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GAME THEORETICAL TREATMENT OF MATERIAL ACCOUNTABILITY PROBLEMS: PART II

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Abstract

In a previous paper, the optimal strategy for an inspection authority which has to safeguard material on the basis of material accountability principles has been determined with game theoretical methods: Sets of reasonable inspection and diversion strategies have been defined, and a saddlepoint of the overall probability of detection for n inventory periods during the reference time under consideration has been determined.

In this paper the problem of the appropriate choice of the number of inventory periods per reference time has been analyzed: it has been shown that the overall probability of detection in the case of one inventory period per reference time is always larger than that in the case of n inventory periods for $n > 1$, and further it has been shown in which way this result is reflected in the expected detection time.

1. Introduction

In the first part of this study on material accountability problems [1], the following question has been analyzed: In an industrial plant the material to be processed is safeguarded on the basis of the material balance principle, i.e. from time to time the so-called book inventory (starting inventory

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plus receipts minus shipments) is compared with the physical inventory. If no material has disappeared, the two inventories should be the same. However, the situation is obscured by measurement errors which are committed inevitably.

A (zero sum) game theoretical analysis has been performed in the first part of the study under the assumptions that during the reference time period in which n physical inventories will be taken

1) the plant operator or people in the plant will divert at least the amount M of material;
2) the probability for at least one false alarm induced by the inspector is not larger than α;
3) the measurements are stochastically independent; and
4) the payoff to the inspector is the total probability of detection.

In this part the question of the appropriate number of inventory periods during a reference time (say one year) has been analyzed for the following reason. The inspection authority has the objective of achieving a probability of detection as high as possible; however, it also has the objective of a short detection time. As the detection time is determined by the frequency of the physical inventories, a statement of the inspector whether or not a diversion has taken place cannot be made before the book and the physical inventory have been compared. It is important to know the impact of the number of inventory periods during the reference time under consideration.

In the following it is shown that the overall probability of detection in the case of one inventory period per reference time is always larger than that in the case of n inventory periods for $n > 1$, and furthermore, in which way this result is reflected in the expected detection time.

2. Analysis of the Influence of the Number of Inventory Periods on the Probability of Detection

2.1 Formulation of the Problem

Let us consider the reference time interval

$$J: = [t_O, t_E] \subset \mathbb{R} \quad ,$$

where $t_O < t_E$. We call

$$Z_n: = \{t_{nO}: = t_O, t_{n1}, \ldots, t_{nn-1}, t_{nn} = t_E, t_{ni} < t_{ni+1} \ \forall i = 0, \ldots, n-1\} \quad , \qquad (2\text{-}1)$$

a partition of J.

At times $t_{ni}, i = 0, \ldots, n$, physical inventories I_{ni}, ($I_O = I_{nO}, I_{nn} = I_E$) are taken and compared with the book inventories B_{ni}, which are the sums of the starting inventories S_{ni-1} at time i-1, and the throughput measurements D_{ni} in the time interval $[t_{ni-1}, t_{ni}]$:

$$B_{ni} = S_{ni-1} + D_{ni} \quad , \qquad i = 1, \ldots, n \quad . \qquad (2\text{-}2a)$$

The starting inventories are linear contributions of the ending physical and book inventories:

$$S_{nO} = I_O$$

$$S_{ni-1} = a_{ni-1} \cdot B_{ni-1} + (1 - a_{ni-1}) \cdot I_{ni-1},$$

$$i = 2,\ldots,n \; , \qquad (2\text{-}2b)$$

where the weights a_{ni-1} are determined by the measurement variances of the physical and book inventories (see Ref. [1], eq. (2-5)):

$$a_{ni-1} = \frac{\text{var } I_{ni-1}}{\text{var } B_{ni-1} + \text{var } I_{ni-1}} \qquad (2\text{-}3a)$$

$$\frac{1}{\text{var } S_{ni-1}} = \frac{1}{\text{var } B_{ni-1}} + \frac{1}{\text{var } I_{ni-1}} \; . \qquad (2\text{-}3b)$$

