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RESERVOIRS WITH SEASONALLY VARYING MARKOVIAN INFLOWS, AND THEIR FIRST PASSAGE TIMES

E. H. LLOYD

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International Institute for Applied Systems Analysis
2361 Laxenburg, Austria

PREFACE

As new techniques for the use of water evolve and as human needs increase, the management of water resources is becoming a task of growing importance. The primary purpose of surface-water reservoirs is to provide a means of regulating the distribution, with respect to time, of surface-water flows and volumes. Stochastic reservoir theory is concerned with the design and operation of storage reservoirs fed by river flows which are considered as stochastic processes.

From 1974 to 1976, stochastic reservoir theory was one of the major research fields of the IIASA Water Project (now the Water Group of the Resources and Environment Area). A number of research papers were published on different aspects of this problem. The present paper by one of the leading authorities on stochastic reservoir theory deals with problems related to the seasonal variability of reservoir inflows.

SUMMARY

In building a realistic mathematical model of a reservoir, one of the most important components is that which describes the time-dependence of the inflowing water. This must exhibit the seasonal variations, as well as the chance fluctuations and the "persistence" structure, that are observed in natural geophysical phenomena. In the present research we work in terms of discrete time units of arbitrary brevity, conventionally called "seasons", and discrete units of water quantity; and we approximate the seasonally varying stochastic and autocorrelation behaviour of the inflow by representing it as a seasonal lag-one Markov chain, i.e. one in which the probability distribution vector $y_{m,r}$ in the r-th season of year m ($r=0,1,\ldots,k$, say) is related to the corresponding vector in the preceding season by the difference equations

$$y_{m,0} = Q_0 y_{m-1,k}$$

and

$$y_{m,r} = Q_r y_{m,r-1} \quad , \qquad r = 1,2,\ldots,k \quad .$$

where $Q_0, Q_1, \ldots, Q_k$ are the k+1 season-to-season transition probability matrices.

The properties of such chains are described, and examples are provided of explicitly formulated matrix models which enable the Q_r to be expressed directly in terms of flow distribution and autocorrelations. (It should be stressed that the methods employed do *not* involve the concept of autoregression.) The major part of the paper is devoted to the problem of obtaining the probability distributions of recurrence times and first passage times in chains of this kind. By a recurrence time is meant the waiting time between consecutive occurrences of given flow values; and by a first passage time, the waiting time between the occurrence of a given

flow value j and the next following occurrence of another flow value i. We also consider recurrences of *sets* of flow values, and first passage times from a given such set to another. Results are obtained both explicitly and in terms of probability generating functions; thereby generalizing to a seasonal situation a problem whose solution is already known in the non-seasonal case.

The methods used are generalizations of the classical renewal theory, on the one hand, and, on the other, of a technique known in random walk theory as the absorbing state method.

The paper closes with a brief resume of stochastic reservoir theory which indicates the relevance to this of the earlier part of the paper. In particular it is shown that, when the inflow process $\{ X_t \}$ is a seasonal Markov chain of the kind described, then, subject to reasonable restrictions on the management policy, the sequence of pairs $\{ Z_t, X_t \}$, where Z_t denotes the reservoir contents at time t, forms a seasonal lag-one *vector* Markov chain to which the main results of the paper may be applied to obtain recurrence times and first passage times for the reservoir contents.

1. INTRODUCTION

A real reservoir is a finite container fed by inflows which in general form a non-stationary continuous-valued multivariate stochastic process in continuous time, subject to withdrawals which depend on current and past contents and inflows as well as on seasonally fluctuating demands of a partly determinate and partly stochastic nature, these having in general some correlation with current and past inflows. The reservoir also loses water by evaporation and seepage, and may be subject to silting which progressively reduces its capacity.

In its present form, "stochastic reservoir theory", as initiated by Moran and extended by Lloyd, neglects evaporation, seepage and silting, and models the inflow-outflow-storage relationship by a discrete-volume process operation in discrete time. Successive values of the storage variable are related to each other by a difference equation expressing the conservation of volume, and the corresponding distribution vectors are determined in terms of the characteristics of the inflow process and the withdrawal policy, the auto- and cross-correlation structure of the (multivariate) inflow process being represented by a (possibly non-homogeneous) multivariate multilag Markov chain, with the object of developing a model with optimal "power-to-weight ratio", that is, one with the best degree of realism that is consistent with a not unduly complicated theory. It appears that a reasonable degree of realism can in fact be achieved.

The present paper is devoted mainly to the theory and properties of finite univariate lag-1 seasonal Markov chains, with special reference to the distribution of the waiting time, from a given configuration, to the next occurrence of a specified configuration. Thus the emphasis is mainly on the properties of the inflow process; but we indicate in general terms how this theory extends to the storage and outflow processes, a detailed discussion of which will be given in a later publication.

2. SEASONAL MARKOV CHAINS

2.1 Notation for Matrices and Vectors

The matrices that occur in the following account will all be square, and will be denoted by capital letters such as A,B,Q, R. It is convenient to label the rows and columns with the indices $0,1,2,\ldots,k$ rather than with the index set $1,2,\ldots,k+1$ that is usually employed in matrix algebra. Typically we denote the (i,j) element of a matrix A by a_{ij} and write

$$A = (a_{ij}) \quad .$$

It is sometimes more convenient to write $a(i,j)$ instead of a_{ij}.

We also use the convention that, for any matrix A, the symbol $(A)_{ij}$ denotes the (i,j) element of A. Thus $(AB)_{ij}$ denotes the (i,j) element of the matrix product AB. The fully displayed version of a matrix $A = (a_{ij})$ with $k+1$ rows and $k+1$ columns (that is a "$(k+1)\times(k+1)$" matrix) is

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} & \cdots & a_{0k} \\ a_{10} & a_{11} & a_{12} & \cdots & a_{1k} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{k0} & a_{k1} & a_{k2} & \cdots & a_{kk} \end{bmatrix} \quad .$$

A "vector" in our convention means a column vector. This will typically be denoted by a small letter, for example

$$x = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_p \end{bmatrix}$$

where the elements x_r may for typographic convenience be alternatively denoted by $x(r)$. Transposition is indicated by a prime, so that the row vector obtained by transposing the vector x described above is

$$x' = (x_0, x_1, \ldots, x_p) \quad ;$$

equivalently,

$$x = (x_0, x_1, \ldots, x_p)' \quad .$$

Thus the first column of the matrix A displayed above would be written in the form

$$(a_{00}, a_{10}, \ldots, a_{k0})' \quad .$$

To avoid the awkwardness of having to describe the row vector

$$(a_{i0}, a_{i1}, \ldots, a_{ik})$$

as the (i+1)-th row, we refer to it simply as the i-row; and similarly for columns.

A particular vector which we use frequently is

$$1 = (1, 1, \ldots, 1)' \quad .$$

Note that

$$1' x = \sum x(r)$$

etc.

2.2 Nonhomogeneous Markov Chains

Let each of the random variables $Y_0, Y_1, Y_2, \ldots$ of the sequence $\{Y_t\}$ be capable of assuming the values $0, 1, \ldots, p$. When $Y_\tau = r$ we say that the system is in state $\mathcal{E}_r$ at epoch τ. Suppose that

$$P(Y_{t+1}=r \mid Y_t=s, Y_{t-1}=s', Y_{t-2}=s'', \ldots)$$

$$=P(Y_{t+1}=r \mid Y_t=s) \quad , \ t=0,1,\ldots,$$

$$r,s,s',s'',\ldots = 0,1,\ldots,p \quad .$$

This makes the sequence $\{Y_t\}$ a lag-one ("lag-1") Markov chain. Define the matrices

$$Q_t = (q_t(r,s)) \quad , \ t = 0,1,\ldots; \ r,s = 0,1,\ldots,p \quad ,$$

of order $(p+1)\times(p+1)$. The time-dependence of these transition matrices makes the chain non-homogeneous. (If the Q_t were all equal, with common value Q, say, the chain would be called homogeneous.)

Little of interest can be said about a completely general nonhomogeneous chain, i.e. one in which the matrices Q_t are unrelated to one another. In geophysical application we encounter a special kind of nonhomogeneity called seasonality which we investigate in the sequel.

It will be convenient to precede that discussion by a brief resume of some properties of finite homogeneous Markov chains.

2.3 First Passages and Recurrence Theory for Homogeneous Markov Chains

Let the sequence of random variables $\{Y_t\}$, defined on $(0,1,\ldots,p)$, be a homogeneous lag-1 Markov chain with transition matrix $Q = (q_{ij})$ where

$$q_{ij} = P(Y_{t+1}=i \mid Y_t=j) \quad , \ t=0,1,\ldots, \ i,j=0,1,\ldots,p \quad .$$

Let $f^{(r)}(i,j)$, $r=1,2,\ldots$ denote the probability that a first passage to $\mathcal{E}_i$ from $\mathcal{E}_j$ requires r jumps, that is

$$f^{(r)}(i,j)=P(Y_{t+r}=i; \ Y_{t+s}\neq i, \ s=1,2,\ldots,r-1 \mid Y_t=j) \quad .$$

This situation will have come about if

either the first arrival at $\mathcal{E}_i$ requires r jumps (the probability of which is $f^{(r)}(i,j)$)

or the first arrival at $\mathcal{E}_i$ requires s jumps (for s=1 or 2 or ⋯ or r-1), and the system then returns to $\mathcal{E}_i$ (not necessarily for the first time) after a further (r-s) jumps (the probability of which is $f^{(s)}(i,j)\ q^{(r-s)}(i,i)$).

We thus derive the renewal equation:

$$q^{(r)}(i,j) = f^{(r)}(i,j) + \sum_{s=1}^{r-1} f^{(s)}(i,j)\ q^{(r,s)}(i,i) \quad ,$$

or, more compactly,

$$q^{(r)}(i,j) = \sum_{s=1}^{r} f^{(s)}(i,j)\ q^{(r,s)}(i,i) \quad , \ r=1,2,... \qquad (2.1)$$

where we define

$$q^{(0)}(i,j) = 1 \quad , \ i,j=0,1,...,p \quad ,$$

and

$$q^{(r)}(i,j) = P(Y_{t+r}=i|Y_t=j) \quad , \ r=1,2,... \quad .$$

The latter quantity is the probability that the system will be in state $\mathcal{E}_i$, not necessarily for the first time, after making r jumps, starting from $\mathcal{E}_j$.

