RM-77-30

MORE COMPUTER PROGRAMS FOR SPATIAL DEMOGRAPHIC ANALYSIS

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Preface

Interest in human settlement systems and policies has been a critical part of urban-related work at IIASA since its inception. Recently this interest has given rise to a concentrated research effort focusing on migration dynamics and settlement patterns. Four sub-tasks form the core of this research effort:

I. the study of spatial population dynamics;

II. the definition and elaboration of a new research area called demometrics and its application to migration analysis and spatial population forecasting;

III. the analysis and design of migration and settlement policy;

IV. a comparative study of national migration and settlement patterns and policies.

As part of the comparative study of migration and settlement, IIASA is developing a set of computer programs for spatial demographic analysis. A first set of programs has already been published (RM-76-58). This paper presents another set --- one focusing on the analysis of stationary and stable multiregional populations.

Related papers in the comparative studies series, and other publications of the migration and settlement study, are listed on the back page of this report.

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Abstract

This report presents the algorithms and lists the FORTRAN IV codes of computer programs for the analysis of multiregional population systems. It is a continuation of the IIASA report RM-76-58. The following topics are included: mobility and fertility analyses of life table and stable populations; methodology and applications of the spatial reproductive value; and the study of the spatial demographic impacts of fertility reduced to replacement level. This report focuses on the interpretation of the output of the computer programs.

Acknowledgements

The numerous reactions to our first set of computer programs (RM-76-58) were extremely helpful for the preparation of this report. In particular, we acknowledge the detailed comments of Tom Carrol, Richard Raquillet and Philip Rees. We also hope that this report will provoke reactions and suggestions that might improve the user-orientation of the computer programs.

During the development of the programs, we have benefited from the assistance of IIASA's Computer Services. We are especially indebted to James Curry and Mark Pearson for their advice and for solving our software problems.

We also are grateful to Jacques Ledent and Richard Raquillet who read an earlier version of this report and suggested several improvements.

The burden of typing the successive drafts of this report was borne by Linda Samide, Elisabeth Grandville, Marina Hornasek and Sonja Selwyn, in chronological order. We appreciate with many thanks the skills and efforts they devoted to this report, in particular the contribution of Sonja who typed the final version.

E R R A T A

Willekens F., and Rogers A.,
Computer Programs for Spatial Demographic Analysis
RM-76-58, July 1976

1. p. 21, eq. 12: $_{10}\ell_2(10) = {}_{10}\ell_{22}(5) + {}_{10}\ell_{12}(5)$

2. p. 21, eq. 13: $_{10}\ell_2(10) = {}_{10}\ell_2(5)\ p_{22}(5) + {}_{10}\ell_1(5)\ p_{12}(5)$

3. p. 28, eq. 28: $\underset{\sim}{e}(x) = \underset{\sim}{T}(x) \left[\underset{\sim}{\overline{\ell}}(x)\ \underset{\sim}{\ell}^{-1}(0)\right]^{-1}$,

   where $\overline{\ell}(x)$ is the diagonal matrix with elements $\underset{\sim}{\{1\}}'\underset{\sim}{\ell}(x)$, or $\sum_j {}_{io}\ell_j(x)$.

4. p. 28, bottom: $\underset{\sim}{e}(10) = \underset{\sim}{T}(10) \left[\underset{\sim}{\overline{\ell}}(10)\ \underset{\sim}{\ell}^{-1}(0)\right]^{-1}$

5. p. 30, top:

$$\begin{bmatrix} 57.145912 & 0.897136 \\ 7.702079 & 63.154427 \end{bmatrix} = \begin{bmatrix} 55.264362 & 0.799076 \\ 7.448485 & 56.251442 \end{bmatrix} \begin{bmatrix} 0.96707 & 0 \\ 0 & 0.89070 \end{bmatrix}^{-1}$$

where the matrix inverse is $\left[\underset{\sim}{\overline{\ell}}(10)\ \underset{\sim}{\ell}^{-1}(0)\right]^{-1}$.

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More Computer Programs for Spatial Demographic Analysis

One of the objectives of the Migration and Settlement Study at IIASA is to develop a package of computer programs for spatial demographic analysis. The reasoning has been that a basic requirement for an effective policy regarding the growth and the distribution of the population is a well-developed understanding of spatial population dynamics. Such an understanding is enhanced if the analyst and the policy-maker are provided with a ready tool for analysis, one which encompasses both the existing methodological knowledge and the computational procedures necessary to implement the methodology. This tool is a set of computer programs.