In the following, it is assumed that the sum of all throughput measurement variances is independent of the partition (2-1),

$$\sum_{i=1}^{n} \text{var } D_{ni} = \Sigma^2 \; . \qquad (2\text{-}4)$$

The difference between the book and the physical inventory at time t_{ni} is called "Material Unaccounted For" (MUF):

$$\text{MUF}_{ni} := B_{ni} - I_{ni} \qquad (2\text{-}5a)$$

$$\text{var MUF}_{ni} := \sigma_{ni}^2 = \text{var } S_{ni-1} + \text{var } D_{ni} + \text{var } B_{ni} \; . \qquad (2\text{-}5b)$$

The safeguards procedure consists of a series of significance tests with respect to the expectation values of the Material Unaccounted For in the different inventory periods:

Let us assume that in the plant either no material is diverted at all (Null hypothesis H_O) or that in the n,i-th inventory period the amount M_{ni} is diverted (Alternative hypothesis H_1):

$$E(I_{ni-1} + D_{ni} - I_{ni}|H_1) = M_{ni}, \quad i = 1,\dots,n \ . \qquad (2\text{-}6)$$

Then the hypotheses of the tests can be written as follows (see Ref. [1]. eq. (2-8)):

$$E(MUF_{ni}|H_O) = 0 \quad , \qquad i = 1,\dots,n \qquad (2\text{-}7a)$$

$$E(MUF_{ni}|H_1) = y_{ni}, y_{ni} = a_{ni} \cdot y_{ni-1} + M_{ni}, y_{ni} = M_{ni} \ . \qquad (2\text{-}7b)$$

Let α_{ni} be the false alarm probability (error of the first kind probability) for the n,i-th test, let α be the total false alarm probability and let $1-\beta_n$ be the total probability of detection (one minus the total error of the second kind probability). Then the calculation gives (see Ref. [1], eqs. (2-11), (2-12)):

$$1 - \beta_n = 1 - \prod_{i=1}^{n} \phi\left(U(1 - \alpha_{ni}) - \frac{y_{ni}}{\sigma_{ni}}\right) \qquad (2\text{-}8a)$$

$$1 - \alpha = \prod_{i=1}^{n} (1 - \alpha_{ni}) \quad , \qquad (2\text{-}8b)$$

where ϕ is the normal distribution function and U its inverse.

In the first part of this study, the following problem has been solved. The operator diverts the amounts $M_{n,i}$, $i = 1,\dots,n$ such that he gets the total amount M:

$$M = \sum_{i=1}^{n} M_{ni} \quad , \tag{2-8c}$$

and that the total probability of detection $1 - \beta_n$ is minimized. The inspection authority chooses its set of false alarm probabilities, $\alpha_{ni}, i = 1,\dots,n$, such that an agreed total false alarm probability α is not exceeded and that $1 - \beta_n$ is maximized. Let us call the optimal values $(\alpha^*_{n1},\dots,\alpha^*_{nn})$ and $(M^*_{n1},\dots,M^*_{nn})$. It is important that $(\alpha^*_{n1},\dots,\alpha^*_{nn})$ is independent of the value of M.

In this part, we want to analyze the influence of the number n of inventory periods on the optimal probability of detection (the value of the two-person zero-sum game).

If we take the logarithm of β_n in eq. (2-8a):

$$F_n(x,y) = \sum_{i=1}^{n} \ell n\phi\left(U(e^{x_{ni}}) - \frac{y_{ni}}{\sigma_{ni}}\right) \tag{2-9a}$$

defined on $X_n \times Y_n$, where

$$X_n: = \{x_n = (x_{n1},\dots,x_{nn}) \in \mathbb{R}^n, \ \sum_{i=1}^{n} x_{ni} = \ell n(1 - \alpha) \quad ,$$
$$0 \geq x_{ni} \geq \ell n(1 - \alpha) \ \forall\, i = 1,\dots,n\} \tag{2-9b}$$

$$Y_n: = \{y_n = (y_{n1},\dots,y_{nn}) \in \mathbb{R}^n, y_{nn} + \sum_{i=1}^{n-1} (1 - a_{ni})\cdot y_{ni} = M\} \tag{2-9c}$$

for given $0 < \alpha < 1$, $M > 0$, this is equivalent to the question, does an optimal partition, Z_m, exist, i.e. a partition Z_m with

$$F_m(x_m^*, y_m^*) \leq F_n(x_n^*, y_n^*) \quad ,$$

where $x_{ni}^* = \ln(1 - \alpha_{ni}^*)$, for all partitions Z_n. In the following, we will show that $Z_1 = \{t_0, t_E\}$ is the only optimal partition.

