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On One Problem of Vector Optimization of Pricing in an Autonomous Wholesale Market

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Abstract

The paper is devoted to mathematical simulations of an autonomous network of wholesale at a local market. The question of optimization of prices at outlets is discussed. The model is constructed on the basis of a vector optimization problem. The main properties of the multicriteria problem are studied. The necessary and sufficient conditions for the existence and uniqueness of a solution are formulated. An effective numerical solution algorithm is developed. The results are illustrated by examples of numerical solutions using real data. Practical applications are presented in the form of specialized algorithms and software.

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Introduction

In this paper we discuss issues of construction of mathematical models of the process in an autonomous market of wholesale trades of uniform goods. The problems considered here deal with earlier investigations conducted by the authors [1]. The problem of price optimization at interconnected outlets under additional restrictions is the focus. The suggested mathematical model is classified as a linear problem of vector optimization. Large number of works (see, for example, [2, 3, 7, 8, 9]) are devoted to problems of vector optimization.

We examine properties of multicriteria optimization problems for the considered mathematical model. The optimal plan is defined; the necessary and sufficient optimality conditions are formulated. The theorem of existence and uniqueness of optimal plan is proved. The finite iterative procedure for the problem solution is developed. The numerical algorithm is based on specific variations of the parameters. Practical applications of the research are presented in the form of algorithms and software for solution of vector optimization problem. The findings are illustrated by examples of numerical solution of price optimization problems at petrol outlets for Ural regional markets.

It is shown that the problem of vector optimization can be reduced to the problem of linear programming. The concept of convolution [7, 8, 9] is applied to components of vector criteria. Transition from the problem with a vector criterion to the equivalent problem with scalar a criterion is realized. For this purpose the so-called “economic” [8] method of convolution is used. As a result of such convolution the linear combination of scalar criteria with positive coefficients is obtained. This scalar criterion in the problem of linear programming has reasonable economic interpretations. This problem for real systems of petrol outlets has large dimension that leads to computational difficulties in numerical solution of the problem using traditional linear programming algorithms [4]. The results of the extensive numerical experiments show that the suggested numerical solution algorithm for the vector optimization problem is faster and more robust than the traditional methods.

Statement of a price optimization problem in an autonomous wholesale market

We study a local autonomous market of wholesale trades of the certain uniform goods (for example, some type of energy resources). It is supposed that this market consists of the finite number of interconnected points. Each point of the market has the following parameters:

- The goods price;
- The upper bound for the goods price.

Besides, the costs of the goods transportation from one point of the market to any other point are known.

The problem of setting the ideal optimal prices at the network points consists in the following. Suppose that the price restrictions, goods realization volumes at all points of the market and costs of the goods transportation from one point of the market to any other point are the fixed values. Under assumption that prices at some points are fixed it is necessary to find maximal possible price values at other points in such a way that the following conditions are fulfilled.

First, the price at each point of the market should not be higher than the allowed restrictions.

Second, for any pair of interconnected points the price at each of them should not exceed the price at another one plus transportation cost between them.

The first condition reflects, in particular, the operation of the market regulating mechanisms.

The second condition creates objective reasons for those consumers who are geographically or economically "fastened" to a concrete outlet of the market to acquire goods exactly at this outlet. Such reasons form a grounded base for the long-term planning of goods deliveries to the outlets of the market.

Terminology and notations

Let $V = \{v_i \in R^2 \mid i = 1,2,\ldots,n\}$ denote a finite set of points from the two-dimensional Euclidian space R^2, and E denote the set of the segments that connect part of pairs of different points from the set V. Thus, $E = \left\{e_{ij} = \left[v_i, v_j\right] \mid 1 \le i \le n,\ \ 1 \le j \le n \ : i \ne j\right\}$. Note, in the general case there exist two different points v_k and v_s in V such that they are not connected directly - set E does not contain elements e_{ks} and e_{sk} (see Fig 1). But it is assumed that V does not have isolated points. It means that for any point v_i in V this set contains at least one point v_j, $i \ne j$, such that segment e_{ij} belongs to E (see Fig 1). Let us note that one can consider all constructions in Euclidian spaces of larger dimension.

Let us consider a flat nonoriented graph *G=(V,E)*. A path that connects two arbitrary nodes of the graph *G* is called a chain [5]. The length of the chain is equal to the quantity of the edges that define the chain. A chain that consists of different nodes is called a simple path [5]. Any chain is identified with the set of its nodes $r_{i_0 i_l} = \left\{v_{i_0}, v_{i_{11}}, v_{i_2}, \ldots, v_{i_l}\right\}$ that are consequently connected by edges of the graph *G*. Thus, for a simple path we have $v_{i_j} \ne v_{i_k}$ for every $j,k = 0,1,2,\ldots,l$, $j \ne k$. The first node v_{i_0} in the chain $r_{i_0 i_l}$ is called the initial node of the chain $r_{i_0 i_l}$, and the last node v_{i_l} - the end node of the chain $r_{i_0 i_l}$. The length of the chain $r_{i_0 i_l}$ is equal to *l*.

Let us make the following assumption for the structure of graph *G*.

Assumption 1. Let us suppose that two arbitrary nodes in graph *G* can be connected by a simple path. Thus, graph *G* is a arcwise connected graph (see [5]) consisting of *n* interconnected nodes $v_i \in V$. Some of nodes are connected directly by edges $e_{ij} \in E$ and others can be connected by some simple paths (see Fig. 1).