A first set of programs for spatial demographic analysis has already been published (Willekens and Rogers, 1976). They include the computation of the multiregional life table and the projection of a multiregional population system forward in time until it stabilizes. This paper focuses on the analysis of stable populations.[1] It consists of seven sections. The first section focuses on the basic input for the analysis: the age and regional distribution of the population. Demographers use three types of population distributions: the observed population distribution, the stationary population distribution, as expressed by the life table, and the stable population distribution. For each type of distribution fertility, mortality and mobility analyses may be performed. This is the task of sections two to five. Sections two and four deal with fertility analysis; section three treats mobility; and section five derives interesting stable population characteristics. The sixth section studies the spatial consequences of a sudden drop of fertility

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[1] A <u>stable</u> population is a population in steady-state equilibrium. It is a zero-growth population only if the stable rate of growth is zero.

to replacement level. The characteristics of spatial ZPG-populations are derived numerically and analytically. The last section of the paper presents a user-oriented description of the computer programs. The actual listings of the programs are contained in the Appendix.

This paper focuses on the interpretation of the output of the computer programs. All numerical illustrations refer to the same real two-region system: Slovenia and the Rest of Yugoslavia. The demographic data on which the computations are based refer to the year 1961 and are given in Rogers (1975a). The same example has been used to illustrate the previous programs (Willekens and Rogers, 1976). The multiregional life table and the stable population computed there are used as input information in this paper.

1. THE POPULATION DISTRIBUTION BY AGE AND REGION

The dynamics of a multiregional population system are governed by fertility, mortality and migration. Age-specific rates of fertility, mortality and migration are the fundamental components of demographic analysis (Rogers and Willekens, 1976c). They determine not only the growth of the population, but also (in the long run) its age composition, spatial distribution, and crude rates.

The observation that a unique combination of age-specific rates results in a particular age and regional composition has induced demographers to read in each population distribution a particular sequence of vital rates. "The demographic history of a population is inscribed in its age distribution" (Keyfitz, et al., 1967, p. 862; see also Namboodiri, 1969). For example, an observed population distribution (population pyramid) reflects periods of high fertility (baby boom) and high mortality (wars). A particularly useful way for understanding how the age and regional structure of a population is determined, is to imagine a particular distribution as describing a population which has been subjected to constant fertility, mortality and migration schedules for a prolonged period of time. The population that develops under such circumstances is called a stable multiregional population.

We may now reverse the procedure and derive the population distribution that would evolve if the actual observed schedules would remain unchanged for a prolonged period of time. This is the stable population associated with the observed demographic behavior. It is obvious that the age-specific rates do not remain constant and therefore that a stable population will never be realized. However, the stable population is a concept that enables us to look behind observed rates to explore what is hidden in the current fertility, mortality, and migration behavior. It shows where the system is heading, in the long run, under the current demographic forces. Keyfitz (1972, p. 347) compares stable population analyses with "microscopic examinations" because they magnify the effects of differences in current rates and therefore show more clearly their true meaning. Rogers (1971, p. 426) and Coale (1972, p. 52) compare these to "speedometer readings" to emphasize their monitoring function and hypothetical nature.

In addition to the observed population distribution and the stable population distribution associated with the observed fertility, mortality and migration schedules, demographers usually consider a third population distribution, namely the distribution of the life table population. The multiregional life table is a device for exhibiting the mortality and mobility history of an arbitrary birth cohort or radix. The representation and interpretation of life table and stable populations will now be discussed in some more detail.

1.1 The Life Table Population

The population distribution that results from applying given mortality and migration schedules to regional radices is represented by $\underset{\sim}{\ell}(x)$ and $\underset{\sim}{L}(x)$ of the life table (Rogers, 1975a). The matrix $\underset{\sim}{\ell}(x)$ represents the distribution of the population of exact age x, whereas $\underset{\sim}{L}(x)$ denotes the distribution of the population in age group x to x + h, with h being 5 (age intervals of 5 years). The matrix $\underset{\sim}{\ell}(x)$ will be used in the continuous models, and $\underset{\sim}{L}(x)$ for the discrete approximations of the continuous models. For example, for a two-region system,

$$\underset{\sim}{L}(x) = \begin{bmatrix} {}_{10}L_1(x) & {}_{20}L_1(x) \\ {}_{10}L_2(x) & {}_{20}L_2(x) \end{bmatrix} . \qquad (1.1)$$