2.2 Optimality of One Inventory Period

In order to prove the optimality of Z_1 we will make use of some properties of the normal distribution function:

Lemma 2.1 Let $Q(x), x \in \mathbb{R}$, be defined by

$$Q(x) = \frac{\phi'(x)}{\phi(x)} \quad . \tag{2-10}$$

Then one has

$$Q''(x) > 0 \text{ for } x \in \mathbb{R} \quad . \tag{2-11}$$

Proof. Let $R(x), x \in \mathbb{R}$, be defined by

$$R(x) := e^{\frac{x^2}{2}} \cdot \int_x^\infty e^{-\frac{t^2}{2}} \, dt \quad .$$

R is called "Mill's Ratio." Then one can show that

$$\left(\frac{1}{R(x)}\right)'' > 0 \text{ for } x \in \mathbb{R}$$

(see, e.g. [3], [4]). As $Q(x) = (R(-x))^{-1}$ the proof is completed. ∎

Lemma 2.2. Let $Q(x)$ be defined by eq. (2-9). Then $Q(U(e^x))$ is strictly convex for $x \leqq 0$. Especially, one has

$$\sum_{i=1}^{n} Q^2(U(e^{x_i})) \leq Q^2(U(e^{\ln(1-\alpha)}))$$

for all $x = (x_1,\ldots,x_n)$; the $<$ sign holds if x is not of the form $x^{(j)} = (x_1^{(j)},\ldots,x_n^{(j)})$, $x_i^{(j)} = 0 \;\forall i \neq j$, $x_j^{(j)} = \ln(1-\alpha)$.

Proof. As we have

$$\frac{d}{dx} Q^2(U(e^x))$$

$$= 2Q' \cdot (U(e^x)) \cdot Q(U(e^x)) \cdot \sqrt{2\pi} \cdot e^{\frac{1}{2}U^2(e^x)} \cdot e^x$$

$$= 2Q' \cdot (U(e^x)) \quad ,$$

we obtain with the help of Lemma 2.1, that

$$\frac{d}{dx} Q^2(U(e^x))$$

is strictly monotonously increasing on $x \leqq 0$. Therefore, $Q^2(U(e^x))$ is strictly convex for $x \leqq 0$, and

$$S(x) := \sum_{i=1}^{n} Q^2(U(e^{x_i}))$$

is also strictly convex on X_n, and with

$$\lambda_j \geq 0, \qquad j = 1,\ldots,n, \qquad \sum_{j=1}^{n} \lambda_j = 1$$

the following inequality holds:

$$Q^2(U(e^{\ln(1-\alpha)})) = \sum_{j=1}^{n} \lambda_j S(x^{(j)}) \geq S\left(\sum_{j=1}^{n} \lambda_j x^{(j)}\right)$$

$$= \sum_{i=1}^{n} Q^2\left(U(e^{\sum_{j=1}^{n} \lambda_j x_i^{(j)}})\right)$$

where the > sign holds if $\lambda_j < 1$ for one j. ■

In addition, we will make use of some properties of the Material Unaccounted For as defined in the previous section:

Lemma 2.3. Let MUF_{ni}, i = 1,...,n, be defined by eq. (2-5a). Then, we have

$$\sum_{i=1}^{n-1} (1 - a_{ni}) \cdot MUF_{ni} + MUF_{nn} = I_O + \sum_{i=1}^{n} D_{ni} + I_E \quad , \tag{2-12}$$

and therefore,

$$E\left(\sum_{i=1}^{n-1} (1 - a_{ni}) \cdot MUF_{ni} + MUF_{nn} \middle| H_1\right) = \sum_{i=1}^{n} M_{ni} \quad . \tag{2-13a}$$