If we take r=1,2,...the successive versions of (2.1) provide a recursive solution for $f^{(r)}$ in terms of $f^{(r-1)}$, $f^{(r-2)}$,...,$f^{(1)}$ and the $q^{(s)}$, so that one may obtain successively

$f^{(1)}$ in terms of $q^{(1)}$
$f^{(2)}$ in terms of $f^{(1)}$, and $q^{(2)}$
$f^{(3)}$ ...

More elegantly we may derive from (2.1) a relation between the generating functions

$$F(i,j;\Theta) = \sum_{s=1}^{\infty} f^{(s)}(i,j)\Theta^{s}$$

and

$$K(i,j;\Theta) = \sum_{r=0}^{\infty} q^{(r)}(i,j)\Theta^{r} = 1 + \sum_{r=1}^{\infty} q^{(r)}(i,j)\Theta^{r} \quad .$$

We have

$$K(i,j;\Theta)-1 = \sum_{r=1}^{\infty} \sum_{s=1}^{r} f^{(s)}(i,j)\; q^{(r-s)}(i,i)\Theta^{r}$$

$$= \sum_{s=1}^{\infty} f^{(s)}(i,j)\Theta^{s} \sum_{r=s}^{\infty} q^{(r-s)}(i,i)\Theta^{r-s}$$

$$= F(i,j;\Theta)\; K(i,i;\Theta)$$

whence

$$F(i,j;\Theta) = \frac{K(i,j;\Theta)-1}{K(i,i;\Theta)} \quad . \tag{2.2}$$

(Equation (2.1) may be compared with the corresponding equation for a simple renewal process in which events of only one kind occur; the equation in that case is

$$q^{(r)} = \sum_{s=1}^{r} f^{(r)}\; q^{(r-s)} \quad , \; r=1,2,\ldots \quad ,$$

in an obvious notation, and equation (2.2) may be compared with the corresponding generating function equation

$$K(\theta) = \frac{K(\theta)-1}{K(\theta)} \quad .)$$

This very nice relationship is not as immediately available for applications as it might appear. Consider for example the question whether the first passage distribution to $\mathcal{E}_i$ from $\mathcal{E}_j$ is a non-defective probability distribution, i.e. whether

$$\sum_{s=1}^{\infty} f^{(s)}(i,j) = 1$$

(whether $F(i,j;1) = 1$). Simply setting $\theta=1$ in (2.2) will not do; it must be remembered that the events

$$Y_{t+r} = i | Y_t = j \quad , \; r=1,2,\ldots \quad ,$$

(which correspond to arrivals at $\mathcal{E}_i$ not necessarily for the first time) are not mutually exclusive, so that the probabilities

$$q^{(r)}(i,j) \quad , \; r=1,2,\ldots$$

do not form a probability distribution, and so

$$\sum_{r=1}^{\infty} q^{(r)}(i,j) \neq 1$$

whence the value of $K(i,j;1)$ is not obvious.

To investigate this further, note that

$$q^{(r)}(i,j) = [Q^r]_{ij}$$

so that

$$K(i,j;\theta) = 1 + [\sum_{r=1}^{\infty} Q^r\theta^r]_{ij}$$

$$= 1 + [Q\theta(I-Q\theta)^{-1}]_{ij} \quad ,$$

provided $|\theta|$ is sufficiently small. When $i=j$, this simplifies to

$$K(i,i;\theta) = [\sum_{r=0}^{\infty} Q^r\theta^r]_{ii} \quad (\text{since } Q^0 = I)$$

$$= [(I-Q\theta)^{-1}]_{ii} \quad .$$

Thus (2.2) becomes

$$F(i,j;\theta) = \frac{[Q(I-Q\theta)^{-1}]_{ij}\theta}{[(I-Q\theta)^{-1}]_{ii}} \quad , \ (i,j=0,1,\ldots,p) \qquad (2.3)$$

To attempt a direct evaluation of $F(i,j;1)$ from this is catastrophic, since, when Q is a stochastic matrix, $I-Q$ is singular, and so $(I-Q)^{-1}$ does not exist.

We can surmount this difficulty however with the aid of the following lemma.

Lemma Suppose Q is the transition matrix of a finite Markov chain, with stationary distribution vector $x = (x_0,x_1,\ldots,x_p)'$, $x > 0$. Then

$$\lim_{\theta\to 1} (1-\theta)(I-Q\theta)^{-1} = x1' \quad . \qquad (2.4)$$

Proof: Q has at least one latent root (eigenvalue) equal to unity, and all other roots are ≤ 1. Without any real loss of generality, we assume that the roots are distinct. Denote them by

$$\lambda_0 = 1,\ \lambda_1,\lambda_2,\ldots,\lambda_p \quad (\lambda_i \neq 1 \text{ when } i\neq 0) \quad .$$

Let the associated normalized latent vectors (eigenvectors) be (u_r,v_r'), $r=0,1,\ldots,p$, so that

$$Qu_r = \lambda_r u_r,\quad v_r'Q = \lambda_r v_r',\quad \text{with } v_r'u_s=\delta_{rs} \quad .$$

Then

$$Q = \sum_{r=0}^{p} \lambda_r u_r v_r'$$

and

$$(I-Q\theta) = \sum_{r=0}^{p} (1-\lambda_r\theta)u_r v_r'$$

whence

$$(I-Q\theta)^{-1} = \sum_{r=0}^{p}\left(\frac{1}{1-\lambda_r\theta}\right) u_r v_r'$$

$$= \frac{1}{1-\theta}\, x1' + \sum_{r=1}^{p}\left(\frac{1}{1-\lambda_r\theta}\right) u_r v_r'$$

since $u_0 = x$ and $v_0' = 1'$. Thus finally,

$$(1-\theta)(I-Q\theta)^{-1} = x1' + \sum_{r=1}^{p}\left(\frac{1-\theta}{1-\lambda_r\theta}\right) u_r v_r' \quad ,$$

whence the result follows.

Corollary (a) Under the same conditions,

$$(1-\theta)^n (I-Q\theta)^{-n} = x1' + \sum_{r=1}^{p}\left(\frac{1-\theta}{1-\lambda_r\theta}\right)^n u_r v_r' \quad , n=1,2,...$$

and (b) for $s = \pm1, \pm2, ...,$

$$Q^s (1-\theta)^n (I-Q\theta)^{-n} = x1' + \sum_{r=1}^{p}\lambda_r^s \left(\frac{1-\theta}{1-\lambda_r\theta}\right)^n u_r v_r' \quad ,$$

(since $Qx = x$ and $Qu_r = \lambda_r u_r$).

If we now write (2.3) in the form

$$F(i,j;\theta) = \frac{(1-\theta)\,[Q(I-Q\theta)^{-1}]_{ij}\,\theta}{(1-\theta)\,[(I-Q\theta)^{-1}]_{ii}}$$

we may employ the lemma to obtain

$$\lim_{\theta\to1} F(i,j;\theta) = (Qx1')_{ij}/(x1')_{ii}$$

$$= (x1')_{ij}/(x1')_{ii} \qquad \text{since } Qx = x$$

$$= x_i/x_i$$

$$= 1 \quad , \qquad\qquad i,j = 0,1,...,p \quad .$$

It follows that $f^{(r)}(i,j)$, $(r=1,2,\ldots)$ is a non-defective probability distribution: the transition $\mathcal{E}_j \longrightarrow \mathcal{E}_i$ is, ultimately bound to occur, i.e., is a recurrent event.

It is also fairly simple to obtain the expected value of $F(i,j,)$ when $i=j$: this is the mean recurrence time $E\{F(i,i)\}$ of the state $\mathcal{E}_i$. We have

$$F(i,i;\theta) = 1 - \frac{1}{K(i,i;\theta)} \qquad \text{by (2.2)}$$

whence, differentiating with respect to θ,

$$F'(i,i;\theta) = \lim_{\theta\to 1}[K'(i,i;\theta)/\{K(i,i;\theta)\}^2] \quad .$$

Now

$$K(i,i;\theta) = [\sum Q^r\theta^r]_{ii} = [(I-Q\theta)^{-1}]_{ii}$$

and so

$$K'(i,i;\theta) = [Q\sum rQ^{r-1}\theta^{r-1}]_{ii} = [Q(I-Q\theta)^{-2}]_{ii} \quad ,$$

whence

$$F'(i,i;1) = \lim \frac{(1-\theta)^2K'(i,i;\theta)}{\{(1-\theta)K(i,i;\theta)\}^2}$$

$$= (x1')_{ii}/\{(x1')_{ii}\}^2 = 1/x_i$$

by the lemma. Thus

$$E\{F(i,i)\} = 1/x_i, \quad i=0,1,\ldots,p,$$

where x_i is the i-element of the stationary distribution vector: the expected interval between successive occurrences of $\mathcal{E}_i$ equals the reciprocal of the probability of being in state $\mathcal{E}_i$ (compare the Poisson process).

The variance and higher moments of F(i,i) are less simple. For example

$$\mathrm{var}\{F(i,i)\} = 2x_i^{-2} \sum_{r=1}^{p} (1-\lambda_r)^{-1} (u_r v_r')_{ii} - x_i^{-1} + x_i^{-2} \quad .$$

In particular cases of course it might be possible to evaluate this explicitly without too much difficulty.

Example Suppose

$$Q = \rho I + (1-\rho)x1'$$

where $0 < \rho < 1$, and where x is a positive distribution vector. It is easily verified that x is in fact the stationary distribution vector of the chain, since $Qx = x$, and it will be shown later that ρ is the lag-1 correlation coefficient of the chain.

Then

$$(I-Q\theta)^{-1} = \{(1-\theta)I + (1-\rho)\theta x1'\}/(1-\theta)(1-\rho\theta)$$

and

$$Q(I-Q\theta)^{-1} = \{\rho(1-\theta)I + (1-\rho)x1'\}/(1-\theta)(1-\rho\theta) \quad ;$$

so that

$$F(i,j;\theta) = \frac{\{\rho(1-\theta)\delta_{ij} + (1-\rho)x_i\}\theta}{(1-\theta) + (1-\rho)\theta x_i} \quad ,$$

whence

$$F(i,j;1) = 1$$

as required.

