}$, since the same effect is obtained at a lower cost without violating the standard by shifting the aerator to the point $\underline{\ell}_k$ (see Figure 4 where u' > u").

Finally the following property is worthy of mention since it allows a nice breakdown of the problem (see next section).

Property C

If there are no economies of scale Problem 2 can be broken down into the following p *independent subproblems* (k=1,...,p).

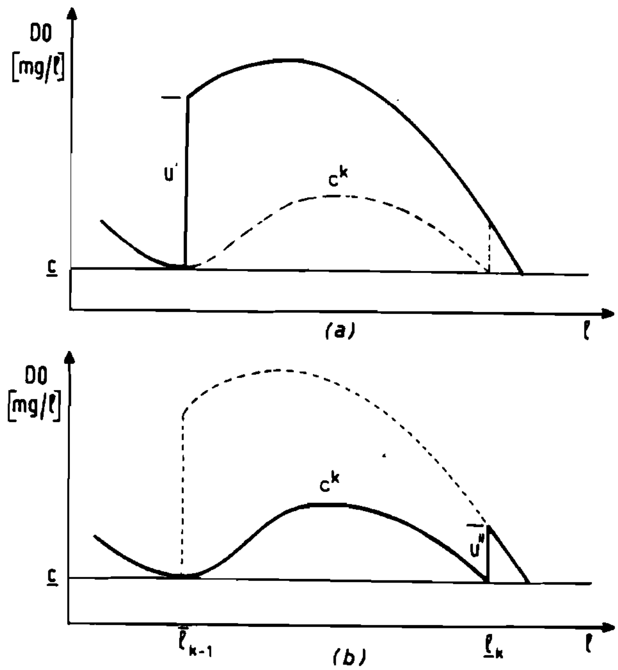

Figure 4. Comparison of two different solutions: the oxygen profile of Figure 4b is obtained at a lower cost than the one of Figure 4a, since u" < u'.

Select N_k , $\left\{u_i\right\}_{i=1}^{N_k}$, $\left\{\ell_i\right\}_{i=1}^{N_k}$ *so that*

$$J_k = \sum_{i=1}^{N_k} C(u_i, c_i, q) = \min \tag{4a}$$

and

$$c(\ell) > \underline{c} \qquad \forall\, \ell \in [\underline{\ell}_k, \overline{\ell}_k] \tag{4b}$$

where $c(\ell)$ *is the solution of Equation (1b) with the initial condition* $c(\underline{\ell}_k) = \underline{c}$.

To prove this property, it is sufficient to show that, in the absence of economies of scale, $c^*(\underline{\ell}_k) = \underline{c}$, since then the result follows immediately from property B. Thus, consider the optimal solution c^* and, in particular, the first aerator placed upstream of the point $\overline{\ell}_{k-1}$, and suppose, absurdly, that its DO increment $(u' + u'')$ is such that (see Figure 5a)

$$c^*(\underline{\ell}_k) > \underline{c} \quad . \tag{5}$$

Since there are no economies of scale, the cost of the aerator that improves the DO level from $\underline{c}$ to $\underline{c} + u' + u''$ is equal to the cost of two aerators in series improving the DO level from $\underline{c}$ to $\underline{c} + u'$ and from $\underline{c} + u'$ to $\underline{c} + u' + u''$. (Note that u' is such that $c(\overline{\ell}_{k-1}) = \underline{c}$.) Since $u'' > u'''$ (see Figure 5a), the cost of in-stream aeration can be reduced, without violating the standard, by shifting the second aerator to the point $\underline{\ell}_k$ and this contradicts assumption (5).

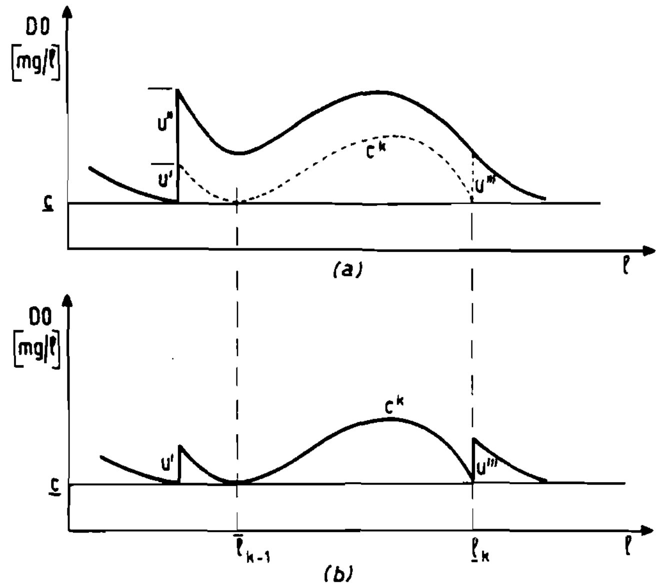

Figure 5. Comparison of two different solutions: the oxygen profile of Figure 5b is less costly than the one of Figure 5a, if there are no economies of scale, since $u''' < u''$.