If $I_O, I_E, D_{1n}, \ldots, D_{nn}$ are stochastically independent, we have

$$\mathrm{var}\left(\sum_{i=1}^{n-1} (1 - a_{ni}) \cdot MUF_{ni} + MUF_{nn}\right) = \sum_{i=1}^{n-1} (1 - a_{ni})^2 \cdot \sigma_{ni}^2 + \sigma_{nn}^2$$

$$= \mathrm{var}\ I_O + \mathrm{var}\ I_E + \sum_{i=1}^{n} \mathrm{var}\ D_{ni} \quad . \qquad (2\text{-}13b)$$

Proof (by induction). For n = 1, eq. (2-12) is true. Let us assume that eq. (2-12) is true for n. Then we have

$$\sum_{i=1}^{n} (1 - a_{n+1i}) \cdot MUF_{n+1i} + MUF_{n+1n+1} = \sum_{i=1}^{n-1} (1 - a_{n+1i}) \cdot MUF_{n+1i}$$

$$+ MUF_{n+1n} - a_{n+1n} \cdot MUF_{n+1n} + MUF_{n+1n+1}$$

$$= I_O + \sum_{i=1}^{n} D_{n+1i} - I_{n+1n} - a_{n+1n} \cdot MUF_{n+1n} + MUF_{n+1n+1}$$

Now, with eqs. (2-2a) and (2-2b):

$$-a_{n+1} \cdot MUF_{n+1n} + MUF_{n+1n+1}$$

$$= -a_{n+1n} \cdot B_{n+1n} + a_{n+1n} \cdot I_{n+1n} + B_{n+1n+1} - I_{n+1n+1}$$

$$= -a_{n+1n} \cdot B_{n+1n} + a_{n+1n} \cdot I_{n+1n} + S_{n+1n}$$

$$+ D_{n+1n+1} - I_{n+1n+1}$$

$$= I_{n+1n} + D_{n+1n+1} - I_{n+1n+1} \quad .$$

Therefore, eq. (2-12) is also true for n+1.

From eq. (2-12) which we write in the following form:

$$\sum_{i=1}^{n-1} (1 - a_{ni}) \cdot MUF_{ni} + MUF_{nn} = I_O + \sum_{i=1}^{n} D_{ni} - I_E$$

$$= \sum_{i=1}^{n} (I_{ni-1} + D_{ni} - I_{ni})$$

we immediately get eq. (2-13a). Because of the independence assumed we finally get

$$\text{var}\left(\sum_{i=1}^{n} (1 - a_{ni}) \cdot MUF_{ni} + MUF_{nn}\right)$$

$$= \sum_{i=1}^{n-1} (1 - a_{ni}) \cdot \sigma_{ni}^2 + \sigma_{nn}^2$$

$$= \text{var } I_O + \text{var } I_E + \sum_{i=1}^{n} \text{var } D_{ni} \quad . \qquad \blacksquare$$

In order to prove the optimality of the partition Z_1, i.e. to prove

$$F_1(x^*,y^*) < F_n(x_n^*,y_n^*) \quad \forall \; n \neq 1 \quad , \tag{2-14}$$

we proceed as follows: Let Z_n, $n > 1$, be an arbitrary, but fixed partition of J. If we prove

$$F_1(x^*,y^*) < F_n(x_n^*,y_n') \quad \forall \; n \neq 1 \tag{2-15}$$

for one specific $y_n' \varepsilon Y_n$ where Y_n is given by (2-9c), then we also have proven (2-14) as

$$F_1(x^*,y^*) - F_n(x_n^*,y_n^*)$$

$$= F_n(x^*,y^*) - \max_{y \in Y_n} F_n(x_n^*,y)$$

$$\leq F_1(x^*,y^*) - F_n(x_n^*,y') < 0$$

For the proof of (2-15) we will use

$$y_n' = (y_{n1},\ldots,y_{nn}) = \Big((1 - a_{n1}) \cdot \sigma_{n1}^2 \cdot \frac{M}{\rho^2},\ldots,$$
$$(1 - a_{nn}) \cdot \sigma_{nn}^2 \cdot \frac{M}{\rho^2}\Big) \quad , \qquad (2\text{-}16a)$$