The mean recurrence time of the state $\mathcal{E}_i$ is

$$F'(i,i;1) = 1/x_i \quad , \; i=0,1,\dots,p,$$

and the mean passage times to $\mathcal{E}_i$ from $\mathcal{E}_j$ $(i\neq j)$ is

$$F'(i,j;1) = 1/(1-\rho)x_i \quad ;$$

a result which, for this chain, depends on i but not on j. The value of $F''(i,i;1)$ is

$$\frac{1}{1-\rho}\sum(u_r v_r')_{ii} = \frac{1}{1-\rho}\{I-x1'\}_{ii} = \frac{1}{1-\rho}(1-x_i)$$

so that

$$\text{var}\quad\{F(i,i)\} = \frac{1}{x_i^2} - \frac{1+\rho}{1-\rho}\,\frac{1}{x_i} \quad .$$

In the case of this particular example it is easy to obtain the passage times distributions explicitly. The generating function for $F(i,j)$ is

$$F(i,j;\theta) = (1-\rho)x_i\theta/\{1-\lambda_i\theta) \quad , \; \lambda_i = 1-(1-\rho)x_i$$

whence

$$P\{F(i,j)=r\} = (1-\rho)x_i\lambda_i^{r-1} \quad , \; r = 1,2,\dots$$

a geometric distribution.

Similarly for the recurrence time distribution of $\mathcal{E}_i$, we have

$$F(i,i;\theta) = \rho(1-\theta)\theta + (1-\rho)x_i\theta$$

$$F(i,i;\theta) = (\beta_i\theta-\rho\theta^2)/(1-\lambda_i\theta) \quad , \; \beta_i = \rho+(1-\rho)x_i = \rho+1-\lambda_i$$

$$= \beta\theta + (1-\lambda)(\lambda-\rho)\{\theta^2+\lambda\theta^3+\dots\} \quad ,$$

a geometric distribution with modified first term:

$$P\{F(i,i)=1\} = \rho+(1-\rho)x_i$$

$$P\{F(i,i)=r\} = (1-\rho)^2 x_i(1-x_i)\{1-(1-\rho)x_i\}^{r-2} \quad , \; r=2,3,\ldots$$

2.4 Seasonal Markov Chains

A seasonal Markov chain, with k+1 seasons, is a nonhomogeneous chain in which the successive transition matrices form a periodic sequence of period k+1.

We call the seasons the 0-season, the 1-season, etc., and denote the corresponding random variables in the m-th year by

$$Y_{m,0},\; Y_{m,1},\ldots,\; Y_{m,k} \quad , \; m = 1,2,\ldots \quad ,$$

respectively. Let $y_{m,r}$ denote the distribution vector of $Y_{m,r}$, $r=0,1,\ldots,k$. Assuming the season-to-season dependence is lag-1 Markovian, the seasonal Structure implies that

$$\left.\begin{aligned}
y_{m,0} &= Q_0 y_{m-1,k} \\
y_{m,1} &= Q_1 y_{m,0} \\
y_{m,2} &= Q_2 y_{m,1} \\
&\;\vdots \\
y_{m,k} &= Q_k y_{m,k-1} \\
y_{m+1,0} &= Q_0 y_{m,k} \\
y_{m+1,1} &= Q_1 y_{m+1,0} \\
&\;\vdots \\
y_{m+1,k} &= Q_k y_{m+1,k-1} \\
y_{m+2,0} &= Q_0 y_{m+1,k} \\
&\text{etc.}
\end{aligned}\right\} \tag{2.5}$$

Such a chain is called a <u>seasonal Markov chain</u>, more specifically in our case a (k+1)-seasonal lag-1 Markov chain.

(The temptation to use the alternative designation "periodic" for such a chain must be resisted, since the work "periodic" has already passed into the technical language of Markov chains, with a completely different meaning. Periodic chains may be seasonal but are not necessarily so. A state $\mathcal{E}_j$ is <u>aperiodic</u> if the transition from $\mathcal{E}_j$ to $\mathcal{E}_j$ in one jump has positive probability, in other words, if

$$P\{Y_{t+1}=j|Y_t=j\} > 0 \quad , t = 0,1,\ldots \quad .$$

If on the other hand, the $\mathcal{E}_j$-to-$\mathcal{E}_j$ transition may only be made in 2 (and therefore also in 4,6,...) jumps, the state $\mathcal{E}_j$ is <u>periodic</u>, with period 2. For such a state

$$P\{Y_{t+1}=j|Y_t=j\} = 0$$

and

$$P\{Y_{t+2}=j|Y_t=j\} > 0 \quad , t = 0,1,\ldots \quad .$$

Periodicity is perhaps theoretically interesting, but in applications it is a nuisance. In our matrices all states will be aperiodic.

Note that, for each j (=0,1,2,...,k), the imbedded sequence $Y_{0,j}$, $Y_{1,j}$, $Y_{2,j}$,... of the "season j" variables is a homogeneous Markov chain, with transition matrix

$$P_j = Q_jQ_{j-1}\cdots Q_0Q_kQ_{k-1}\cdots Q_{j+1} \tag{2.6}$$

so that

$$y_{m,j} = P_j^m y_{0,j} \quad , m = 1,2,\ldots \quad ; j = 0,1,\ldots,k \quad .$$

If P_j is ergodic (i.e., aperiodic and ireducible: a return to each state is possible in one transition, and each state can be reached from each other state in a finite number of transitions) then the vector $y_{m,j}$ converges to a fixed limiting "stationary" distribution vector $y^{(j)} = \{y^{(j)}(0), y^{(j)}(1),\ldots,y^{(j)}(p)\}'$, defined by the homogeneous linear algebraic equations represented by

$$y^{(j)} = P_j y^{(j)} \quad (j = 0,1,\ldots,p) \tag{2.7}$$

together with the normalization condition

$$1' y^{(j)} = 1 \quad .$$

We then have

$$y^{(j)} = Q_j y^{(j-1)} \quad , \quad (j = 1,2,\ldots,k)$$

and

$$y^{(0)} = Q_0 y^{(k)}$$

as in (2.5).

2.5 Correlation in a Seasonal Lag-1 Markov Chain

In a 2-season year the system (2.5) reduces to the two vector equations

$$y_{m,1} = Q_1 y_{m,0} \quad , \quad y_{m,0} = Q_0 y_{m-1,1} \quad .$$

We speak of the two seasons as the 0-season and the 1-season. The (homogeneous) chain of 0-season variables satisfies the difference equation

$$y_{m,0} = Q_0 Q_1 y_{m-1,0} \quad , \quad m = 1,2,\ldots \quad ,$$

with stationary distribution vector $y^{(0)} = Q_0 Q_1 y^{(0)}$; and correspondingly for the 1-season variables we have

$$y_{m,1} = Q_1 Q_0 y_{m-1,1} \quad , \quad m = 1,2,\ldots \quad ,$$

with stationary distribution vector $y^{(1)} = Q_1 Q_0 y^{(1)}$; and we have

$$y^{(1)} = Q_1 y^{(0)} \quad , \quad y^{(0)} = Q_0 y^{(1)} \quad .$$

We require two lag-1 interseason correlation coefficients, namely

$$\text{corr } (Y_{m,0}, Y_{m,1}) = \rho_{01}$$

and

$$\text{corr } (Y_{m,1}, Y_{m+1,0}) = \rho_{10}$$

where

$$\rho_{01} = \{E(Y_{m,0} Y_{m,1}) - E(Y_{m,0})\, E\,(Y_{m,1})\}/\sigma_0 \sigma_1$$

denotes the correlation between a 0-season and the <u>following</u> 1-season (under stationary conditions) and

$$\rho_{10} = \{E(Y_{m,1} Y_{m+1,0}) - E(Y_{m,1})\, E\,(Y_{m+1,0})\}/\sigma_0 \sigma_1$$

the correlation between a 1-season and the following 0-season, the σ's denoting appropriate seasonal standard deviations:

$$\sigma_r^2 = E(Y_{m,r}^2) - E^2(Y_{m,r}) \quad , \quad r = 0,1.$$

We may conveniently express these in terms of the vector

$$\nu' = (0,1,2,\ldots,p) \quad :$$

we have

$$E(Y_{m,0}Y_{m,1}) = \sum_i \sum_j ij\,P(Y_{m,1}=j \mid Y_{m,0}=i)\,P(Y_{m,0}=i)$$

$$= \nu' Q_1 y^{(0)\delta} \nu \quad ,$$

$$E(Y_{m,0}) = \nu' y^{(0)} \quad , \qquad E(Y_{m,1}) = y^{(1)'} \nu \quad ,$$

and

$$\sigma_r^2 = \nu' y^{(r)\delta} \nu - (\nu' y^{(r)})^2$$

$$= \nu' (I - y^{(r)} 1') y^{(r)\delta} \nu \quad , \quad r = 0,1.$$

(Here $y^{(r)\delta}$ is the diagonal matrix whose i-th diagonal element is the i-th element of $y^{(r)}$, $i = 0,1,\ldots,p$, so that

$$y^{(r)\delta} = \text{diag}\{y^{(r)}(0),\ y^{(r)}(1),\ldots,\ y^{(r)}(p)\} \quad , \ r=0,1.)$$

Thus

$$\rho_{01} = \frac{\nu'(Q_1 - y^{(1)}1')y^{(0)\delta}\nu}{\sqrt{\{\nu'(I-y^{(0)}1')y^{(0)\delta}\nu\}\{\nu'(I-y^{(1)}1')y^{(1)\delta}\nu\}}}$$

$$= \{\nu'(Q_1 - y^{(1)}1')y^{(0)\delta}\nu\}/\sigma_0\sigma_1$$

and

$$\rho_{10} = \{\nu'(Q_0 - y^{(0)}1')y^{(1)\delta}\nu\}/\sigma_0\sigma_1 \quad . \qquad (2.8)$$

For the imbedded homogeneous chain $\ldots, Y_{m,0}, Y_{m+1,0}, Y_{m+2,0}, \ldots$ of 0-season variables we have

$$y_{m,0} = P_0 y_{m-1,0} \quad , \quad m = 1,2,\ldots$$

where $P_0 = Q_0Q_1$, and, under stationary conditions, the correlation coefficient ρ_0 between consecutive 0-seasons is

$$\left.\begin{aligned}\rho_0 &= \mathrm{corr}(Y_{m-1,0},\ Y_{m,0}) \\ &= \nu'(P_0 - y^{(0)}1')y^{(0)}\delta_\nu / \nu'(I - y^{(0)}1')y^{(0)}\delta_\nu \quad ; \\ \text{similarly} & \\ \rho_1 &= \mathrm{corr}(Y_{m-1,1},\ Y_{m,1}) \\ &= \nu'(P_1 - y^{(1)}1')y^{(1)}\delta_\nu / \nu'(I - y^{(1)}1')y^{(1)}\delta_\nu \quad . \end{aligned}\right\} \quad (2.9)$$