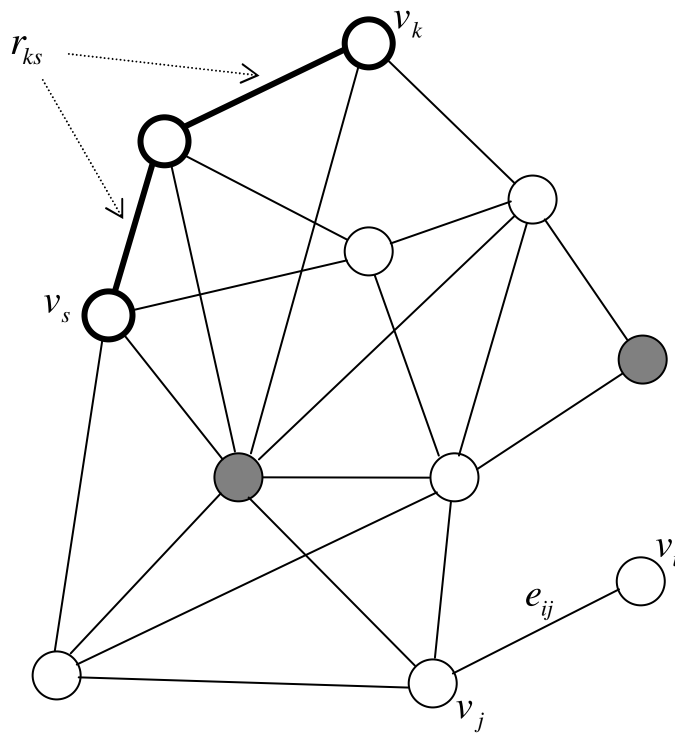

Fig. 1. An example of a flat graph with 11 interconnected nodes.

We assign numerical characteristics $0 \le p_i \le s_i$ and $c_{ij} > 0$ to all nodes v_i and edges e_{ij} of the graph *G*.

The graph *G* is viewed as a model of a local autonomous market of wholesale trades of a certain uniform goods. Really, let us suppose that the market consists of *n* interconnected outlets. Herewith only one connection can be established between two different points. This connection is interpreted as a channel of goods transportation from one point to another point (for example, car transportation, railway transportation, power transmission lines). We shall consider that the market does not contain isolated outlets. It means that each point is connected with at least one of others. Thus, under these assumptions such market can be interpreted as coherent arcwise connected nonoriented graph *G*. Each node of this graph is associated with some outlet, and a graph edge is interpreted as the channel of goods transportation between outlets (see Fig. 1). Here p_i is the price of the good, c_{ij} are the costs of transportation connection, and s_i denotes an upper bound for price p_i at node i.

According to the statement of the problem, the p_i values can be fixed at some points. The p_i values at other points can be arbitrary non-negative values which do not exceed the upper bounds s_i. It means that we fix the values of all parameters c_{ij}, s_i and the values of a part of parameters $p_k = s_k < +\infty$ $(k = m+1,\ldots,n,\ m \le n)$. All other parameters p_i $(i = 1,\ldots,m,\ m \le n)$ are variables, their values can change in corresponding ranges: $0 \le p_i \le s_i$. The nodes corresponding to variable prices will be called free nodes, and the nodes corresponding to fixed prices - basic nodes. Here we can consider the first *m* ($1 \le m \le n$) nodes as the free nodes (otherwise, the nodes of the graph can be renumbered).

Let us make the following assumptions regarding the transportation costs c_{ij}. First, there is a $c_0 > 0$ such that for all $i, j = 1,2,\ldots,n$ one has $c_{ij} \ge c_0$. Second, $c_{ij} = c_{ji}$.

Statement of a problem of vector optimization

Let us study the following problem of vector optimization on the graph *G*:

$$P \to \max_{P \in U} \tag{1.1}$$

Here $P = (p_1, p_2, \ldots, p_n)^T \in R^n$ is the system of prices, and the set $U \subseteq R^n$ of admissible solutions is defined by linear inequalities

$$p_i \le p_j + c_{ij} \quad \forall i = 1,2,\ldots,m, \forall j = 1,2,\ldots,n, \tag{1.2}$$

$$p_i \le s_i \quad \forall i = 1,2,\ldots,m, \tag{1.3}$$

$$p_i \ge 0 \quad \forall i = 1,2,\ldots,n, \tag{1.4}$$

$$p_j = s_j < +\infty \quad \forall j = m+1,\ldots,n. \tag{1.5}$$

The optimization problem (1.1) belongs to the class of linear problems of vector optimization (see, for example, [2]).

It is necessary to notice that in multicriteria optimization problems ideal solutions providing the maximum values to all criteria simultaneously can be found very rarely. But if it is found, then, naturally, this solution should be considered as an optimal one. Definitions of vector preferences with the help of effective solutions or Pareto optimal solutions lead to common and well-designed methods for solving problems of vector optimization. In what follows, vector inequalities $P \geq P^*$ are understood coordinate-wise.

Definition 1.1. The vector $P^* = \left(p_1^*, p_2^*, \dots, p_n^*\right)^T \in U$ is called a Pareto optimal solution of problem (1.1)-(1.5), if there is no $P = \left(p_1, p_2, \dots, p_n\right)^T \in U$ such that $P \geq P^*, P \neq P^*$.

From formulas (1.2)-(1.5) it follows that the objective space in problem (1.1) is convex and compact. Hence, the set U_P of the Pareto optimal solutions is not empty. It is known [9] that the Pareto set U_P has a property of external stability, which is formulated in the following definition.

Definition 1.2. We call the Pareto set U_P externally stable if for any $P \in U$ there exists an estimate $P^* \in U_P$ such that $P^* \geq P$.