where

$$a_{nn} := 0$$
$$\rho^2 := \text{var } I_O^2 + \text{var } I_E^2 + \Sigma^2 \qquad (2\text{-}16b)$$

and where Σ^2 is given by (2-4). Because of eq. (2-13b), the vector y_n' given by (2-16a) fulfills the condition $y_n' \varepsilon Y_n$. Therefore, with

$$F_1(x^*,y^*) = F_1(x,y) = F_1(\ell n(1 - \alpha),M)$$
$$= \ell n \phi(U(1 - \alpha) - \frac{M}{\rho})$$

it only remains to be proven that

$$\ell n \phi(U(1 - \alpha) - \frac{M}{\rho}) < F_n(x_n^*,y_n')$$

where y_n' is given by (2-16a).

Theorem 2.4. Let $\Delta(M)$ be defined by

$$\Delta(M) := \ln\phi(U(1-\alpha) - \frac{M}{\rho}) - \sum_{i=1}^{n} \ln\phi\left(U(e^{x^*_{ni}}) - (1 - a_{ni})\cdot\sigma_{ni} \cdot \frac{M}{\rho^2}\right) , \tag{2-18}$$

where $x^*_n = (x^*_{n1},\ldots,x^*_{nn}) \varepsilon X_n$ is given by Theorem (3-12), Ref. [1]. Then we have

$$\Delta(M) < 0 \quad \forall \; M > 0 \quad . \tag{2-19}$$

Proof. In the following, we will omit the index n. In order to prove the inequality we will show

$$\Delta(0) = 0, \qquad \Delta'(0) < 0, \qquad \Delta''(M) < 0 \quad .$$

(i) $\Delta(0) = \ln(1-\alpha) - \sum_{i=1}^{n} x^*_i = 0$ as $x^* \varepsilon X_n$,

defined by (2-9b).

(ii) According to Theorem (3.12), Ref. [1], we have

$$0 > x^*_i > \ln(1-\alpha) \quad \forall \; i = 1,\ldots,n \quad . \tag{2-20}$$

Therefore, with (2-10),

$$\frac{d}{dM}\,\Delta(M)\,|\,M = 0$$

$$= -\frac{1}{\rho}\,Q(U(1-\alpha)) + \sum_i (1-a_i)\cdot\frac{\sigma i}{\rho^2}\cdot Q(U(e^{x_i^*}))$$

$$\leq\; -\frac{1}{\rho}\,Q(U(1-\alpha)) + \left(\sum_i (1-a_i)^2\,\frac{\sigma_i^2}{\rho^4}\right)^{\frac{1}{2}}\cdot\left(\sum_i Q^2(U(e^{x_i^*}))\right)^{\frac{1}{2}}$$

$$= \frac{1}{\rho}\left[-Q(U(1-\alpha)) + \left(\sum_i Q^2(U(e^{x_i^*}))\right)^{\frac{1}{2}}\right] < 0 \quad ,$$

where Schwarz's inequality has been used, eqs. (2-13b) and (2-16b), and Lemma 2.2.

(iii)

$$\frac{d^2}{dM^2}\,\Delta(M) = \frac{1}{\rho^2}\,Q'(U(1-\alpha) - \frac{M}{\rho})$$

$$-\frac{1}{\rho^4}\cdot\sum_i (1-a_i)^2\sigma_i^2\cdot Q'\left(U(e^{x_i^*}) - (1-a_i)\cdot\sigma_i\cdot\frac{M}{\rho^2}\right)$$

$$\leq \frac{1}{\rho^2}\cdot Q'(U(1-\alpha) - \frac{M}{\rho}) - \frac{1}{\rho^4}\sum_i (1-a_i)^2\sigma_i^2\cdot\min_k Q'\left(U(e^{x_k^*})\right.$$

$$\left.-(1-a_k)\cdot\sigma_k\cdot\frac{M}{\rho^2}\right)$$

$$= \frac{1}{\rho^2}\left[Q'(U(1-\alpha) - \frac{M}{\rho}) - \min_k Q'\left(U(e^{x_k^*})\right.\right.$$