2.6 A Tractable Family of Seasonal Transition Matrices

Consider the matrices

$$Q_j = \alpha_j I + (1-\alpha_j)u_j 1' \quad , \quad j = 0,1,\dots,k \qquad (2.10)$$

where $0 < \alpha_j < 1$, and u_j is a distribution vector; that is, $u_j > 0$, and $1'u_j = 1$. Then the Q_j satisfy all the requirements of transition matrices, namely

$$1'Q_j = 1 \quad , \quad Q_j > 0 \qquad (j = 0,1,\dots,k) \quad .$$

These matrices have the following convenient properties:

(i) $Q_j^r = \alpha_j^r I + (1-\alpha_j^r)u_j 1' \quad , \quad r = \pm 1,\ \pm 2,\dots \quad ,$ (2.11)

(ii) $Q_j Q_k = \beta_{jk} I + (1-\beta_{jk})v_{jk} 1'$

where $\beta_{jk} = \alpha_j \alpha_k$, and

$$(1-\beta_{jk})v_{jk} = (1-\alpha_j)u_j + \alpha_j(1-\alpha_k)u_k \quad , \qquad (2.12)$$

(iii) $Q_j Q_k Q_\ell = \beta_{jk\ell} I + (1-\beta_{jk\ell})v_{jk\ell} 1'$

where $\beta_{jk\ell} = \alpha_j \alpha_k \alpha_\ell$, and

$$(1-\beta_{jk\ell})v_{jk\ell} = (1-\alpha_j)u_j + \alpha_j(1-\alpha_k)u_k + \alpha_j\alpha_k(1-\alpha_\ell)u_\ell \quad ,$$

etc.

If we take the Q_j defined in (2.10) to be the inter-season transition matrices in (2.5) we find that the transition matrix P_j of the imbedded homogeneous "annual" chains $Y_{0,j}, Y_{1,j}, Y_{2,j}, \ldots$ $(j = 0,1,\ldots,k)$ become

$$P_j = \rho_j I + (1-\rho_j) y^{(j)} 1'$$

where

$$\rho_j = \alpha_0\alpha_1\cdots\alpha_k \qquad (j = 0,1,\ldots,k)$$

is the correlation between season j in one year and season j in the preceding year. We note that ρ_j is independent of j; and

$$\begin{aligned}(1-\rho_j)y^{(j)} = {} & (1-\alpha_j)u_j+\alpha_j(1-\alpha_{j-1})u_{j-1}+\cdots+\alpha_j\alpha_{j-1}\cdots\alpha_1(1-\alpha_0)u_0 \\ & + \alpha_j\alpha_{j-1}\cdots\alpha_0(1-\alpha_k)u_k+\cdots \\ & + \alpha_j\alpha_{j-1}\cdots\alpha_0\alpha_k\alpha_{k-1}\cdots \quad \cdots\alpha_{j+2}(1-\alpha_{j+1})u_{j+1} \quad .\end{aligned}$$

For a 2-season year it follows that

$$P_0 = \alpha_0\alpha_1 I + (1-\alpha_0\alpha_1)y^{(0)}1'$$

and

$$P_1 = \alpha_0\alpha_1 I + (1-\alpha_0\alpha_1)y^{(1)}1'$$

where

$$(1-\alpha_0\alpha_1)y^{(0)} = (1-\alpha_0)u_0 + \alpha_0(1-\alpha_1)u_1$$

and

$$(1-\alpha_0\alpha_1)y^{(1)} = (1-\alpha_1)u_1 + \alpha_1(1-\alpha_0)u_0$$

so that

$$(1-\alpha_0)u_0 = y^{(0)} - \alpha_0 y^{(1)}$$

and (2.13)

$$(1-\alpha_1)u_1 = y^{(1)} - \alpha_1 y^{(0)} \quad .$$

It follows that $P_0 - y^{(1)}1' = \alpha_0\alpha_1(I-y^{(0)}1')$, whence, by (2.9), the correlation coefficient between consecutive 0-seasons is

$$\rho_0 = \alpha_0\alpha_1 \qquad (=\rho,\ \text{say});$$

and similarly that between consecutive 1-seasons is

$$\rho_1 = \alpha_0\alpha_1 \qquad (=\rho).$$

We may therefore write P_0 and P_1 in the form

$$P_r = \rho I + (1-\rho)y^{(r)}1' \quad , \quad r = 0,1,$$

where $y^{(0)}$ and $y^{(1)}$ are the stationary distribution vectors of the 0-season and the 1-season, respectively, and ρ is the correlation coefficient between corresponding dates in consecutive years. (Thus ρ is the lag-1 "annual" correlation coefficient. It is easily shown that the lag-h annual correlation coefficient is ρ^h, $h = 1,2,\ldots$.)

Similarly we may write Q_0 and Q_1 in the forms

$$Q_0 = \alpha_o I + (y^{(0)}-\alpha_0 y^{(1)})1'$$

(2.14)

$$Q_1 = \alpha_1 I + (y^{(1)}-\alpha_1 y^{(0)})1' \quad .$$

Thus, in computing the inter-season correlation coefficients (2.8) we have

$$Q_1 - y^{(1)}1' = \alpha_1(I - y^{(0)}1')$$

so that

$$\rho_{01} = \alpha_1\sqrt{\left\{\frac{\nu'(I-y^{(0)}1')y^{(0)}\delta_\nu}{\nu'(I-y^{(1)}1')y^{(1)}\delta_\nu}\right\}}$$

$$= \alpha_1\sigma_0/\sigma_1$$

and similarly

$$\rho_{10} = \alpha_0\sigma_1/\sigma_0$$

so that

$$\rho_{01}\rho_{10} = \alpha_0\alpha_1 = \rho \quad . \tag{2.15}$$

Finally, then, we may write the season-to-season transition matrices directly in terms of the stationary distribution vectors $y^{(0)}$ and $y^{(1)}$, the standard deviations σ_0 and σ_1 corresponding to these distributions, and the correlation coefficients ρ_{01} and ρ_{10}; thus

$$\left.\begin{aligned} Q_0 &= \frac{\rho_{10}\sigma_0}{\sigma_1}(I-y^{(1)}1') + y^{(0)}1' \quad , \\ Q_1 &= \frac{\rho_{01}\sigma_1}{\sigma_0}(I-y^{(0)}1') + y^{(1)}1' \quad . \end{aligned}\right\} \tag{2.16}$$

The extension of these results to a k-season year is straightforward.

3. PASSAGE TIMES IN A SEASONAL MARKOV CHAIN

3.1 The Method of Renewals

To illustrate the method without introducing excessive notational complexities we work with a 2-season year, each year containing a "0-season" and a "1-season". The successive variables in the chain are then

$$\ldots, Y_{r,0}, Y_{r,1}, Y_{r+1,0}, Y_{r+1,1}, Y_{r+2,0}, \ldots$$

We shall describe the event "$Y_{r,0} = j$" as the occurrence of the state $\mathcal{E}_j$ in a 0-season, j=0,1,...,p; and similarly for a 1-season. We have

$$\begin{aligned} y_{r+1,1} &= Q_1 y_{r+1,0} \\ Y_{r+1,0} &= Q_0 y_{r,1} \qquad r = 0,1,\ldots \end{aligned} \tag{3.1}$$

We shall be interested in the first passage times F(i,j;s) to $\mathcal{E}_i$ from an occurrence of $\mathcal{E}_j$ in an s-season (s=0,1), for i,j=0,1,...,p. This includes as a special case (i=j) the <u>recurrence time</u> F(i,i;s) of state $\mathcal{E}_i$, starting from an initial occurrence of $\mathcal{E}_i$ in an s-season.

Thus for example the statement "F(i,j;0)=3" means that, starting from $\mathcal{E}_j$ in a 0-season, the first passage to $\mathcal{E}_i$ required 3 transitions (or "took 3 units of time" = 3 seasons, at the rate of one jump per unit of time), the arrival at $\mathcal{E}_i$ occurring, naturally, in a 1-season. This is illustrated in the following table.

	Year	m		m+1		m+2	
	Season	0	1	0	1	0	
State occupied	Initially	$\mathcal{E}_j$					
	After 1 jump		not $\mathcal{E}_i$				
	After 2 jumps			not $\mathcal{E}_i$			
	After 3 jumps				$\mathcal{E}_i$		

(Illustrating "F(i,j;0) = 3").

Consider the first passage time F(i,j;0), assuming that the "initial" occurrence of $\mathcal{E}_j$ belonged to year m (the actual value of m does not matter). Then

$$P\{F(i,j;0)=2n\} = P\{Y_{m,1}\neq i,\ Y_{m+1,0}\neq i,\ldots,\ Y_{m+n-1,1}\neq i,\ Y_{m+n,0}=i\,|\,Y_{m,0}=j\} \quad ,$$

$$i,j=0,1,\ldots,p;\ n=1,2,3,\ldots,$$

$$= f^{(2n)}(i,j;0) \quad , \qquad (3.2)$$

say, and

$$P\{F(i,j;0) = 2n+1\} = P\{Y_{m,1}\neq i, Y_{m+1,0}\neq i,\ldots,Y_{m+n,0}\neq i, Y_{m+n,1}=i\,|\,Y_{m,0}=j\}$$

$$i,j=0,1,\ldots,p;\ n=0,1,2,\ldots \quad ,$$

$$= f^{(2n+1)}(i,j;0) \quad . \qquad (3.3)$$

The problem is to evaluate these probabilities in terms of the seasonal matrices Q_0 and Q_1. Define the random variables T(i,j;s) as follows:

> T(i,j;s) = n if the chain is in state $\mathcal{E}_i$ (not necessarily for the first time) after n transitions from an "initial" occurrence of $\mathcal{E}_j$ in an s-season; s = 0,1; n = 1,2,... , (i,j = 0,1,...,p).