For unit regional radices, an element ${}_{i0}L_j(x)$ denotes the number of people of region j in age group x to x + 4, who were born in region i, per unit birth in i.[2] For arbitrary radices

$$\{\underset{\sim}{Q}^a\} = \begin{bmatrix} Q_1^a \\ Q_2^a \end{bmatrix} , \qquad (1.2)$$

the number of people in region j between ages x and x + 5 and born in i is ${}_{i0}L_j(x)\ Q_i^a$, and in general $\underset{\sim}{L}(x)\{\underset{\sim}{Q}^a\}$. $\underset{\sim}{L}(x)$ and its elements are computed for unit radices. The absolute number of people in each age group and region is found by multiplying $\underset{\sim}{L}(x)$ by the given vector of radices $\{\underset{\sim}{Q}^a\}$.

Note that $\underset{\sim}{L}(x)$ represents the relative population distribution by place of residence and place of birth. Instead of being expressed in percentages (fractions of the total), or in some other manner, the population is given in unit births. This is a logical procedure in demography since it separates the fertility component from the survivorship (mortality and migration) component. It will become clear later that this is

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[2] An equivalent interpretation, which is more suited for life table construction is the "person-years lived" interpretation. In this sense ${}_{i0}L_j(x)$ is the number of years expected to be lived in region j between ages x to x + 5 by a person born in region i.

also a very convenient way of "norming" in spatial population analysis.

Table 1 gives the distribution of the observed, life table and stable populations of the one-sex (female), two-region system: Slovenia-Rest of Yugoslavia, 1961. The observed population is given by place of residence. The life table population is computed by applying the 1961 schedules of mortality and migration to unit radices. The computation is part of the construction of multiregional life tables. (Table 1b is identical to Table 8 of Willekens and Rogers (1976, p. 25)[3]. To derive the population by place of residence, and the aggregate population, one must introduce the radices $\{\underset{\sim}{Q}^a\}$.

Table 1. population distribution by age and region
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Table 1a. observed population (by place of residence)
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	slovenia	r.yugos.
0	67800.	847900.
5	74100.	905200.
10	70700.	808100.
15	60100.	617400.
20	62900.	725500.
25	66500.	774000.
30	67100.	728400.
35	62900.	633300.
40	39500.	392400.
45	47900.	437100.
50	51300.	453800.
55	46100.	389300.
60	39600.	325800.
65	29500.	230600.
70	21700.	180000.
75	14400.	120900.
80	7100.	61200.
85	3600.	39300.
total	832800.	8670200.

[3]The multiregional life table is computed using the Rogers-Ledent Method (see Willekens and Rogers, 1976, pp. 33-36). As a consequence, the numerical results shown in this paper deviate slightly from those of Rogers (1975) and Rogers and Willekens (1976b), which used the so-called Option 1 method.

Table 1b. life table population

initial region of cohort slovenia

	total	slovenia	r.yugos.
0	4.922968	4.890209	0.032759
5	4.840654	4.748097	0.092557
10	4.831723	4.694913	0.136810
15	4.821670	4.609031	0.212639
20	4.805876	4.456621	0.349255
25	4.787288	4.303031	0.484256
30	4.765104	4.187569	0.577535
35	4.736290	4.107964	0.628327
40	4.696336	4.048200	0.648136
45	4.630364	3.978028	0.652337
50	4.527514	3.880381	0.647134
55	4.376385	3.743020	0.633365
60	4.146075	3.540382	0.605693
65	3.760088	3.203668	0.556420
70	3.139220	2.665276	0.473945
75	2.327638	1.964904	0.362734
80	1.399027	1.160924	0.238103
85	0.962249	0.720453	0.241796
total	72.476471	64.902672	7.573801

initial region of cohort r.yugos.

	total	slovenia	r.yugos.
0	4.734203	0.003152	4.731051
5	4.460945	0.008093	4.452853
10	4.448177	0.011574	4.436604
15	4.433303	0.020462	4.412841
20	4.410328	0.037170	4.373158
25	4.379682	0.051365	4.328318
30	4.343516	0.059095	4.284421
35	4.302426	0.064056	4.238370
40	4.249906	0.067394	4.182512
45	4.178973	0.069071	4.109902
50	4.073606	0.069602	4.004004
55	3.914996	0.070336	3.844660
60	3.667310	0.070187	3.597124
65	3.285549	0.066386	3.219163
70	2.729682	0.056808	2.672875
75	2.034671	0.042728	1.991944
80	1.301538	0.026060	1.275479
85	1.296852	0.016784	1.280068
total	66.245674	0.810323	65.435349