Definition 1.3. We call a $P^* = \left(p_1^*, p_2^*, \dots, p_n^*\right)^T \in U$ an ideal solution of problem (1.1) (an ideal optimal plan of problem (1.1)) if for every $P = \left(p_1, p_2, \dots, p_n\right)^T \in U$ it holds that

$$P^* \geq P \tag{1.6}$$

The goal is to find the ideal optimal solution (see Definition 1.3) that maximizes prices at all outlets simultaneously under the specified conditions (1.2)-(1.5). Conditions for the existence and uniqueness of an ideal solution are formulated below.

Main properties of the optimization problem

We notice that if $s_i = +\infty$ for each $i = 1,2,\ldots,n$, then problem (1.1) degenerates under condition $m = n$. In this case the optimal values of the parameters p_i are reached at infinity ($p_i = +\infty,\ i = 1,2,\ldots,n$). This solution does not have any practical interest. Therefore, hereinafter we suppose that

there is a k : $1 \le k \le n$ such that $0 \le p_k \le s_k < +\infty$ (2.1)

Let us note that if the graph *G* has basic nodes, then for each of them, k, condition (2.1) is fulfilled.

Lemma 2.1. Under condition (2.1) the set $U \subseteq R^n$ defined by the system of linear inequalities (1.2.)-(1.4) is bounded, that is there is a $\bar{p} > 0$ such that $p_i \le \bar{p}$ for all $i = 1,2,\ldots,n$.

Proof. Let us consider an arbitrary node v_{i_0}. According to Assumption 1 there exists a simple path $r_{i_0 i_l} = \left\{ v_{i_0}, v_{i_{11}}, v_{i_2}, \ldots, v_{i_l} \right\}$ connected this node with a node v_{i_l} for which condition (2.1) (with $k = i_l$) is fulfilled, $0 \le p_{i_l} \le s_{i_l} < +\infty$. Such node can be either one of the basic nodes of the graph *G*, if such nodes exist, or at least one of free nodes. In the first case it is supposed that the considered chain contains one basic node v_{i_l} as the end node of the chain.

Obviously, the length of this chain satisfies inequality $l \le n-1$. Hence, taking into account (1.2), one can obtain the following relations

$$p_{i_0} \le p_{i_1} + c_{i_0 i_1} \le p_{i_2} + c_{i_1 i_2} + c_{i_0 i_1} \le p_{i_3} + c_{i_2 i_3} + c_{i_1 i_2} + c_{i_0 i_1} \le \ldots \le p_{i_l} + \sum_{j=0}^{l-1} c_{i_j i_{j+3}},$$

$$p_{i_0} \le s_{i_l} + (n-1)c_{\max}. \qquad (2.2)$$

Here $c_{\max} = \max_{i,j=1,\ldots,n} \left\{ c_{ij} \right\}$. Let us introduce the following parameter $\bar{p} = s_{\max} + (n-1)c_{\max}$, where $s_{\max} = \max_{i=1,\ldots,n} \left\{ s_i \mid s_i < +\infty \right\}$ for all $i = 1,2,\ldots,n$. The inequality (2.2) leads to relations $p_i \le \bar{p}$. Lemma is proved •

We derive a necessary condition of the optimality of a plan P^* from relations (1.2).

Lemma 2.2. (Necessary condition of optimality)

Let $P^* = \left(p_1^*, p_2^*, \ldots, p_n^*\right)^T \in U$ be an ideal optimal plan for problem (1.1). Then for any $i = 1,2,\ldots,m$ such that $p_i^* < s_i$ there is such $j = 1,2,\ldots,n$ $(j \neq i)$ that

$$p_i^* = p_j^* + c_{ij}. \tag{2.3}$$

Proof. Assume the contrary.

Let there be $i = 1,2,\ldots,m$ such that $p_i^* < s_i$ and $p_i^* < p_j^* + c_{ij}$ for all $j = 1,2,\ldots,n,\ j \neq i$.

The coherence of the graph G suggests that $0 < \Delta p_i < +\infty$, where $\Delta p_i = \min\left\{ s_i - p_i^*,\ \min_{j \neq i}\left\{ p_j^* + c_{ij} - p_i^* \right\}\right\}$. Let us consider a vector $P_\Delta = \left(p_1^*, \ldots, p_{i-1}^*, p_i^* + \Delta p_i, p_{i+1}^*, \ldots, p_n^*\right)^T$. It is easy to see that $P_\Delta \in U$. This contradicts to the optimality of P^* since $\hat{p}_i = p_i^* + \Delta p_i > p_i^*$. Lemma is proved •

Let us introduce the following definitions.

Definition 2.1. A chain $r_{i_0 i_l} = \left\{ v_{i_0}, v_{i_{11}}, v_{i_2}, \ldots, v_{i_l} \right\}$ which connects the nodes v_{i_0} and v_{i_l} of the graph *G* is called a limited path if $p_{i_k} = p_{i_{k+1}} + c_{i_k i_{k+1}}$ for all $0 \le k \le l-1$.

One can indicate the following useful properties of limited path.

- Any limited path $r_{i_0 i_l} = \left\{ v_{i_0}, v_{i_{11}}, v_{i_2}, \ldots, v_{i_l} \right\}$ is a simple path.

Indeed, for a limited path we have the following relations

$$p_{i_0} = p_{i_1} + c_{i_0 i_1} = p_{i_2} + c_{i_1 i_2} + c_{i_0 i_1} = p_{i_3} + c_{i_2 i_3} + c_{i_1 i_2} + c_{i_0 i_1} = \ldots = p_{i_l} + \sum_{j=0}^{l-1} c_{i_j i_{j+3}}.$$

If we assume on the contrary to condition of the simple path that $v_{i_0} = v_{i_l}$ then the equality $p_{i_0} = p_{i_0} + \sum_{j=0}^{l-1} c_{i_j i_{j+3}}$ leads to $\sum_{j=0}^{l-1} c_{i_j i_{j+3}} = 0$. It contradicts to the positiveness of prices of connections. •

- The limited path $r_{i_0 i_l} = \{v_{i_0}, v_{i_{11}}, v_{i_2}, \ldots, v_{i_l}\}$ is an oriented chain (see [5]).