THE ALGORITHM

In order to apply dynamic programming to each of the subproblems derived from Problem 2, it is necessary (Chang and Yeh, 1973; Koivo and Phillips, 1975; Ortolano, 1972) to discretize constraint (4b) and restrict the decision process to only that finite number of positions where an aerator can be placed. For this purpose, it is worth noting that an aerator working downstream of another one is remarkably less efficient if the two aerators are too close (Price et al., 1973). Hence, the distance between two aerators must be greater than or equal to a critical distance ℓ_c, such that the interval $[\underline{\ell}_k, \overline{\ell}_k]$ can be subdivided into $M = (\overline{\ell}_k - \underline{\ell}_k)/\ell_c$ subintervals. Each subinterval is delimited by an initial and final point, all of which are ordered from 0 to M and indexed by j. Then it is assumed that an aerator can be placed at any one of these points, and the compliance of the DO level with the standard is checked only at these points.

Thus the continuous model (1) can be replaced by a discrete model. If c_j is the DO concentration downstream of the jth interval and U_j is the difference between the DO levels upstream and downstream of the jth point (recall that the DO concentration is discontinuous at the points where there is an aerator), then such a model gives the following difference equation:

$$c_{j+1} = \phi_j(c_j + U_j) + \theta_j \, , \tag{6}$$

where ϕ_j and θ_j are coefficients derived from the solution of Equation (1b). For the sake of simplicity in notation Equation (6) can be written in the form

$$c_{j+1} = \psi(j, c_j + U_j) \quad ,$$

where $U_j = 0$ means that no aerator is located at point j, while the contrary is true when $U_j > 0$. Thus a cost given by

$$H_j(U_j) = \begin{cases} 0 & \text{if } U_j = 0 \\ C(U_j, c_j, q) & \text{if } U_j > 0 \end{cases}$$

can be associated to each point j and $H_M = 0$ since $U_m = 0$ $(c(\overline{\ell}_p) = \underline{c})$.

If $U = \left[U_0 \ldots U_{M-1}\right]$ is the new decision vector, Problem 4 becomes:

Select U *so that*

$$H(U) = \sum_{j=0}^{M-1} H_j(U_j) = \min \tag{7a}$$

and

$$c_{j+1} = \psi(j, c_j + U_j) \qquad j = 0,\ldots,M-1 \tag{7b}$$

$$c_j \geq \underline{c} \qquad j = 1,\ldots,M \tag{7c}$$

$$c_0 = \underline{c} \ . \tag{7d}$$

Problem 7 is a multistage decision problem which can easily be solved by dynamic programming. The computational effort necessary to solve the problem can be greatly reduced if one takes advantage of the fact that the aerators will be located at points ℓ_j such that $c(\ell_j) = \underline{c}$. If $H_h(c_h)$ is the minimum aeration cost downstream of the hth point when c_h is the DO level at the end of the hth interval, i.e.

$$H_h(c_h) = \min_{\{U_j\}_{j=h}^{M-1}} \left[\sum_{j=h}^{M-1} H_j(U_j) \right]$$

subject to

$$c_{j+1} = \psi(j, c_j + U_j) \qquad j = h,\ldots,M-1$$

$$c_j \geq \underline{c} \qquad j = h,\ldots,M \ ,$$

and assume

$$H_h(c_h) = \infty$$

for $c_h > \underline{c}$, then the possibility of allocating an aerator at the hth point must be considered only if $c_h = \underline{c}$ (see Property A). Hence, the dynamic programming functional equation is

$$H_h(c_h) = \begin{cases} H_{h+1}(\psi(h,c_h)) & \text{if } c_h > \underline{c} \\ \min\limits_{U_h} \left[\mathcal{H}(U_h) + H_{h+1}(\psi(h,c_h + U_h)) \right] & \text{if } c_h = \underline{c} \end{cases} \quad . \tag{8}$$

Equation (8) can be solved recursively for $h = M-1, M-2, \ldots, 0$ if the boundary condition $H_M(c_M) = 0$ for $c_M \geq \underline{c}$ is taken into account.

Since

$$\min \left[\mathcal{H}(U) \right] = H_0(\underline{c})$$

Problem 7 is solved when $H_0(\underline{c})$ is computed by means of Equation (8). Finally the number N of aerators to be actually installed, their locations $\left\{ \ell_i \right\}_{i=1}^{N}$, and their induced DO increments $\left\{ u_i \right\}_{i=1}^{N}$, and consequently their powers π_i, are computed by backtracking, as is usual with dynamic programming.