$$\left.\left. - (1-a_k)\cdot\sigma_k\cdot\frac{M}{\rho^2}\right)\right] < 0 \quad ;$$

where in the last equation Lemma 2.3 was used, and the < sign in the last inequality follows from

$$U(e^{\ell n(1-\alpha)}) < U(e^{x_k^*}) \text{ for } k = 1,\dots,n \quad \text{(see 2-20)},$$

and

$$(1-a_k)\cdot\sigma_k \leq \rho \qquad \text{for } k = 1,\dots,n$$

because of

$$\sum_i (1-a_i)^2\cdot\sigma_i^2 = \rho^2 \quad .$$

■

2.3 Additional Remarks

In the last section it has been shown that the partition Z_1 is better than the partition Z_n for $n > 1$. Under the assumptions

(i) $\text{var}(I_i) = 0$ for $i = 0,1,\dots,j$ and $j\varepsilon\mathbf{N}$, and

(ii) $\text{var}(D_{ji}) = \frac{1}{j}\Sigma^2$ for $i = 1,\dots,j$ and $j\varepsilon\mathbf{N}$

one also can show that the partition Z_n is better than the partition Z_{n+m} for $n,m\varepsilon\mathbf{N}$.

Under some assumptions, e.g.

(i) $\text{var}(I_i) = \sigma_I^2$ for $i = 0,1,\dots,j$ and $j\varepsilon\mathbf{N}$, and

(ii) $\text{var}(D_{ji}) = \frac{1}{j}\Sigma^2$ for $i = 1,\dots,j$ for $j\varepsilon\mathbf{N}$

one can show at least

$$\lim_{j\to\infty} (1 - \beta_j) = \alpha \quad . \qquad (2\text{-}21)$$

Therefore, for each $n\varepsilon\mathbf{N}$ there exists an m_n with the property that the partition Z_n is better than the partition Z_{n+m} for all $m \geq m_n$.

In Fig. 1, the results of some numerical calculations for a realistic case are given; the data have been taken from Ref. [2]. The special cases mentioned above, as well as numerical calculations, indicate that one has in general $Z_n > Z_{n+m}$ for $n,m\varepsilon\mathbf{N}$ even though it was not possible to prove this.

It is clear that the number of inventory periods per reference time determines the detection time, i.e. the time where the Material Unaccounted For is greater than the significance threshold for the first time. Now this gives two effects. First, one would assume that the shorter one inventory period is, the shorter the detection time is. Second, according to the foregoing results, with an increasing number of inventory periods per reference time, the probability of detection decreases. Therefore, detection may depend on the values of the parameters of which of the two effects is stronger.

If one wants to define the expected detection time (measured in units of inventory periods) one has the difficulty that one does not know which detection time one should take in the case that no detection at all takes place during the reference time. One possibility would be to define the expected detection time as

$$E_1T := \sum_{i=1}^{n} i \cdot P_i \cdot \prod_{j=1}^{i-1} (1 - P_j) + (n + 1) \cdot \prod_{i=1}^{n} (1 - P_i) \qquad \text{(2-22a)}$$

where

$$P_i := \text{prob}\,\{MUF_{ni} > S_{ni}\} \quad \text{for } i = 1,\ldots,n. \qquad \text{(2-22b)}$$

Another possibility would be to take the expected detection time under the condition that a detection will take place during the reference time:

$$E_CT := \frac{\sum_{i=1}^{n} i \cdot P_i \cdot \prod_{j=1}^{i-1} (1 - P_j)}{1 - \prod_{i=1}^{n} (1 - P_i)} \,, \qquad \text{(2-23)}$$

where P_i again is given by eq. (2-22b).

In Figs. 2a and 2b, the results of the calculations for both the detection times (measured in fractions of the reference time) on the basis of the data used earlier are represented. In both cases one sees that there exists a minimum which may be explained with the arguments given above.

3. Conclusion

It has been shown that in the case that the overall probability of detection for a reference period of time is the criterion of optimization, it is best to have only one inventory period in the reference time. However, there are considerations which indicate that not only the probability of detection but also the detection time should be taken as a criterion of optimization. As the detection time is regulated by the number of inventory periods per reference time--a detection cannot take place before the end of an inventory period--one might want to have more than one inventory period per reference time at the expense of the probability of detection. Therefore, a reasonable procedure would be first to decide about the minimum probability of detection, and thereafter to choose as many inventory periods as are compatible with this minimum.