This means that chain $r_{i_l i_0} = \{v_{i_l}, v_{i_{l-1}}, v_{i_{l-2}}, \ldots, v_{i_0}\}$ is not a limited path. The proof of this property is analogous to the proof of the previous one. •

Definition 2.2. A limited path $r_{i_0 i_l} = \{v_{i_0}, v_{i_{11}}, v_{i_2}, \ldots, v_{i_l}\}$ is called a basic limited path if $p_{i_l} = s_{i_l}$ and $p_{i_j} < s_{i_j}$ for all $j = 0,1,2,\ldots,l - 1$.

On the basis of Lemmas 2.1, 2.2 one can formulate and prove the conditions of existence and uniqueness for an ideal optimal plan P^* in the problem (1.1).

Theorem 2.1. (Existence and uniqueness of the ideal solution)

Let the conditions of the lemma 2.1 be met. Then problem (1.1) has the unique ideal solution (the ideal optimal plan P^*).

Proof. We prove the theorem in two stages. First, we prove the existence of the stationary point in the problem (1.1) that meets the necessary condition of optimality (2.3). Then we prove the uniqueness of such point and show that this point is the ideal solution.

1. Existence of a stationary point.

Let us construct the points sequence $\Sigma = \left\{P^{(m)} = \left(p_1^{(m)}, p_2^{(m)}, \ldots, p_n^{(m)}\right)^T\right\}_{m=0}^{\infty} \subset R^n$ in the following way. The initial point is defined as $P^{(0)} = (0,0,\ldots,0)^T \in U$. All other elements of the sequence are defined by the recursive relation

$$P^{(m+1)} = P^{(m)} + \Delta P^{(m)}, \quad m = 0,1,2,\ldots,$$

where $\Delta P^{(m)} = (0,\ldots,0,\Delta p_k,0,\ldots,0)^T$ for all $m \geq 0$ and

$$\Delta p_k = \max_{k=1,\ldots,m}\left\{\max_{j=1,\ldots,n}\left\{\min\left\{s_k - p_j^{(m)}, p_j^{(m)} + c_{jk} - p_k^{(m)}\right\}\right\}\right\}, \quad \Delta p_k \geq 0. \tag{2.4}$$

These formulas have the following sense. Each successor $P^{(m+1)}$ of the sequence is a result of such a variation of the antecedent $P^{(m)}$ that leads to increasing of the value of only one coordinate of the point $P^{(m)}$. According to (2.4) the number k of this coordinate and the increment value Δp_k are defined in the following way. Number k corresponds to such free node v_k of the graph G where the current value of its $p_k^{(m)}$ can be increased more than in other nodes. Herewith the value Δp_k is defined as the maximum possible value that meets the condition $P^{(m+1)} = P^{(m)} + \Delta P^{(m)} \in U$.

As a result, we obtain the sequence satisfying the condition $P^{(m)} \in U$ and $P^{(m+1)} \geq P^{(m)}$ for all $m \geq 0$. Due to the completeness of the space R^n and the boundedness of the objective set $U \subseteq R^n$ there exists the limit $\lim_{m\to\infty} P^{(m)} = P^* \in U$. Herewith, $\lim_{m\to\infty} \Delta P^{(m)} = (0,0,\ldots,0)^T$ due to the definition of the sequence Σ.

Let us prove that P^* is the unique ideal optimal plan of the problem (1.1).

2. Optimality and uniqueness of the plan P^*.

Let us prove that plan P^* meets the necessary condition of optimality (2.3) in the problem (1.1). If it is not so then according to the rule (2.4) one can find such $\Delta P > 0$ that $P^* + \Delta P > P^*$. It contradicts to the fact that P^* is the limit of the sequence Σ.

Hence, a limited path $r_{i_0 i_l} = \left\{v_{i_0}, v_{i_{11}}, v_{i_2}, \ldots, v_{i_l}\right\}$, connecting a node v_{i_0} with a node v_{i_l}, does exist for each free node v_{i_0} $(1 \leq i_0 \leq m)$ of the graph G, where $p_{i_0}^* < s_{i_0}$.

One can prove that such limited path $r_{i_0 i_l} = \left\{v_{i_0}, v_{i_{11}}, v_{i_2}, \ldots, v_{i_l}\right\}$ can be chosen as a basic limited path. If it is not so then either P^* is not the limit of the sequence Σ or the limited path $r_{i_0 i_l}$ is not a simple path.

Really, properties of the oriented chain $r_{i_0 i_l}$ lead to inequalities

$$p^*_{i_s} < p^*_{i_{s-k}} + \sum_{j=s}^{s-k+1} c_{i_j i_{j-1}} \text{ for all } s,k = 1,2,\ldots,l.$$

In particular,

$$p^*_{i_l} < p^*_{i_{l-1}} + c_{i_s i_{l-1}}. \tag{2.5}$$

If v_{i_l} is a free node and $p^*_{i_l} < s_{i_l}$ then a limited path $r_{i_0 i_l}$ can be continued. It means that one more node $v_{i_{l+1}}$ can be added to this limited path in such a way that the resulting chain $r_{i_0 i_{l+1}} = \{v_{i_0}, v_{i_{11}}, v_{i_2}, \ldots, v_{i_l}, v_{i_{l+1}}\}$ will be a limited path. Otherwise, from the inequality (2.5) we obtain automatically the contradiction with the definition of P^* as the limit of the sequence Σ. The same arguments for the node $v_{i_{l+1}}$ contradict either to the fact that P^* is the limit of the sequence Σ or to the fact that a limited path $r_{i_0 i_l}$ is a simple path because the graph G has a finite number of nodes.