EXTENSION TO THE CASE OF A RIVER BASIN

The general case of a river basin, with the assumption of piecewise constant flow rate, is now considered. Obviously, in all the reaches where the flow rate is constant the anlysis developed in the previous section can be applied, so that all the intervals $[\underline{\ell}_k, \overline{\ell}_k]$ can be found. The new aspect in this case is the existence of confluence points where two streams come together and form a larger one. Since the aeration cost is proportional to the flow rate, there may be an economic advantage in installing an aerator in one of the two upstream branches (just before the confluence point) even if the corresponding DO level there is greater than the standard. These confluence points must also be considered as possible points of artificial in-stream aeration. As a consequence, the functional equation (8) must be suitably modified at such points to account for the possibility of locating an aerator in correspondence with any DO level.

If the cost function $C(u,c,q)$ is convex with respect to u, the optimal solution is characterized by at most one aerator, located just upstream of the confluence point on the branch having the lower oxygen content. In fact it can be proved that the same effect as is produced by an aerator located on the more oxygenated branch, can be obtained at a lower cost by shifting it downstream of the confluence point.

To clarify the decomposition procedure, consider the example shown in Figure 6 where four reaches are determined by the points where the flow rate has stepwise changes. Assume that in reaches 1 and 2 the DO standard is first violated at points $\underline{\ell}_1^{(1)}$ and $\underline{\ell}_1^{(2)}$ respectively, and that the segments to be considered as possible points of artificial in-stream aeration end at points $\bar{\ell}_1^{(1)}$ and $\bar{\ell}_1^{(2)}$. Moreover assume that the aeration cost does not exhibit economies of scale. Then, from Property C, the optimal solution turns out to be characterized by $c^*(\bar{\ell}_1^{(1)}) = c^*(\bar{\ell}_1^{(2)}) = \underline{c}$, so that the optimal DO levels at points L_1 and L_2 upstream of the confluence point can be evaluated a priori.

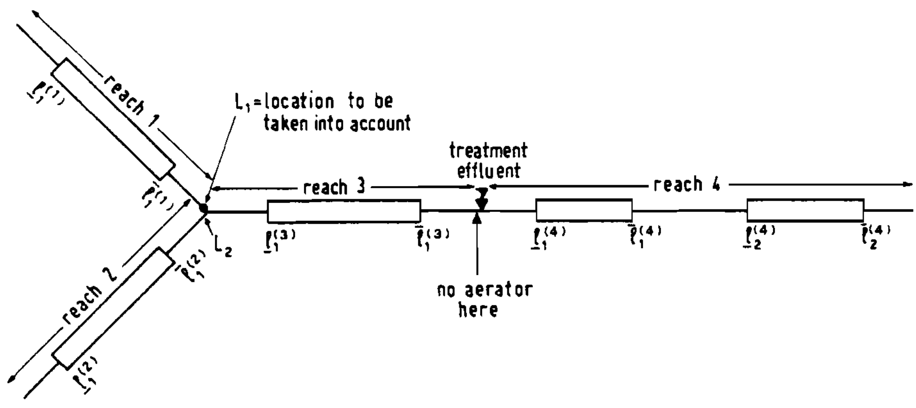

Figure 6. The optimal allocation problem for a river basin.

Now assume that $c^*(L_1) < c^*(L_2)$. Then the property described above allows us to state that only L_1 has to be considered as a possible aeration point. The DO concentration at the upstream end of reach 3 is dependent upon the presence of an aeration device at point L_1. Thus, in order to determine the possible points for artificial aeration in reach 3, it is necessary to consider the case corresponding to the minimum value of $\underline{\ell}_1^{(3)}$, i.e. to assume that no aerator is present at point L_1. Finally, the position upstream of the treatment plant effluent must not be taken into account if one is sure that the DO level in the river is reduced by the deficit of the effluent.

Once the segments of the river that are to be considered as possible points for artificial instream aeration have been determined, it is possible to decide on the optimal aeration system by solving five independent subproblems. (Point L_1 must be considered with segment $[\underline{\ell}_1^{(3)}, \bar{\ell}_1^{(3)}]$.) In view of this breakdown, and because of the absence of economies of scale, it is possible to apply the simple algorithm presented in the foregoing section, while in the presence of economies of scale the problem can be solved only by applying more complex techniques (for instance nonserial dynamic programming (Bertelè and Brioschi, 1971)).