Here again the season containing the terminal state $\mathcal{E}_i$ is not prescribed; and the statement "T(i,j;s) = n" is consistent with the system's being brought to $\mathcal{E}_i$ at the n-th transition, regardless of the number of previous visits to $\mathcal{E}_i$ since leaving the initial state $\mathcal{E}_j$. Let

$$P\{T(i,j;s) = n\} = q^{(n)}(i,j;s) \quad , \quad s = 0,1; \quad n = 1,2,\ldots \quad .$$

We have

$$q^{(2n)}(i,j;0) = P\{Y_{m+n,0} = i | Y_{m,0} = j\}$$

$$= [(Q_0Q_1)^n]_{ij} \quad , \qquad n = 1,2,\ldots \quad . \qquad (3.4)$$

Likewise

$$q^{(2n+1)}(i,j;0) = [Q_1(Q_0Q_1)^n]_{ij} \quad , \qquad n = 0,1,2,\ldots \quad . \qquad (3.5)$$

Similarly one finds

$$q^{(2n)}(i,j;1) = [(Q_1Q_0)^n]_{ij} \quad , \qquad n = 1,2,\ldots \qquad (3.6)$$

and

$$q^{(2n+1)}(i,j;1) = [Q_0(Q_1Q_0)^n]_{ij} \quad , \qquad n = 0,1,\ldots \quad . \qquad (3.7)$$

The required relation between the f's and the q's is the renewal equation. For homogeneous Markov chains this equation is well known (see Section 2.2). In the seasonal version, for our 2-season year, the appropriate generalization of this is:

$$q^{(r)}(i,j;0) = \sum_{s=1}^{r} f^{(s)}(i,j;0)q^{(r-s)}(i,i;\pi_s) \quad , \qquad (3.8)$$

$$r = 1,2,\ldots \quad , \; i,j = 0,1,\ldots,p \quad ,$$

where

$$\pi_s = \begin{cases} 0 \text{ when } s \text{ is even} \\ 1 \text{ when } s \text{ is odd} \end{cases}$$

and $\quad q^{(0)}(i,j;\pi_r) = 1$ for each r.

Similarly, if the initial state $\mathcal{E}_j$ is taken as occurring in a 1-season we find

$$q^{(r)}(i,j;1) = \sum_{s=1}^{r} f^{(s)}(i,j;1)q^{(r-s)}(i,i;\pi_{s+1}) \quad . \qquad (3.9)$$

Equation (3.8) states that, starting from $\mathcal{E}_j$ in a 0-season, an occupation of $\mathcal{E}_i$ after r transitions must either be the first occupation of $\mathcal{E}_i$ (s=r in (3.8)), or else the first occupation of $\mathcal{E}_i$ occurred after s jumps (and therefore in a π_s-season), with s=1 or 2 or $\cdots$ or r-1, and that, after a further r-s transitions the system must have returned to $\mathcal{E}_i$, not necessarily for the first time, having started this loop in a π_s-season. Equation (3.9) has a similar justification.

Developing (3.8) we have

$$q^{(1)}(i,j;0) = f^{(1)}(i,j;0)$$

$$q^{(2)}(i,j;0) = f^{(1)}(i,j;0)\, q^{(1)}(i,i;1) + f^{(2)}(i,j;0)$$

$$q^{(3)}(i,j;0) = f^{(1)}(i,j;0)\, q^{(2)}(i,i;1) + f^{(2)}(i,j;0)q^{(1)}(i,i;0) + f^{(3)}(i,j;0)$$

$$q^{(4)}(i,j;0) = f^{(1)}(i,j;0)\, q^{(3)}(i,i;1) + f^{(2)}(i,j;0)q^{(2)}(i,i;0)$$

$$+ f^{(3)}(i,j;0)q^{(1)}(i,i;1) + f^{(4)}(i,j;0)$$

$$\text{etc.} \qquad (3.10)$$

This triangular system is a ready-made algorithm for the numerical evaluation of the $f^{(n)}(i,j;0)$, n=1,2,... in terms of the known (see (3.4),(3.5),(3.6), and (3.7)) values of the $q^{(r)}(i,j;s)$: the first equation gives $f^{(1)}$, the second gives $f^{(2)}$ in terms of $f^{(1)}$, the third gives $f^{(3)}$ in terms of $f^{(1)}$ and $f^{(2)}$, and so on. A corresponding system exists for the $f^{(n)}(i,j;1)$.

To obtain an analytical solution of the equation (3.10) we work in terms of generating functions (as usual in the case of an infinite system of equations).

For $s = 0,1$, let

$$K(i,j;s;\theta) = \sum_{n=0}^{\infty} q^{(n)}(i,j;s)\theta^{n} \quad ,$$

with

$$q^{(0)}(i,j;s) = 1 \quad .$$

This is the generating function of the $q^{(n)}(i,j;s)$, $n=1,2,\ldots$. It is convenient to split it into two parts, one containing only odd powers of θ and the other containing only even powers of θ. We therefore define the "partial generating functions"

$$K_0(i,j;s;\theta) = \sum_{n=0}^{\infty} q^{(2n)}(i,j;s)\theta^{2n} \quad , \qquad \text{(even powers)},$$

$$K_1(i,j;s;\theta) = \sum_{n=0}^{\infty} q^{(2n+1)}(i,j;s)\theta^{2n+1} \quad , \text{(odd powers)}.$$

Then

$$K(i,j;s;\theta) = K_0(i,j;s;\theta) + K_1(i,j;s;\theta) \quad .$$

Likewise, for $s = 0,1$, define the generating function of the first passage probabilities as

$$F(i,j;s;\theta) = \sum_{n=1}^{\infty} f^{(n)}(i,j;s)\theta^{n}$$

$$= F_0(i,j;s;\theta) + F_1(i,j;s;\theta)$$

where F_0 contains the even powers and F_1 the odd powers.

Now multiply the equations (3.10) by $\theta, \theta^2, \theta^3, \ldots$, respectively. Adding the first, the third, the fifth,..., we find

$$\begin{aligned} K_1(i,j;0;\theta) &= \theta f^{(1)}(i,j;0)K_0(i,i;1;\theta) \\ &\quad + \theta^2 f^{(2)}(i,j;0)K_1(i,i;0;\theta) \\ &\quad + \theta^3 f^{(3)}(i,j;0)K_0(i,i;1;\theta) \\ &\quad + \theta^4 f^{(4)}K_1(i,i;0;\theta) + \cdots \\ &= K_0(i,i;1;\theta)\{\theta f^{(1)}(i,j;0) + \theta^3 f^{(3)}(i,j;0) + \cdots\} \\ &\quad + K_1(i,i;0;\theta)\{\theta^2 f^{(2)}(i,j;0) + \theta^4 f^{(4)}(i,j;0) + \cdots\} \end{aligned}$$

so that

$$K_1(i,j;0;\theta) = K_1(i,i;0;\theta)F_0(i,j;0;\theta) + K_0(i,i;1;\theta)F_1(i,j;0;\theta) .$$

Similarly

$$K_0(i,j;0;\theta)-1 = K_0(i,i;0;\theta)F_0(i,j;0;\theta) + K_1(i,i;1;\theta)F_1(i,j;0;\theta) . \tag{3.11}$$

Thus the required partial generating functions are given by the following:

$$\begin{bmatrix} F_0(i,j;0;\theta) \\ \\ F_1(i,j;0;\theta) \end{bmatrix} = \begin{bmatrix} K_1(i,i;0;\theta) & K_0(i,i;1;\theta) \\ \\ K_0(i,i;0;\theta) & K_1(i,i;1;\theta) \end{bmatrix}^{-1} \begin{bmatrix} K_1(i,j;0;\theta) \\ \\ K_0(i,j;0;\theta)-1 \end{bmatrix} . \tag{3.12}$$

Analogous results hold for $F_0(i,j;1;\theta)$ and $F_1(i,j;1;\theta)$. This formal solution can be made a little more explicit by expressing the $K_s(i,j;r;\theta)$ in terms of the season-to-season transition matrices Q_0 and Q_1. We have

$$K_0(i,j;0;\theta) = \sum_{n=0}^{\infty} q^{(2n)}(i,j;0)\theta^{2n}$$

$$= 1 + \sum_{n=1}^{\infty} [(Q_0Q_1)^n]_{ij}\theta^{2n}$$

$$= 1 + [Q_0Q_1(I-Q_0Q_1\theta^2)^{-1}]_{ij}\theta^2 \quad .$$

When i=j this simplifies to

$$K_0(i,i;0;\theta) = [(I-Q_0Q_1Q^2)^{-1}]_{ii} \quad .$$

Likewise

$$K_1(i,j;0;\theta) = \sum_{n=0}^{\infty} q^{(2n+1)}(i,j;0)\theta^{2n+1}$$

$$= \sum_{n=0}^{\infty} [Q_1(Q_0Q_1)^n]_{ij}\theta^{2n+1}$$

$$= \theta[Q_1(I-Q_0Q_1\theta^2)^{-1}]_{ij} \quad .$$

The corresponding expressions for $K_0(i,j;1;\theta)$ and $K_1(i,j;1;\theta)$ are

$$K_0(i,j;1;\theta) = 1 + [Q_1Q_0(I-Q_1Q_0\theta^2)^{-1}]_{ij}\theta^2 \quad ,$$

$$K_0(i,\, i;1;\theta) = [(I-Q_1Q_0\theta^2)^{-1}]_{ii} \quad ,$$

$$K_1(k,j;1;\theta) = \theta[Q_0(I-Q_1Q_0\theta^2)^{-1}]_{ij} \quad .$$

(Subsequent computations may on occasion be simplified by noting that

$$Q_1(I-Q_0Q_1\theta^2)^{-1} = (I-Q_1Q_0\theta^2)^{-1}Q_1 \quad .)$$

[We may verify that the ordinary non-seasonal (homogeneous Markov chain) renewal equation is a special case of the above, with $Q_0 = Q_1 (=Q)$, and $F_s(i,j;0;\theta) = F_s(i,j;1;\theta)$ $(=F_s(i,j;\theta)$ say) for $s = 0,1$. In the latter case (3.12) reduces to

$$\theta[Q(I-Q^2\theta^2)^{-1}]_{ii}F_0(i,j;\theta) + [(I-Q^2\theta^2)^{-1}]_{ii}F_1(i,j;\theta) = \theta[Q(I-Q^2\theta^2)^{-1}]_{ij}$$

$$[(I-Q^2\theta^2)^{-1}]_{ii}F_0(i,j;\theta) + \theta[Q(I-Q^2\theta^2)^{-1}]_{ii}F_1(i,j;\theta) = [(I-Q^2\theta^2)^{-1}]_{ij}-1 \quad ,$$

whence, adding, we obtain the generating function $F(i,j;\theta)$ of the first passage time distribution to $\mathcal{E}_i$ from $\mathcal{E}_j$ in the homogeneous chain as

$$F(i,j;\theta) = F_0(i,j;\theta) + F_1(i,j;\theta)$$

$$= \frac{[(I+Q\theta)(I-Q^2\theta^2)^{-1}]_{ij}-1}{[(I+Q\theta)(I-Q^2\theta^2)^{-1}]_{ii}}$$

$$= \frac{[(I-Q\theta)^{-1}]_{ij}-1}{[(I-Q\theta)^{-1}]_{ii}} \quad , \quad \text{(cf. Section 2.2).}$$

3.2 First Passage Times $F(i,j;s)$, with $i \neq j$: the Method of Absorbing States

Whilst the method described in Section 3.1 applies whether or not $i = j$, there is an alternative method that is sometimes simpler for the cases $i \neq j$. This makes use of the "absorbing state" technique that is well known in the theory of random walks.