Now we prove that the plan P^* meets the condition (1.6). Let us suppose the opposite. Let us consider such a point $\tilde{P} = \left(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n\right)^T \in U$ where there is $1 \le i_0 \le m$ such that $\tilde{p}_{i_0} > p^*_{i_0}$.

Now let us consider the basic limited path $r_{i_0 i_l} = \{v_{i_0}, v_{i_{11}}, v_{i_2}, \ldots, v_{i_l}\}$. Then the conditions $p^*_{i_k} = p^*_{i_{k+1}} + c_{i_k i_{k+1}}$, $\tilde{p}_{i_k} \le \tilde{p}_{i_{k+1}} + c_{i_k i_{k+1}}$ (for all $0 \le k \le l-1$) lead to inequalities $\tilde{p}_{i_s} > p^*_{i_s}$ (for all $s = 0,1,\ldots,l$). We obtain the contradiction with the fact that $\tilde{p}_{i_l} \le s_{i_l} = p^*_{i_l}$.

The proof of the uniqueness result for the plan P^* is analogous.

The theorem is proved. •

Let us make the following remarks.

First, the proof of the theorem 2.1 suggests a constructive numerical algorithm for solving a vector optimization problem. A modification of this algorithm is described in details below.

Second, from the proof of this theorem one can obtain a sufficient condition of optimality in the problem (1.1). A sufficient condition of optimality for plan P^* consists in the existence of a basic limited path for each free node as an initial node of this path.

The following theorem can be proved analogously to the proof of the theorem 2.1.

Theorem 2.2. The problem (1.1) is equivalent to the linear programming problem

$$f(P) = \sum_{i=1}^{n} h_i p_i \quad \to \quad \max_{P \in U}, \tag{2.6}$$

for arbitrary positive vector $H = (h_1, h_2, \ldots, h_n)^T \in R^n$ such that $h_i > 0$ for all $i = 1,2,\ldots,n.$

Koopmans [11] was the first who obtained the analogous result of reducing a specialized linear problem with a vector criterion to a standard linear programming problem by means of convolution technique. Charnes and Cooper [12] obtained the analogous result with the help of duality theorems for linear programming problems. Later, the equivalence of a linear vector optimization problem and a linear programming problem whose objective function is presented as a sum of scalar coordinates with positive coefficients was established in the paper by Bod, Focke, Isermann, Podinovskii [9].

The scalar criterion of the problem (2.6) has the following substantive interpretation. If $h_1 = \ldots = h_n = \frac{1}{n}$ then the value $f(P)$ is the average price on goods at an autonomous market. Then the problem (2.6) is a problem of maximization of the average market price.

The ideal solution of the problem (1.1) can be found as the solution of the standard problem of linear programming (2.6). The traditional methods for solving linear programming problems, for example, simplex-method, can be used here.

Let us consider a canonical form of the linear programming problem (2.6). We calculate the dimension of this problem in some special situations. Let us make the following assumptions.

- All nodes of the graph are free;

- The upper bound for price *p* exists at each node *v* ($m = n$, $0 \le p_i \le s_i < +\infty$ for all $i = 1,\ldots,n$);

Under these assumptions the system (1.2) contains $l = 2\times\sum_{k=1}^{n-1} k$ inequalities, and the system (1.3) contains *n* inequalities. Then the objective set *U* in the problem (1.1) is defined (without taking into account conditions (1.4)) with the help of $l + n$ inequalities of "$\le$" type. Hence the mentioned canonical problem of linear programming includes the system of $l + n$ equations with $l + 2n$ unknowns.

When $n = 5$ the matrix dimension of this system equals 25x30. If $n = 10$ then the dimension equals 100x110. If $n = 20$ then it is 410x420. If $n = 30$ then such dimension is more than 1000x1000.

In many situations of practical importance the quantity of nodes can reach several tens, in some situations – several hundreds. In such cases the numerical solution of the multicriteria optimization problem by using linear programming algorithms, in particular, – simplex-method [4], requires operations over matrices of large dimension. As a rule, the realization of these operations deals with computational difficulties (for example, ill-stipulation of the system, computing inaccuracy, high requirements to computer resources).

Using specifics of the problem we work out a sufficiently easy-to-actualize, robust and fast numerical procedure for solving the optimization problem (1.1).

Solution algorithm of the vector optimization problem

The suggested numerical algorithm is a special iterative procedure that allows to determine the solution of the problem (1.1) by the finite number of steps.

We make the following simple observations. Let us choose an arbitrary node. Let k $(1 \le k \le m)$ be the index of the selected node (see Fig. 2). Let $\{k_1, k_2, \ldots, k_s\}$ denote the set of indices of the free nodes connected with this node (see Fig. 2). Let us select from the system (1.2)-(1.3) the inequalities that include the parameter p_k. As a result, we obtain the following system of inequalities

$$p_k \le p_{k_j} + c_{kk_j} \quad \forall j = 1,2,\ldots,s \tag{3.1}$$

$$p_{k_j} \le p_k + c_{k_j k} \quad \forall j = 1, 2, \ldots, s \tag{3.2}$$

$$p_k \le s_k \tag{3.3}$$

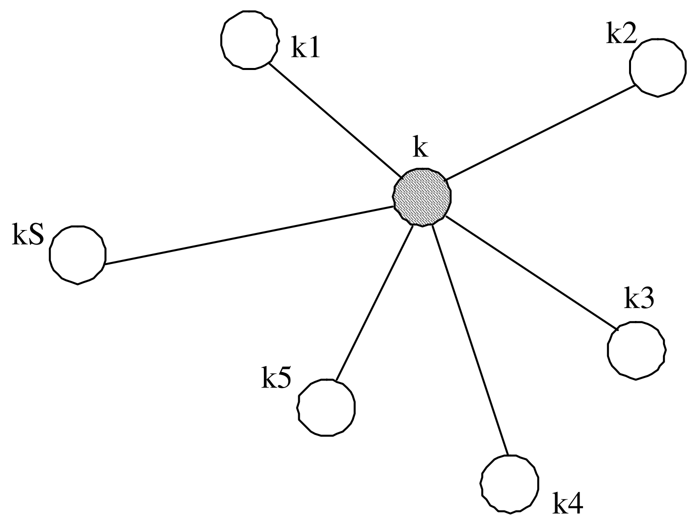

Fig. 2. The *k*-th node of the graph and its “neighborhood”.