Table 1c. stable population (growth rate = 0.006099)

initial region of cohort slovenia

	total	slovenia	r.yugos.
0	4.848469	4.816206	0.032263
5	4.624202	4.535784	0.088418
10	4.477031	4.350264	0.126767
15	4.333519	4.142408	0.191111
20	4.189585	3.885117	0.304468
25	4.048025	3.638548	0.409476
30	3.908240	3.434558	0.473682
35	3.767926	3.268065	0.499862
40	3.623918	3.123786	0.500133
45	3.465690	2.977436	0.488255
50	3.286923	2.817112	0.469812
55	3.081772	2.635767	0.446004
60	2.831896	2.418189	0.413707
65	2.491113	2.122477	0.368636
70	2.017308	1.712745	0.304564
75	1.450846	1.224749	0.226097
80	0.845838	0.701883	0.143955
85	0.564292	0.422495	0.141796
total	57.856602	52.227596	5.629005

initial region of cohort r.yugos.

	total	slovenia	r.yugos.
0	4.662560	0.003104	4.659456
5	4.261472	0.007731	4.253742
10	4.121640	0.010724	4.110917
15	3.984471	0.018390	3.966080
20	3.844761	0.032403	3.812357
25	3.703362	0.043433	3.659930
30	3.562462	0.048469	3.513994
35	3.422768	0.050959	3.371809
40	3.279432	0.052004	3.227427
45	3.127837	0.051698	3.076139
50	2.957391	0.050530	2.906860
55	2.756870	0.049529	2.707340
60	2.504885	0.047940	2.456946
65	2.176724	0.043982	2.132742
70	1.754133	0.036506	1.717628
75	1.268236	0.026633	1.241604
80	0.786897	0.015756	0.771142
85	0.760513	0.009843	0.750670
total	52.936424	0.599635	52.336788

1.2 The Stable Population

The stable population by place of residence and place of birth, per unit radices, is given by

$$\underset{\sim}{\ell}^{(r)}(x) = e^{-rx}\,\underset{\sim}{\ell}(x) \qquad (1.3a)$$

and

$$\underset{\sim}{L}^{(r)}(x) = e^{-r(x+2.5)}\,\underset{\sim}{L}(x) \quad , \qquad (1.3b)$$

or

$$\begin{bmatrix} {}_{10}L_1^{(r)}(x) & {}_{20}L_1^{(r)}(x) \\ {}_{10}L_2^{(r)}(x) & {}_{20}L_2^{(r)}(x) \end{bmatrix} = \begin{bmatrix} e^{-r(x+2.5)}\,{}_{10}L_1(x) & e^{-r(x+2.5)}\,{}_{20}L_1(x) \\ e^{-r(x+2.5)}\,{}_{10}L_2(x) & e^{-r(x+2.5)}\,{}_{20}L_2(x) \end{bmatrix} \qquad (1.4)$$

where r is the annual growth rate of the stable population, i.e. the intrinsic growth rate. The rate r only depends on the observed schedules and is independent of the observed population distribution. It is computed as follows:

$$r = \frac{1}{h}\,\ell n\ \lambda$$

with h being the age interval (5 years), and λ the eigenvalue of the population growth matrix. The value of λ is computed by the subroutine PROJECT in Willekens and Rogers (1976 , p. 50).

The absolute number of people in each age group by place of residence is

$$\{\underset{\sim}{K}(x)\} = e^{-r(x+2.5)}\,\underset{\sim}{L}(x)\{\underset{\sim}{Q}\} \quad , \qquad (1.5)$$

where $\{\underset{\sim}{Q}\}$ is the stable distribution of births and will be determined in section five of this report. Expression (1.5) is the numeral evaluation of the continuous formula (Rogers and Willekens, 1976b, p. 22).

$$\{\underset{\sim}{k}(x)\} = e^{-rx} \underset{\sim}{\ell}(x)\{\underset{\sim}{Q}\} \quad . \tag{1.6}$$

At this point it is useful to stress that:

i. The life table population distribution is a special case of (1.3) with $r = 0$.