CONCLUDING REMARKS

In this paper the problem of optimal in-stream aeration, extensively dealt with in the literature (see, for other references, Bertelè and Brioschi, 1971; Fioramonti et al., 1973), has been formulated as an optimal control problem for a system described by differential equations. The following two properties of the optimal solution have been proved for a constant flow rate:

1. All the aerators must be located at points where the dissolved oxygen level is equal to the standard.
2. It is not necessary to take the whole river into account but only suitable segments of it, which can be easily predetermined.

On this basis the original problem has been reduced to a simpler set of subproblems under the assumption that economies of scale can be neglected. Such problems are solved by a simple recursive scheme derived through dynamic programming. The more general case of a main stream and its tributaries has been considered and the differences with the former case briefly pointed out.

Notation

The following symbols are used in this paper:

c	dissolved oxygen (DO) concentration [mg/ℓ]
$c(\ell)$	dissolved oxygen (DO) concentration at point ℓ [mg/ℓ]
$\underline{c}$	DO stream standard [mg/ℓ]
c_i	DO concentration just upstream of the point ℓ_i [mg/ℓ]
c_{in}	DO concentration at the initial point of the river stretch [mg/ℓ]
c_s	DO saturation level [mg/ℓ]
C_i	cost of the ith aerator [any monetary unit]
k	re-aeration coefficient [km^{-1}]
ℓ	ordinate along the river stretch [km]
ℓ_c	minimum allowed distance between two aerators [km]
ℓ_i	location of the ith aerator in the optimal solution [km] $(i=1,\ldots,N)$
ℓ_j	possible location for an aerator $(j=0,\ldots,M)$ [km]
$\underline{\ell}_k$	points where $c(\ell) = \underline{c}$ [km]
$\overline{\ell}_k$	points in which the DO minimum is a tangent to the stream standard $\underline{c}$ [km]
L	length of the river stretch [km]
$\mathcal{L}$	$\{\ell: 0 \leq \ell \leq L\}$ the river stretch
M	number of subintervals of the interval $[\overline{\ell}_k, \underline{\ell}_k]$
N	optimal number of aeration units
N_k	optimal number of aeration units in the kth subproblem
π_i	power of an aeration device able to generate the DO increment u_i [W]
$q(\ell)$	flow rate at point ℓ [m^3/s]
q_i	flow rate at point ℓ_i [m^3/s]
T	water temperature [^oC]
$u(\ell)$	artificially induced aeration [mg/ℓ]
u_i	DO increment induced by the ith aerator [mg/ℓ]
U_j	DO increment in position ℓ_j [mg/ℓ]
$v(\ell)$	vector function representing the sources and the sinks of the components of z
$w(\ell)$	sources and sinks of dissolved oxygen
$z(\ell)$	nth order vector describing the degradation of organic compounds

REFERENCES

Bertelè, U., and F. Brioschi (1971), *Nonserial Dynamic Programming*, Academic Press, New York and London.

Chang, S., and W.W. Yeh (1973), Optimal Allocation of Artificial Aeration along a Polluted Stream using Dynamic Programming, *Water Res. Bull.*, 9, 5, 985-997.

Fioramonti, W., et al. (1973), Optimal Design of Induced Aeration in Polluted Streams Via Dynamic Programming, *Proc. IFAC-UNESCO Workshop on System Analysis and Modelling Approaches in Environmental Systems*, Zakopane, Poland, 333-344.

Koivo, A.J., and G. Phillips (1975), *Optimization of Dissolved Oxygen Concentration in Polluted Streams Using Instream Aeration*, VIth IFAC Conference, Boston, Mass.

Liebman, J.C., and W.R. Lynn (1966), The Optimal Allocation of Stream Dissolved Oxygen, *Water Resources Res.*, 2, 3, 581-591.

Ortolano, L. (1972), Artifical Aeration as a Substitute for Waste Water Treatment, in R. Dorfman, H.D. Jacoby, and H.A. Thomas, Jr., eds., *Models for Managing Regional Water Quality*, Harvard University Press, Cambridge, Mass.

Price, K.S., et al. (1973), Surface Aerator Interactions, *Journal of the Environmental Division, ASCE*, 99, EE3, 283-299.

Revelle, C.S., et al. (1968), Linear Programming Applied to Water Quality Management, *Water Resources Res.*, 4, 1, 1-9.

Susag, R.H., et al. (1966), Mechanical Surface Aeration of Receiving Waters, *Journal Water Pollution Control Federation*, 38, 1, 53-68.

Whipple, W., Jr., et al. (1970), Instream Aerators for Polluted Rivers, *Journal of Sanitary Engineering Division, ASCE*, 96, SA5, 1153-1165.

Whipple, W., Jr., and S.L. Yu (1971), Alternative Oxygenation Possibilities for Large Polluted Rivers, *Water Resources Res.*, 7, 3, 566-579.