The inequalities (3.1)-(3.3) show that for current values of the variables $p_{k_1}, p_{k_2}, \ldots, p_{k_s}$ the maximal value $\overline{p_k}$ of the variable p_k can be calculated according to the following rule

$$\overline{p_k} = \min\left\{s_k, \min_{1 \le j \le s}\left(p_{k_j} + c_{kk_j}\right)\right\}. \tag{3.4}$$

The relations (3.1)-(3.4) indicate the following tendency. The increasing of one variable gives the possibility to increase other variables. This tendency entirely corresponds to the purposes of the considered problem solution (1.1).

These observations form a basis of the iterative numerical procedure for calculation of the vector $P^* = \left(p_1^*, p_2^*, \ldots, p_n^*\right)^T$ - the ideal optimal plan of the problem (1.1). The main idea consists in an iterative maximum increment of price values p_k $(k = 1, \ldots, n)$ at nodes of the graph from the initially admissible values until this increment is possible.

We suggest the following iterative numerical procedure for calculation of the ideal optimal plan $P^* \in U$.

1. Let us fix the initial approximation P_0 for the point P^* as $P_0 = (0,0,\ldots,0)^T$. Obviously, that $P_0 \in U$. Assume $k = 0$.
2. The transformation (3.4) is performed for the current point $P_k = \left(p_1^{(k)}, p_2^{(k)},\ldots, p_m^{(k)}\right)^T$. It means that the values of all coordinates of this point are consequently recalculated by the formula (3.4). Herewith the value of each following coordinate is calculated taking into account the "new" values of all preceding coordinates. When the above cycle through coordinates of the point is completed then the following value is calculated

$$\Delta p_k = \max_{1 \le i \le m} \left(\overline{p_i^{(k)}} - p_i^{(k)} \right) \tag{3.5}$$

3. The next approximation P_{k+1} for the point P^* is defined by formula $P_{k+1} = \left(p_1^{(k+1)}, p_2^{(k+1)},\ldots, p_m^{(k+1)}\right)^T = \left(\overline{p_1^{(k)}}, \overline{p_2^{(k)}},\ldots, \overline{p_m^{(k)}} \right)^T$.
4. The condition of the termination of calculation is checked

$$\Delta p_{k+1} \le \varepsilon , \tag{3.6}$$

Here $\varepsilon \ge 0$ is a defined in advance accuracy of the P^* calculation.

If the condition (3.5) is met then we accept $P^* = P_{k+1}$. Otherwise, the procedure of calculating the following approximation of the ideal optimal point described above is repeated from the second step.

One can prove that any required accuracy (condition (3.6)) can be reached in a finite number of iterations.

It is necessary to notice that iterative procedures are often used for solving economic problems. They are applied, like in this work, to the problem of pricing. Here one can refer to the Walras "tatonnement" process [3, 10]. It is a well-known iterative procedure in economic theory [13] for defining balance prices at a competitive market.

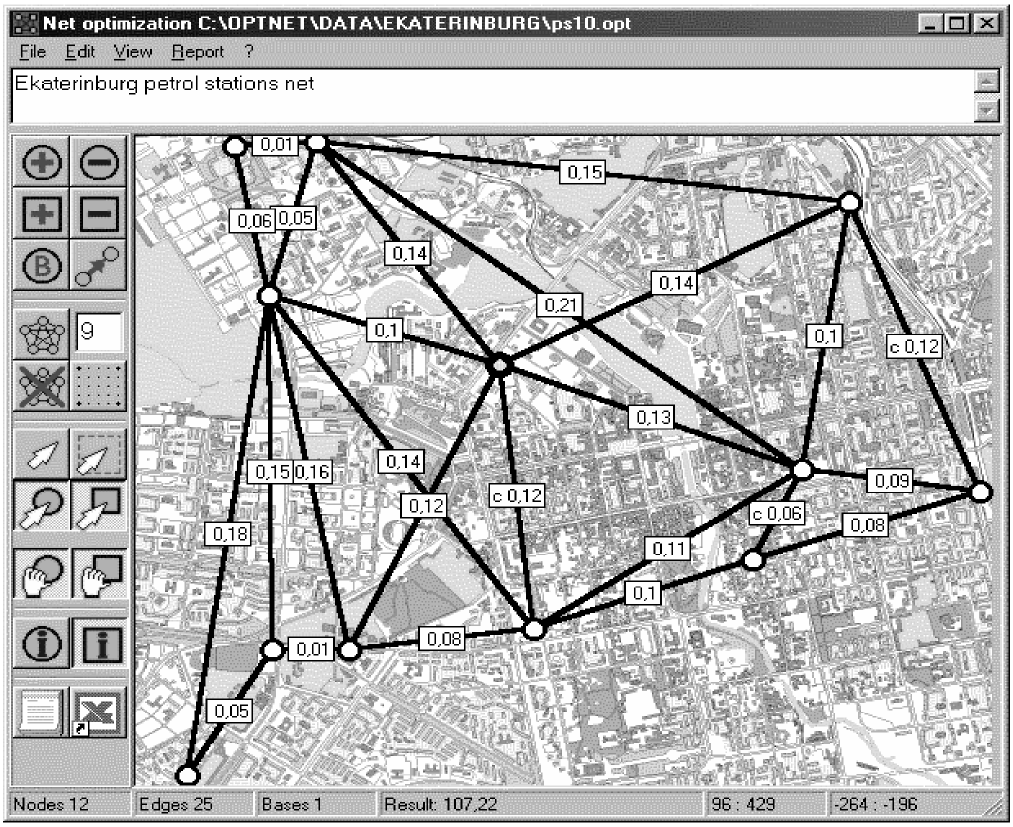

Fig.3. The network of petrol stations in the central part of Ekaterinburg.