ii. Any stationary population, i.e. stable population with zero growth rate, is distributed according to a life table. population. Its relative distribution (in terms of unit births) is therefore independent of how fertility is reduced to replacement level.

iii. The column totals in Table 1b are the number of people in the life table population per baby born. Adopting the "person-years lived" interpretation of $\underset{\sim}{L}(x)$, the totals would be the life expectancies at birth by place of birth and place of residence,

$$\underset{\sim}{e}(0) = \sum_x \underset{\sim}{L}(x) \quad . \tag{1.7}$$

For example, the total life expectancy of a baby girl born in Slovenia is 72.48 years. A total of 64.90 years are expected to be lived in Slovenia and 7.57 years in the Rest of Yugoslavia.

iv. The column totals in Table 1c are the number of people in the stable population per baby born. If the growth rate r is positive, then the stable population is growing and the share of the births in the total population is greater than in the stationary population. Therefore, for $r > 0$

$$\sum_x e^{-r(x+2.5)} \underset{\sim}{L}(x) < \sum_x \underset{\sim}{L}(x)$$

or

$$\underset{\sim}{e}^{(r)}(0) < \underset{\sim}{e}(0) \quad . \tag{1.8}$$

For example, for each baby born in Slovenia, there are 57.86 persons living in Yugoslavia who were born in Slovenia. Of these 52.23 are living in Slovenia and 5.63 in the Rest of Yugoslavia. Analogous to the expectation of life at birth-interpretation of $\underset{\sim}{e}(0)$, the matrix $\underset{\sim}{e}^{(r)}(0)$ may be considered as the discounted life expectancy matrix, with r being the rate of discount (Willekens, 1977). The meaning and relevance of this approach will be discussed in section four.

The three types of age distribution are the cornerstones for further study. Fertility analysis is performed by applying age-specific fertility rates to the age distributions. In mobility analysis, age-specific outmigration rates are used instead. The next two sections deal with these topics in greater detail.

2. FERTILITY ANALYSIS

The fertility analysis proceeds by applying the fertility schedule to the three types of age distributions. Let the diagonal matrix $\underset{\sim}{m}(x)$ contain the annual regional fertility rates of the women at exact age x, and let $\underset{\sim}{F}(x)$ be the diagonal matrix of annual regional fertility rates of age group x to x + 4, e.g.

$$\underset{\sim}{F}(x) = \begin{bmatrix} F_1(x) & 0 \\ 0 & F_2(x) \end{bmatrix} . \qquad (2.1)$$

The integration of the matrices of age-specific fertility rates over all ages is the gross reproduction rate matrix, i.e.

$$\underset{\sim}{GRR} = \int_0^{\omega} \underset{\sim}{m}(x)\,dx \doteq 5 \sum_x \underset{\sim}{F}(x) .$$

The GRR-matrix is a diagonal matrix with the regional gross rates of reproduction as its elements. Age-specific fertility rates for Slovenia and the Rest of Yugoslavia are given in Table 2. The column totals denote the regional gross reproduction rates.

Table 2. fertility analysis
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observed rates
==============

age	slovenia	r.yugos.
0	0.000000	0.000000
5	0.000000	0.000000
10	0.000071	0.000067
15	0.015857	0.026458
20	0.070652	0.087978
25	0.063218	0.074260
30	0.041103	0.044290
35	0.022862	0.023532
40	0.007797	0.012051
45	0.000710	0.002151
50	0.000292	0.000714
55	0.000000	0.000000
60	0.000000	0.000000
65	0.000000	0.000000
70	0.000000	0.000000
75	0.000000	0.000000
80	0.000000	0.000000
85	0.000000	0.000000
grr	1.112810	1.357505

The regional crude birth rates may be derived by multiplying the age-specific fertility rates by the observed population distribution, in fractions of the total, and summing over all age groups. Denoting the regional distribution of the people aged x to x + 4 by the diagonal matrix $\mathbf{K}(x)$, the regional crude birth rates are given by the vector $\{\mathbf{b}^0\}$:

$$\{\mathbf{b}^0\} = \left[\sum_x \mathbf{F}(x)\ \mathbf{K}(x)\right]\left[\sum_x \mathbf{K}(x)\right]^{-1} \{\mathbf{1}\} \quad . \tag{2.2}$$

The product $\mathbf{F}(x)\ \mathbf{K}(x)$ is of course the observed regional number of births to a mother aged x to x + 4.