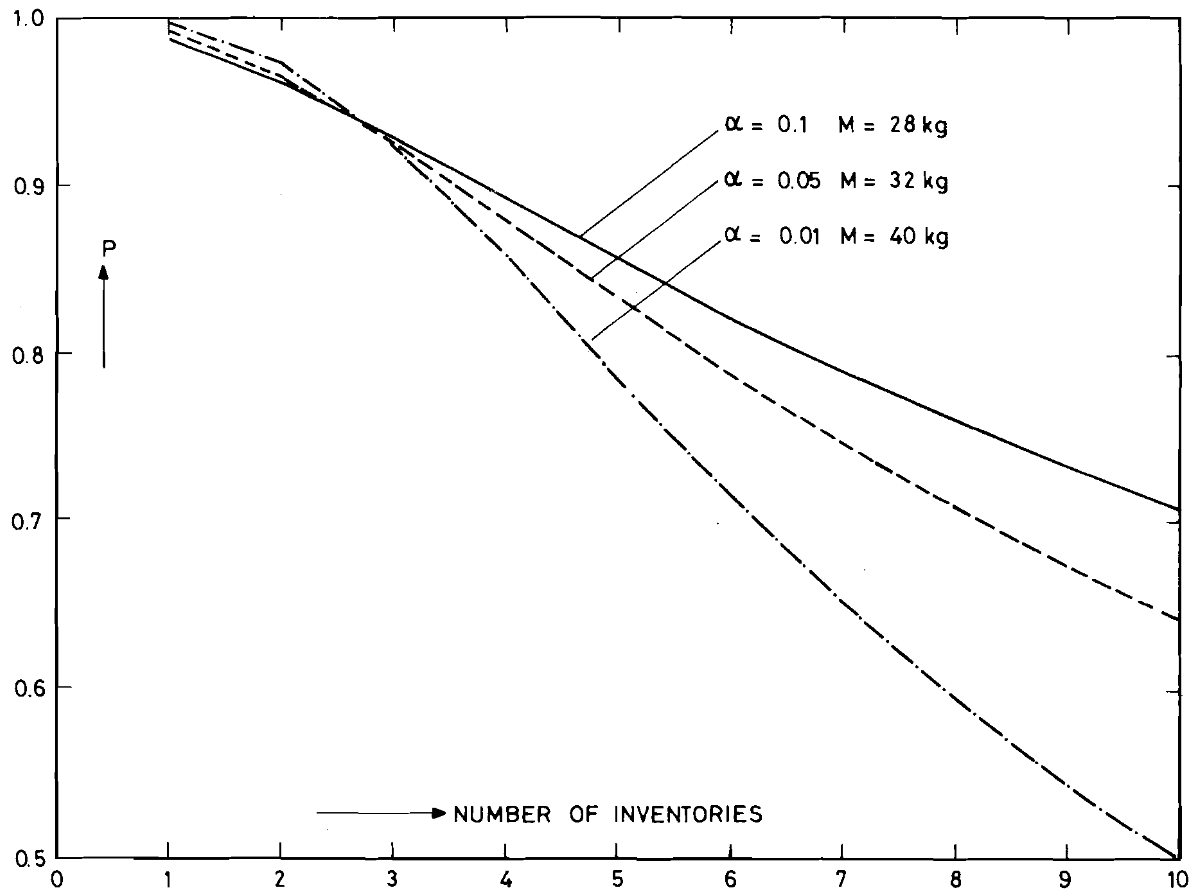

FIGURE 1: PROBABILITY OF DETECTION P AS A FUNCTION OF THE NUMBER OF INVENTORY PERIODS PER YEAR WITH α AND M AS PARAMETERS; DATA FROM REF[2]:
$var(I_i) = 0.013\ kg^2$ for $i = 0, 1, \ldots$; $\sum_i var(D_i) = 63.26\ kg^2$.