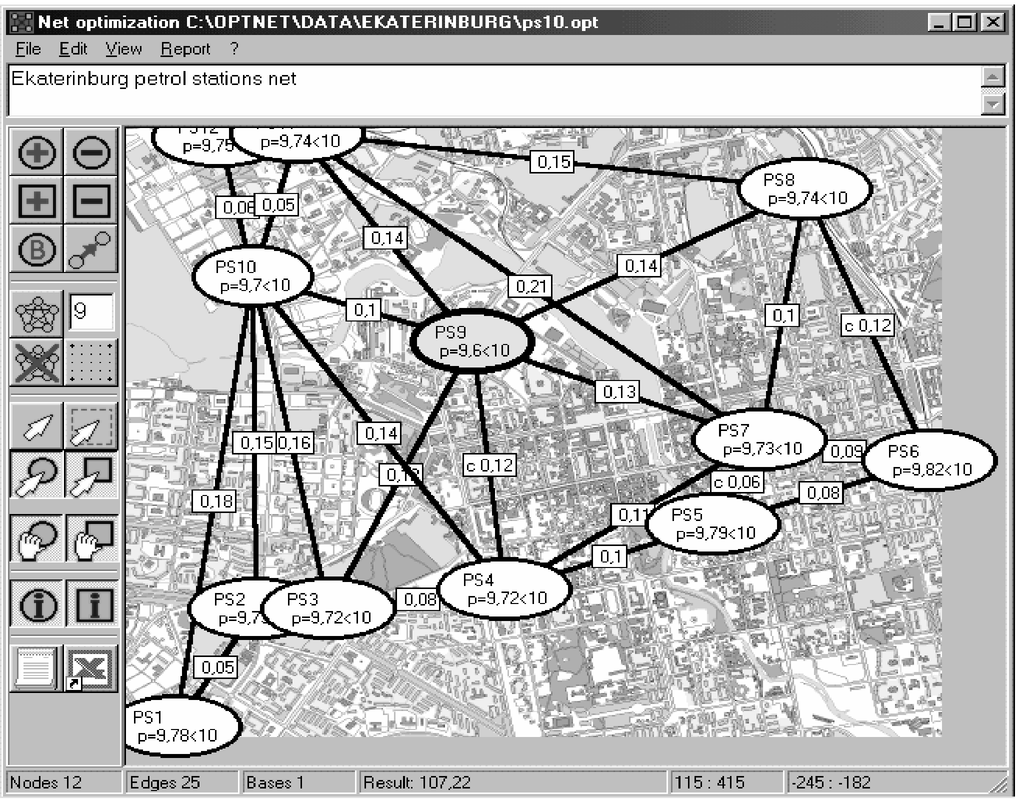

Fig.4. Results of prices optimization for a network of petrol stations.

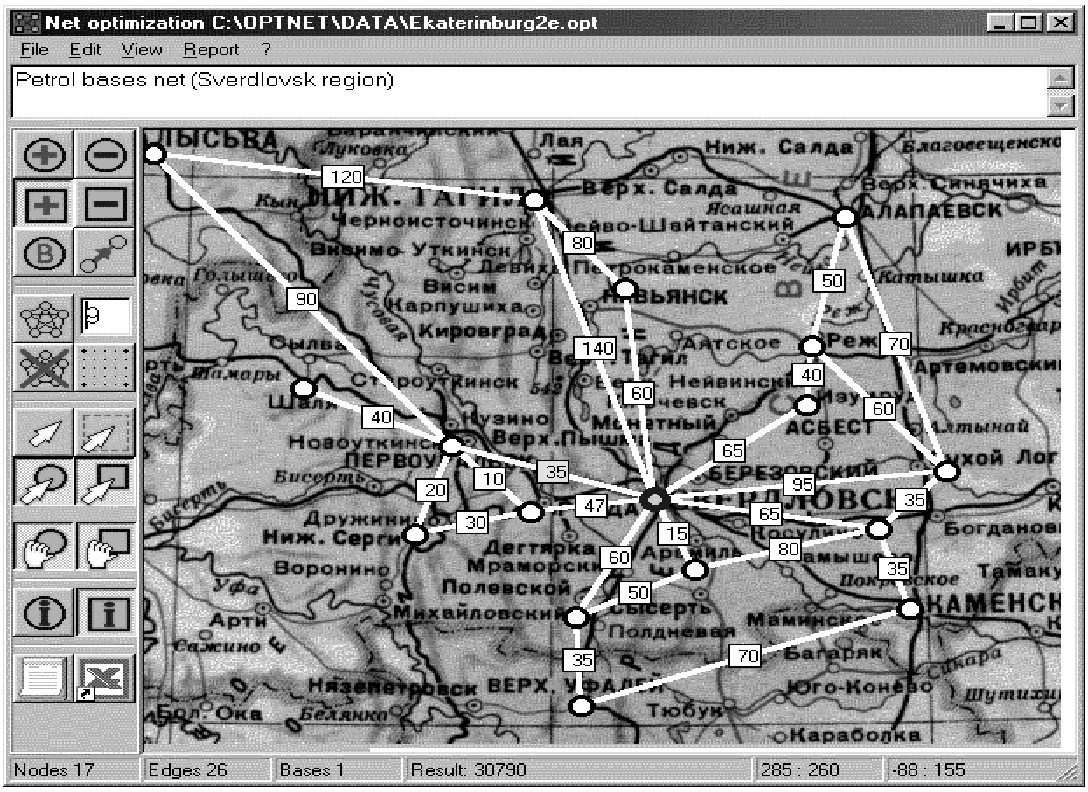

Fig.5. A fragment of the network of petrol distribution terminals in Sverdlovsk region.