By relabelling the states, if necessary, we can ensure that the state to which the first passage time is required is the state $\mathcal{E}_0$. We consider, then $F(0,j;0)$, that is the first passage time to $\mathcal{E}_0$, starting from $\mathcal{E}_j$ in a 0-season $(j\neq 0)$. We provide our seasonal chain $\{Y_{m,j}\}$ with a companion chain $\{\tilde{Y}_{m,j}\}$, which has seasonal transition matrices $\tilde{Q}_0$ and $\tilde{Q}_1$ where the original chain has Q_0 and Q_1. $\tilde{Q}_0$ and $\tilde{Q}_1$ differ from Q_0 and Q_1 in one column only, in each case: $\tilde{Q}_0$ and $\tilde{Q}_1$ each has as its first column the vector $e_0 = (1,0,0,\ldots,0)'$; in all other respects $\tilde{Q}_0$ is identical with Q_0, and $\tilde{Q}_1$ with Q_1.

This means that, in the $\tilde{Y}$-chain, $\mathcal{E}_0$ is an absorbing state: once the $\tilde{Y}$-chain has moved into this state it stays there. For our purposes it is more appropriate to rephrase this as follows: if after a certain number of transitions the $\tilde{Y}$-chain is in state $\mathcal{E}_0$, then all further transitions are from $\mathcal{E}_0$ to $\mathcal{E}_0$. Thus, if the $\tilde{Y}$-chain is observed to be in state $\mathcal{E}_0$ after n transitions from the initial 0-season occurrence of $\mathcal{E}_j$, it follows that the first arrival of $\tilde{Y}$ at $\mathcal{E}_0$ must have occurred after n or fewer transitions, so that

$$P\{\tilde{F}(0,j;0) \leq n\} = P\{\tilde{T}(0,j;0) = n\} \tag{3.13}$$

where $\tilde{F}$ and $\tilde{T}$ denote, respectively, the relevant first passage time and the not-necessarily-first arrival time in the $\tilde{Y}$-chain.

However, since all transition probabilities other than those out of $\mathcal{E}_0$ have the same value for the Y-chain and the $\tilde{Y}$-chain, it follows that $F(0,j;0)$ and $\tilde{F}(0,j;)0$ have the same distribution. Let

$$P\{\tilde{T}(0,j;0) = r\} = \tilde{q}^{(r)}(0,j;0) \quad , \quad r = 1,2,\ldots \quad .$$

Then

$$\left.\begin{aligned} \tilde{q}^{(2n)}(0,j;0) &= [(\tilde{Q}_0\tilde{Q}_1)^n]_{0j} \quad , \quad n = 1,2,\ldots \\ &\text{and} \\ \tilde{q}^{(2n+1)}(0,j;0) &= [\tilde{Q}_1(\tilde{Q}_0\tilde{Q}_1)^n]_{0j} \quad , \quad n = 0,1,\ldots \end{aligned}\right\} \tag{3.13a}$$

as in (3.4) and (3.5).

We have thus obtained the cumulative distribution of the first passage time in the original chain as

$$P\{F(0,j;0) \leq r\} = \tilde{q}^{(r)}(0,j;0) \quad , \quad r = 1,2,\ldots \qquad (3.14)$$

where the $\tilde{q}^{(r)}$ are defined in (3.13a); and the corresponding point distribution is

$$\begin{aligned} f^{(r)}(0,j;0) &= P\{F(0,j;0)\leq r\} - P\{F(0,j;0)\leq r-1)\} \\ &= \tilde{q}^{(r)}(0,j;0) - \tilde{q}^{(r-1)}(0,j;0) \quad . \end{aligned}$$

Thus

$$\begin{aligned} f^{(2n)}(0,j;0) &= [(\tilde{Q}_0\tilde{Q}_1)^n - \tilde{Q}_1(\tilde{Q}_0\tilde{Q}_1)^{n-1}]_{0j} \\ &= [(\tilde{Q}_0-I)\tilde{Q}_1(\tilde{Q}_0\tilde{Q}_1)^{n-1}]_{0j} \quad , \quad n = 1,2,\ldots, \qquad (3.15) \end{aligned}$$

and similarly

$$\begin{aligned} f^{(2n+1)}(0,j;0) &= [\tilde{Q}_1(\tilde{Q}_0\tilde{Q}_1)^n - (\tilde{Q}_0\tilde{Q}_1)^n]_{0j} \\ &= [(\tilde{Q}_1-I)(\tilde{Q}_0\tilde{Q}_1)^n]_{0j} \quad , \quad n = 0,1,\ldots \quad . \qquad (3.16) \end{aligned}$$

From (3.15) and (3.16) we may obtain the generating function

$$F(0,j;0) = \sum_1^\infty f^{(r)}(0,j;0)\theta^r$$

as

$$[\{(\tilde{Q}_1-I) + (\tilde{Q}_0-I)\tilde{Q}_1\theta\}(I-\tilde{Q}_0\tilde{Q}_1\theta^2)^{-1}]_{0j}\theta \quad , \quad j = 1,2,\ldots,p \quad .$$

Equations (3.15) and (3.16) are convenient formulae for computational purposes. They can however be written in terms of computationally more attractive powers of matrices of lower order than the $\tilde{Q}_s$. Partition the $\tilde{Q}_s$ as follows:

$$\tilde{Q}_s = \begin{bmatrix} 1 & a_s' \\ 0 & B_s \end{bmatrix}, \quad s = 0,1. \tag{3.17}$$

where $(1, \underline{a}_s')$ denotes the first row of Q_s. Then

$$\tilde{Q}_0\tilde{Q}_1 = \begin{bmatrix} 1 & a_0' \\ 0 & B_0 \end{bmatrix} \begin{bmatrix} 1 & a_1' \\ 0 & B_1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & a' \\ 0 & B \end{bmatrix}$$

say, where

$$a' = a_1' + a_0'B_1 \qquad B = B_0B_1 \quad ; \tag{3.18}$$

and

$$(\tilde{Q}_0\tilde{Q}_1)^r = \begin{bmatrix} 1 & b_r' \\ 0 & B^r \end{bmatrix}$$

where b_r' is a row vector whose explicit form is irrelevant. In terms of these submatrices it will be found that (3.15) and (3.16) reduce as follows:

$f^{(2n)}(0,j;0)$ = (j-1)-element of the row vector $a_0'B_1B^{n-1}$,

$j = 1,2,\ldots,p$

$= a_0'B_1B^{n-1}e_{j-1}$, $(e_{j-1} = (\underbrace{0,0,\ldots 0}_{j-2},1,0,\ldots 0)'$

and

$f^{(2n+1)}(0,j;0)$ = (j-1)-element of the row vector $a_1'B^n$

$= a_1'B^n e_{j-1}$, $\qquad j = 1,2,\ldots,p$.

These formulae involve products and powers of matrices of order $(p\times p)$, whereas the matrices in (3.15) and (3.16) are of order $(p+1)\times(p+1)$.

The above version of the first passage time distribution also lends itself to the evaluation of moments: for example the expected value of $F(0,j;0)$ is

$$E\{F(0,j;0)\} = \sum_{1}^{\infty} r f^{(r)}(0,j;0)$$

$$= \sum_{n=0}^{\infty} (2n+1) f^{(2n+1)}(0,j;0) + \sum_{n=1}^{\infty} 2n\, f^{(2n)}(0,j;0)$$

$$= \left\{a_1' \sum_{0}^{\infty}(2n+1)B^n + a_0' B_1 \sum_{1}^{\infty} 2n\, B^{n-1}\right\} e_{j-1}$$

$$= \left\{a_1'(I+B)(I-B)^{-2} + 2a_0' B_1 (I-B)^{-2}\right\} e_{j-1} \quad , \qquad (3.20)$$

$$j = 1,2,\dots,p \quad .$$

In a similar way we may obtain higher moments.

The generating function of $F(0,j;0)$ is

$$\phi(\theta|0,j;0) = \sum_{r=1}^{\infty} \theta^r f^{(r)}(0,j;0)$$

$$= (a_1' + a' B_1 \theta)(I-B\theta^2)^{-1} e_{j-1}\theta \quad . \qquad (3.21)$$

We have dealt with first passages from an occurrence of $\mathcal{E}_j$ in a 0-season. Analogous results hold when the initial occurrence is in a 1-season.

3.3 First Entry Time into a Set of States, Starting from a Specified State in a Specified Season

Let $F(\mathcal{E},j;s)$, denote the first entry time to the set $\mathcal{E}$ of states, starting from an occurrence of $\mathcal{E}_j$ in an s-season, where the set $\mathcal{E}$ does not contain the starting state $\mathcal{E}_j$, and let

$$f^{(r)}(\mathcal{E},j;s) = P\{F(\mathcal{E},j;s)=r\} \quad , \quad r = 1,2,\ldots \quad .$$

Then, if the set $\mathcal{E}$ consists of the states $\mathcal{E}_{i(1)}$, $\mathcal{E}_{i(2)},\ldots,$ $\mathcal{E}_{i(a)}$, we have

$$f^{(r)}(\mathcal{E},j;s) = \sum_{n=1}^{a} f^{(r)}(i(n),j;s)$$

where the terms under the summation sign are the first passages probabilities discussed in Section 3.2.