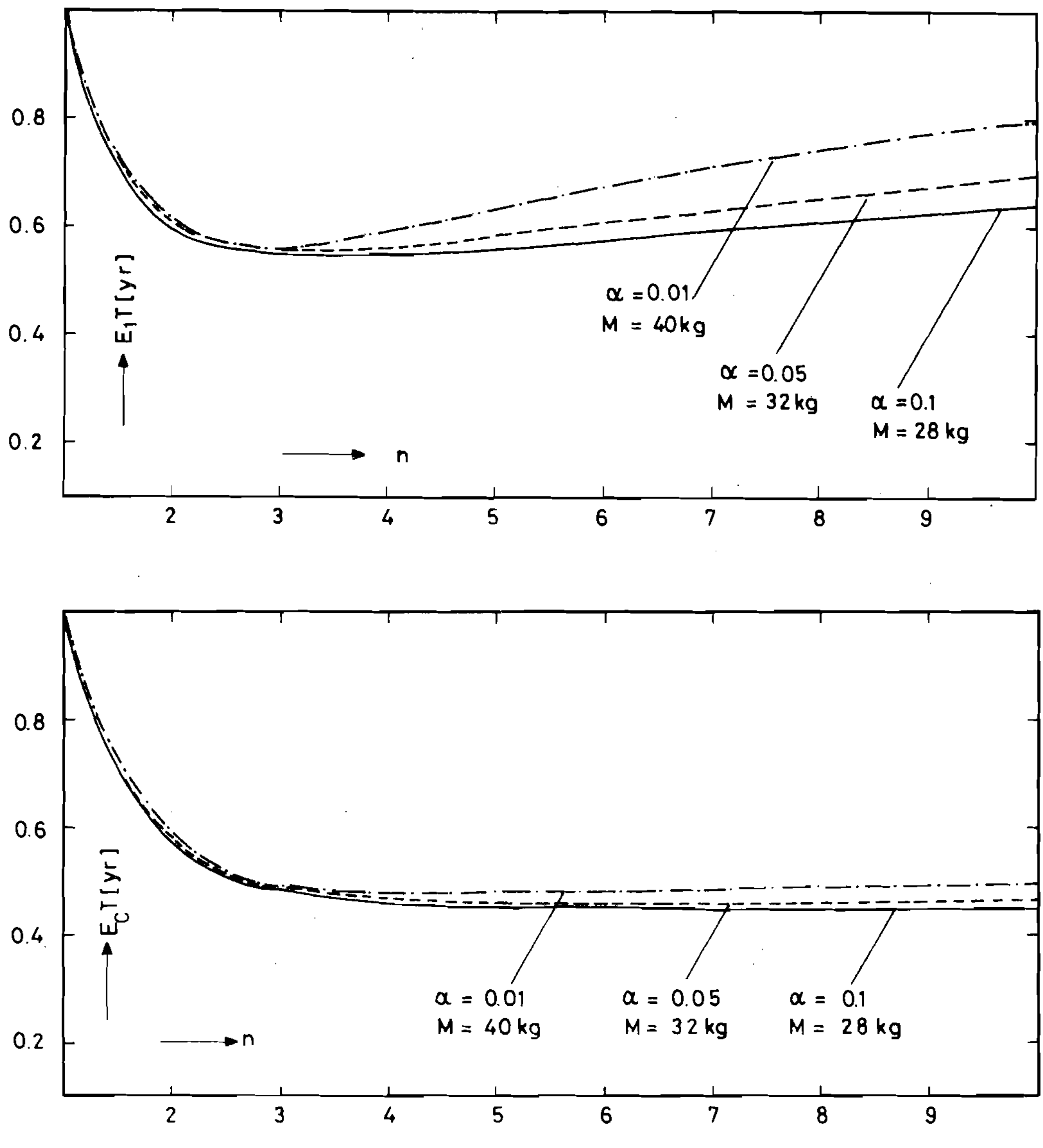

FIGURE 2: EXPECTED DETECTION TIMES E_1T, eq (2-22) AND E_CT, eq (2-23), AS A FUNCTION OF THE NUMBER OF INVENTORY PERIODS PER YEAR WITH α AND M AS PARAMETERS; DATA FROM REF. [2].

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