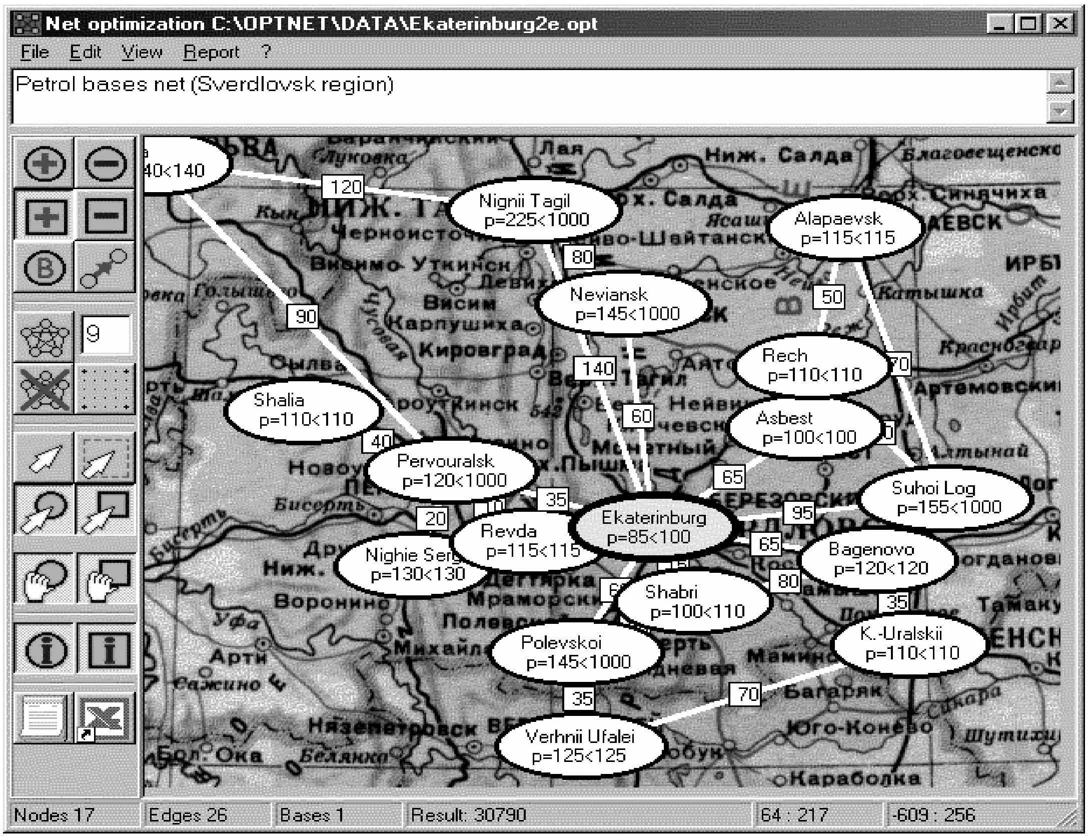

Fig.6. Results of prices optimization at petrol distribution terminals.

Results of numerical simulation

Real data about local petrol markets in Ekaterinburg (Fig. 3,4) and Sverdlovsk region are used in the numerical simulation.

The structure of investigated markets (outlets and connections between them) is illustrated below. The Figures 3-6 present the values of prices of connections between the outlets of the market, the optimal prices on goods at them. The presented results are obtained with the help of the algorithm described above.

Figures 3,4 reflect the results of simulation at the microlevel (the scale of Ekaterinburg). The results of simulation at the macrolevel (the scale of Sverdlovsk region) are presented on Figures 5,6.

The network of petrol stations (PS) located in the central part of Ekaterinburg is shown on Fig. 3. There are twelve stations in this network. The prices of connections between PS are also presented on Fig. 3. The Figure 4 illustrates the results of prices optimization in PS network. Each ellipse on this Figure contains the name of the corresponding PS (for example, "PS4"), the value of the maximum price on petrol at the PS and the upper bound for price ("p=9,72<10"). The outlet "PS9" is the unique basic node of the graph, all other outlets are free nodes.

Conclusions

The model of price optimization at an autonomous market is considered in the paper. The proposed model can be used for solving actual management problems such as deliveries controlling problem. The real values of the parameters of this model are liable to random disturbance. The parameters of the model result from statistical data. The main factors are estimated by means of econometric analysis.

A specialized software is developed for practical approbation of the algorithm for solving multicriteria optimization problem. Traditional numerical methods for solving linear programming problems (in particular, simplex-method) are implemented in this software. The algorithm described in the paper has been tested on the real data of petrol stations (petrol distribution terminals) of Sverdlovsk region in Russia. The results of a extensive numerical experiments show the adequacy of the obtained results and the real data.

Comparative analysis demonstrates that the proposed algorithm is more effective than the simplex-method in the problems of high dimension. The results of the wide numerical experiment also show the great robustness of the numerical procedure with respect to disturbances of the model parameters.

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