Some reduction in the order of the matrices involved may be obtained by an extension of the partitioning method used in the previous section. Suppose, for example that $\mathcal{E}$ consists of the states $\mathcal{E}_0$ and $\mathcal{E}_1$ (after a relabelling of states if necessary), in a seasonal chain which has two seasons in the "year". We shall take the starting season to have been a 0-season. We now provide our chain $\underset{\sim}{Y}_{m,j}$ with a companion chain $\tilde{Y}_{m,j}$ having seasonal matrices $\tilde{Q}_0,\tilde{Q}_1$ in each of which the first two columns coincide with the first two columns of the unit matrix (thereby making $\mathcal{E}_0$ and $\mathcal{E}_1$ absorbing states in the $\tilde{Y}$-chain). In all other respects $\tilde{Q}_0$ coincides with Q_0, and $\tilde{Q}_1$ with Q_1.

By an obvious extension of the methods used in deriving (3.15) and (3.16) we find

$$P\{F(0,j;0)\leq r\} = \tilde{q}^{(r)}(0,j;0) + \tilde{q}^{(r)}(1,j;0)$$

where the $\tilde{q}^{(r)}$ functions are defined in terms of the matrices $\tilde{Q}_0$ and $\tilde{Q}_1$ in exactly the same way as the $q^{(r)}$ are defined in terms of Q_0 and Q_1, in (3.13).

If follows that

$$f^{(2n)}(\mathcal{E},j;0) = \text{the sum of the } (0,j) \text{ and the } (1,j) \text{ elements of } (\tilde{Q}_0\tilde{Q}_1)^n - \tilde{Q}_1(\tilde{Q}_0\tilde{Q}_1)^{n-1} \qquad (3.22)$$

as in (3.15), with results for $f^{(2n+1)}(\mathcal{E},j;0)$ similarly related to (3.16).

The order-reduction process is now put into effect.

Partition Q_0 and Q_1 as

$$Q_s = \begin{bmatrix} I & A_s \\ 0 & B_s \end{bmatrix} , \quad s = 0,1$$

where A_s is a 2-rowed matrix. Then (3.22) reduces to

$$\left.\begin{aligned} f^{(2n)}(\mathscr{E},j;0) &= 1'A_0B_1B^{n-1}e_{j-2} \\ &\text{and} \\ f^{(2n+1)}(\mathscr{E},j;0) &= 1'A_1B^{n}e_{j-2} \end{aligned}\right\} \quad (3.23)$$

where

$$B = B_0B_1$$

and

$$e_{j-2} = (\underbrace{0,0,\ldots,0}_{j-3},1,0,\ldots 0)' \quad .$$

(The first of these expressions, for example, is the sum of the elements in the (j-2)-column of the matrix $A_0B_1B^{n-1}$.)

The results in (3.23) are expressed in terms of powers and products of matrices of order $(p-1)\times(p-1)$. Similar techniques applied to first entry times into a set of h states would be expressed in terms of matrices of order $(P+1-h)\times(p+1-h)$.

First-entry times into a set of states are important in reservoir applications.

3.4 First-Entry Times into a Set $\mathscr{A}$ of States, Starting from a Set $\mathscr{B}$ of States, Where $\mathscr{A}\cap\mathscr{B} = 0$.

In a simple (non-seasonal, lag-1) Markov chain $\{X_t\}$ defined on $(0,1,\ldots,h)$, with

$$P(X_{t+1}=r|X_t=s) = d_{rs} \quad , \qquad (3.24)$$

the generalization of (3.21) to $P(X_{t+1} \in \mathscr{A} | X_t = s)$, where $\mathscr{A}$ is a specified set of states not containing $\mathscr{E}_s$, is simply

$$P(X_{t+1} \in \mathscr{A} | X_t = s) = \sum_{r \in \mathscr{A}} d_{rs} \quad ,$$

a simplified version of the corresponding concept for a seasonal chain, as used in Section 3.3.

The corresponding generalization at "the other end", namely to $P(X_{t+1}=r | X_t \in \mathscr{B})$, where $\mathscr{B}$ is a given set of states, is of quite a different nature. If, for example, $\mathscr{B}$ consists of $\mathscr{E}_0$ and $\mathscr{E}_1$, we have

$$\begin{aligned} P(X_{t+1}=r | X_t \in \mathscr{B}) &= P(X_{t+1}=r, X_t \in \mathscr{B}) / P(X_t \in \mathscr{B}) \\ &= \{q_{r0}\, x_t(0) + q_{r1}\, x_t(1)\} / \{x_t(0) + x_t(1)\} \end{aligned}$$

where

$$x_t(r) = P(X_t = r) \quad , \quad r = 0,1,\ldots,p \quad .$$

In general, then

$$P(X_{t+1} \mathscr{A} | X_t \in \mathscr{B}) = \sum_{r \in \mathscr{A}} \sum_{s \in \mathscr{B}} q_{rs} x_t(s) / b_t$$

where

$$b_t = \sum_{s \in \mathscr{B}} x_t(s) \quad .$$

This probability is therefore time-dependent; but, provided the process is ergodic, converges to a constant with increasing t, since then the $x_t(s)$ converge to their stationary values.

3.5 "Recurrence" Times of a Set of States

In Section 3.4 we dealt with the first entry times to a set $\mathscr{A}$ of states from a set $\mathscr{B}$ of states, the sets $\mathscr{A}$ and $\mathscr{B}$ having no states in common. The first entry time to the set $\mathscr{A}$ from the same set $\mathscr{A}$--i.e., the first arrival time at an unspecified one of the states of $\mathscr{A}$, starting from an unspecified one of the states of $\mathscr{A}$--may be called the recurrence time of the set .

These recurrence times are rather messy to evaluate. For example, let the set $\mathscr{A}$ consist of the 3 states $\mathscr{E}_0$, $\mathscr{E}_1$, $\mathscr{E}_2$. The probability of first passage to $\mathscr{A}$, from an occurrence of $\mathscr{E}_0$ in the season s, at the n-th transition is

$$\pi_0 = f^{(n)}(0,0;s) + f^{(n)}(1,0;s) + f^{(n)}(2,0;s) \quad .$$

The first of these three terms is a true recurrence time probability, to be evaluated by the methods of Section 3.1; the other terms are true first passage times in the sense of Section 3.2.

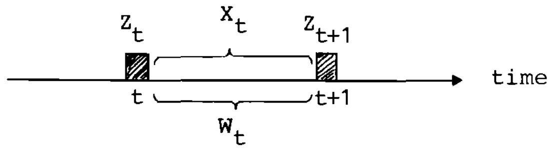
This gives the first entry time into $\mathscr{A}$, from a specified state $\mathscr{E}_0$ of $\mathscr{A}$. Similarly for π_1 and π_2.

For what we have called the recurrence time probability of the set $\mathscr{A}$, we must take the weighted average of π_0, π_1, and π_2, the weights being the ratio of the absolute probabilities of $\mathscr{E}_0$, $\mathscr{E}_1$, and $\mathscr{E}_2$, respectively, to the sum of these probabilities, as in Section 3.4. The resulting transition probability is in general time-dependent, but as in Section 3.4, will converge to fixed value as $t \to \infty$.

4. STOCHASTIC RESERVOIR THEORY

4.1 The General Background

We work with discrete time, and take the grid spacing Δt of the discrete time scale as our unit of time; and with discrete volumes, taking the quantum Δv of the discrete volume scale as our unit of volume. In these units, we consider a finite reservoir of capacity c. In the following diagram



Z_t and Z_{t+1} represent the volume of water in the reservoir at times t, t+1, respectively, X_t the inflow available during (t,t+1), and W_t the desired withdrawal during that interval, where X_t may be a vector, if the inflow process is multivariate, and where W_t is a specified function of Z_t, $Z_{t-1},\ldots$, Z_{t-k}, and X_t, $X_{t-1},\ldots$, X_{t-m} (for given values of k and m), and of various environmental factors. (This function is the so-called "release policy".) Subject to suitable assumptions concerning the sequencing of X_t and W_t we have the following "continuity equation" expressing the conservation of water quantity:

$$\begin{aligned} Z_{t+1} &= \min(Z_t+X_t-W_t,c) - \min(Z_t+X_t-W_t,0) \quad , \\ &= \alpha_{t+1}(Z_t,X_t,W_t)\text{, say,} \quad t = 0,1,\ldots \quad . \end{aligned} \tag{4.1}$$

The reservoir might be unable to accommodate the whole of the available inflow X_t, in which case the unacceptable part, or <u>spillage</u>, in (t,t+1) is

$$S_t = \max\ (Z_t+X_t-W_t-c,0) \quad . \tag{4.2}$$

Likewise the reservoir might not contain enough water to satisfy the whole of the withdrawal demand W_t, in which case the actual amount supplied in (t,t+1) is the <u>yield</u>

$$L_t = \min(Z_t+X_t,W_t) \quad . \tag{4.3}$$

Once the structure of the inflow process and of the release policy are given, (4.1) becomes a difference equation for $\{Z_t\}$, from which one may derive equations for the distribution of Z_t, generally in terms of multivariate distribution vectors involving the joint distribution of Z_t, X_t and a set of earlier values of Z and of X.

4.2 Reservoirs with Univariate Seasonal Lag-1 Markovian Inflows and Simple Seasonal Release Policies

The principal result of this section is to show that in reservoirs of this kind the sequence of pairs (Z_t, X_t) of storage and inflow can be represented in terms of a univariate lag-1 seasonal Markov chain, so that the results of Section 3 apply to the passage times of this reservoir.

Suppose the year divided into (k+1) working intervals (of size $\Delta t = 1$ in our units) called "seasons". The inflow process $\{X_t\}$ is assumed to be a seasonal lag-1 Markov chain with a finite set of states corresponding to $X_t = j$, $j = 0,1,\dots,n$. As in Section 2 it is convenient to provide an alternative notation, in which

$$X_t = X_{m,j}$$

if

$$t = (k+1)m+j \quad , \quad m = 0,1,\dots, \; 0 \leq j \leq k \quad .$$

We then regard m as the indicator of the year and j that of the season corresponding to the epoch t. In this notation, let $D_0, D_1, \dots, D_k$ denote the transition matrices of the seasonal inflows, so that for each value of m,

$$x_{m,j} = D_j x_{m,j-1} \quad , \quad j = 1,2,\dots,k$$

with

$$x_{m,0} = D_0 x_{m-1,k}$$

where $x_{m,j}$ is the distribution vector of the random variable $X_{m,j}$, (cf. (2.5)).

The (r,s) element of a transition matrix D_j will be denoted by $d_j(r,s)$.

The withdrawal policy W_t is assumed to be a simple one in the sense that W_t depends only Z_t and X_t only, the dependence being possibly one that varies from season to season. We express this in the form

$$W_t = w_j(Z_t, X_t) \quad ,$$

the subscript j representing the season corresponding to the epoch t. The continuity equation (4.3) then reduces to the form

$$Z_{t+1} = \alpha_j(Z_t, X_t) \quad . \tag{4.5}$$

It follows that the sequence of pairs ... (Z_t, X_t), (Z_{t+1}, X_{t+1}), ... form a bivariate seasonal Markov chain, in the sense that

$$P\{(Z_{t+1}=u, X_{t+1}=v) \mid (Z_t=r, X_t=s), (Z_{t-1}=r', X_{t-1}=s'), (Z_{t-2}=r'', X_{t-2}=s''), \ldots\}$$

$$= P\{(Z_{t+1}=u, X_{t+1}=v) \mid (Z_t=r, X_t=s)\} \quad , \quad t = 0,1,\ldots$$

$$r,s = 0,1,\ldots,n \quad . \tag{4.6}$$

To see this, insert (4.5) in the former of these expressions. It becomes

$$P\{\alpha_j(r,s)=u, \ X_{t+1}=v \mid (Z_t=r, X_t=s), \ (Z_{t-1}=r', X_{t-1}=s'), \ldots\}$$

$$= P\{\alpha_j(r,s)=u, X_{t+1}=v \mid X_t=s\} \tag{4.7}$$

since X_{t+1}, being by hypothesis a lag-1 Markov chain, is affected by X_t but not by any of the other conditioning variables. This expression reduces to

$$\delta\{\alpha_j(r,s), u\} P(X_{t+1}=v \mid X_t=s) \tag{4.8}$$

where the δ-function takes the value zero when $\alpha_j(r,s) \neq u$, and unity otherwise.

For formal purposes however it is more convenient to replace the pair of variables X_t, Z_t by a single variable Y_t which "carries" the two variables X_t and Z_t. We define

$$Y_t = (n+1)Z_t + X_t \quad , \quad t = 0,1,\ldots \tag{4.9}$$

where n is the largest permissible value of X_t. Consider the equation

$$y = (n+1)y_Z + y_X \quad , \quad y_X = 0,1,\ldots,n \quad . \qquad y_Z = 0,1,\ldots,c \quad . \tag{4.10}$$

Not only is y uniquely determined by y_Z and y_X, but the converse is also true. For any given integer y, $0 \leq y \leq (n+1)(c+1)-1$, y_Z is the integral part $[y/(n+1)]$ of $y/(n+1)$, that is y_Z is the unique integer such that

$$y_Z \leq y/(n+1) < y_Z + 1 \quad ,$$

and

$$y_X = y - (n+1)y_Z \quad .$$

It follows that

$$Y_t = y \text{ if and only if } Z_t = y_Z \text{ and } X_t = y_X \quad ,$$

where y, y_Z, and y_X are related as in (4.10). It then follows from (4.7) that Y_t is a seasonal lag-one univariate Markov chain, with transition probabilities

$$P(Y_{t+1}=u|Y_t=v) = P(Z_{t+1}=u_Z,\ X_{t+1}=u_X|Z_t=v_Z,\ X_t = v_X)$$

(where u, u_Z, u_X and also v, v_Z, v_X are related as in (4.10))

$$= \delta\{\alpha_j(v_Z,v_X),u_Z\}\, d_{j+1}(u_X,v_X) \quad . \tag{4.11}$$

as in (4.8).

Here "j" is the season label of epoch t, that is $j = t - [t/(k+1)]$.

This establishes the result announced in the opening paragraph of this section.

We conclude with a simple example of the matrix (4.11).

Example 4.11

Suppose we have a 2-season year, then the sequence Y_t becomes

$$Y_{00}, Y_{01}, Y_{10}, Y_{11}, Y_{20}, Y_{21}, \ldots, Y_{m0}, Y_{m1}, \ldots$$

the coresponding distribution vectors $y_{m,j}$ being related to each other thus:

$$y_{m1} = Q_1 y_{m0} \quad ,$$

$$y_{m+1,0} = Q_0 y_{m1} \quad , \quad m = 0,1,\ldots \quad . \qquad (4.12)$$

Suppose the inflows are 3-valued: $X_t = 0,1,2$; with transition matrices D_0, D_1, so that the inflow distribution vector x_t of X_t is related to x_{t+1} thus:

$$x_{m,1} = D_1 x_{m0}$$

$$x_{m+1,0} = D_0 x_{m,1} \quad , \quad m = 0,1,\ldots$$

where

$$D_j = ({}_j d_{rs}) = \begin{bmatrix} {}_j d_{00} & {}_j d_{01} & {}_j d_{02} \\ {}_j d_{10} & {}_j d_{11} & {}_j d_{12} \\ {}_j d_{20} & {}_j d_{21} & {}_j d_{22} \end{bmatrix} \quad , \; j = 0,1 \quad .$$

It is convenient to partition D, into its columns:

$$D_j = ({}_jd_0 \vdots {}_jd_1 \vdots {}_jd_2)$$

and to define auxiliary matrices

$$D_{j(0)} = ({}_jd_0 \vdots 0 \vdots 0) \quad ,$$

$$D_{j(1)} = (0 \vdots {}_jd_1 \vdots 0) \quad ,$$

$$D_{j(2)} = (0 \vdots 0 \vdots {}_jd_2) \quad .$$

Suppose, for example, the withdrawal policy is

$$W_t = 1 \quad .$$

Then, in partitioned form, the transition matrices Q_0 and Q_1 of the $\{Y_t\}$ process are given by

$$Q_j = \begin{bmatrix}
D_{j(0)} + D_{j(1)} & D_{j(0)} & 0 & & 0 & 0 & 0 \\
D_{j(2)} & D_{j(1)} & D_{j(0)} & \cdots & 0 & 0 & 0 \\
0 & D_{j(2)} & D_{j(1)} & \ddots & 0 & 0 & 0 \\
0 & 0 & D_{j(2)} & \ddots & 0 & 0 & 0 \\
0 & 0 & 0 & \ddots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\cdot & \cdot & \cdot & \ddots & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \ddots & D_{j(0)} & 0 & 0 \\
\cdot & \cdot & \cdot & \ddots & D_{j(1)} & D_{j(0)} & 0 \\
\cdot & \cdot & \cdot & \ddots & D_{j(2)} & D_{j(1)} & D_{j(0)} \\
0 & 0 & 0 & \cdots & 0 & D_{j(2)} & D_{j(1)} + D_{j(2)}
\end{bmatrix}$$

The ordering of the Z and X states and the detailed layout is exemplified in the following display of Q_j in the case c = 2,

Z_{t+1}	X_{t+1} \ Z_t	0			1			2		
	X_t	0	1	2	0	1	2	0	1	2
0	0	d_{00}	d_{01}		d_{00}					
	1	d_{10}	d_{11}		d_{10}					
	2	d_{20}	d_{21}		d_{20}					
1	0			d_{02}		d_{01}		d_{00}		
	1			d_{12}		d_{11}		d_{10}		
	2			d_{22}		d_{21}		d_{20}		
2	0						d_{02}		d_{00}	d_{02}
	1						d_{12}		d_{11}	d_{12}
	2						d_{22}		d_{21}	d_{22}

Here we have written d_{rs} instead of ${}_jd_{rs}$, for clarity; blanks represent zero entries.

Equation (4.12) is a 2-season version of (2.5). The notation and concepts are then those of the univariate seasonal lag-one Markov chain of Section 2, and the passage time analysis of Section 3 applies directly.

In particular, the first passage time to storage level Z=r from a storage level Z=s (s≠r) is the first entry time to the entry set (Z=r,X=0), (Z=r, X=1),...,(Z=r, X=n) from the set (Z=s, X=0),...,(Z=s, X=n) as explained in Section 3.4.

4.3 Reservoirs with More General Inflow and Release Policies

Inflow processes of the type we have discussed can be generalized in two ways: they can be multilag, instead of lag-1; and they can be multivariate, instead of univariate.

Likewise the release policies can be generalized, by allowing W_t to depend not only on Z_t and X_t but also on Z_{t-1}, $Z_{t-2}, \ldots,$ Z_{t-k}, and on X_{t-1}, $X_{t-2}, \ldots, X_{t-p}$, for arbitrary values of the integers k and p.

All these kinds of generalizations fit in with the methods presented in earlier sections, all being formally equivalent to a suitably defined univariate lag-1 Markov chain $\{Y_t\}$.

As an example, consider the following: the inflow is a univariate seasonal lag-1 Markov chain (as in section 4.2) and the withdrawal policy W_t depends on Z_t, Z_{t-1}, X_t and X_{t-1}:

$$W_t = w_j(Z_t, Z_{t-1}, X_t, X_{t-1})$$

where the suffix j represents the season corresponding to t. The continuity equation coresponding to (4.5) is now

$$Z_{t+1} = \alpha_j(Z_t, Z_{t-1}, X_t, X_{t-1}) \quad .$$

It is easy to see that the sequence of quadruples $\{(Z_t, Z_{t-1}, X_t, X_{t-1})\}$ forms a quadrivariate seasonal lag-1 Markov chain, which can be represented as a univariate chain $\{Y_t\}$ by the obvious extension of (4.9), namely

$$Y_t = (c+1)^2(n+1)Z_t + (c+1)(n+1)Z_{t-1} + (n+1)X_t + X_{t-1} \quad ,$$

$$(X_{t-1}, X_t = 0, 1, \ldots, n \quad , \quad Z_{t-1}, Z_t = 0, 1, \ldots, c \quad)$$

When the inflow is a bivariate lag-1 Markov chain, it can be represented as a (suitably labelled) univariate lag-1 chain by the methods used for the pair (Z_t, X_t) in (4.9) and this can then be proceeded with by the methods explained.

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