The application of the age-specific fertility rates to the life table population and to the stable population has received much attention in the demographic literature.

2.1 The Generalized Net Maternity Function

The generalized net maternity (GNM) function is defined as the product (Rogers, 1975a, p. 93)

$$\boldsymbol{\phi}(x) = \mathbf{m}(x)\ \boldsymbol{\ell}(x) \quad , \tag{2.3}$$

where

$$\boldsymbol{\phi}(x) = \begin{bmatrix} {}_1\phi_1(x) & {}_2\phi_1(x) \\ {}_1\phi_2(x) & {}_2\phi_2(x) \end{bmatrix}$$

$$= \begin{bmatrix} m_1(x)\ {}_{10}\ell_1(x) & m_1(x)\ {}_{20}\ell_1(x) \\ m_2(x)\ {}_{10}\ell_2(x) & m_2(x)\ {}_{20}\ell_2(x) \end{bmatrix} .$$

An element ${}_i\phi_j(x)$ denotes the expected number of children to be born during a unit time interval in region j to a woman of exact age x, who was born in region i, and who is part of a stationary (life table) population. The fertility rates applied to this stationary population are the observed fertility rates.

Since the actual population data are usually given for five-year age groups, one normally evaluates (2.3) with the numerical approximation

$$\underset{\sim}{\bar{\phi}}(x) = \underset{\sim}{F}(x)\ \underset{\sim}{L}(x) \tag{2.4}$$

in which the integral $\int_0^5 \underset{\sim}{M}(x+t)\ \underset{\sim}{\ell}(x+t)\,dt$ is replaced by the product of $\underset{\sim}{F}(x)$ and $\underset{\sim}{L}(x)$. The numerical evaluations or the integrals of the generalized net maternity function are given in Table 3. They are obtained by multiplying the fertility rates of Table 2 by the age composition of the life table population (Table 1b). For example, $\underset{\sim}{\bar{\phi}}(20)$ is:

$$\underset{\sim}{\bar{\phi}}(20) = \begin{bmatrix} 0.070652 & 0 \\ 0 & 0.087978 \end{bmatrix} \begin{bmatrix} 4.456621 & 0.037170 \\ 0.349255 & 4.373158 \end{bmatrix}$$

$$= \begin{bmatrix} 0.314869 & 0.002626 \\ 0.030727 & 0.384742 \end{bmatrix} .$$

The GNM function gives the number of offspring by age of a population which is distributed according to the life table (stationary) population, and which is subjected to the observed regional fertility schedules. The total number of offspring per unit birth is

$$\underset{\sim}{NRR} = \sum_x \underset{\sim}{\bar{\phi}}(x) \quad . \tag{2.5}$$

An element

$$ {}_{i}NRR_{j} = \sum_{x} {}_{i}\bar{\phi}_{j}(x) $$

denotes the total number of children expected to be born in region j to a woman who was born in region i, and who is a member of a life table population.[4] The matrix $\underset{\sim}{NRR}$ is the <u>net reproduction rate matrix</u>, and is the multiregional generalization of the Net Reproduction Rate (NRR) (Rogers, 1975a, p. 106). The elements of $\underset{\sim}{NRR}$ are the totals in table 3.

The matrix $\underset{\sim}{NRR}$ gives the regional distribution of the offspring per unit birth in each region. It has been computed using unit radices. From the discussion of the life table in the previous section it is clear that a birth cohort of $\{\underset{\sim}{Q}_1\}$ would lead to a regional number of offspring, after a generation, of

$$ \{\underset{\sim}{Q}_2\} = \underset{\sim}{NRR}\{\underset{\sim}{Q}_1\} \quad . \qquad (2.6) $$

The GNM function contains additional useful information for fertility analysis. Define the n-th moment of the GNM function (2.3) as (Rogers, 1975a, p. 106)

$$ \underset{\sim}{R}(n) = \int_{\alpha}^{\beta} x^n \, \underset{\sim}{\phi}(x) \, dx \qquad (2.7) $$

where α and β are the lowest and highest reproductive ages respectively, and, for example,

$$ \underset{\sim}{R}(n) = \begin{bmatrix} {}_{1}R_{1}(n) & {}_{2}R_{1}(n) \\ {}_{1}R_{2}(n) & {}_{2}R_{2}(n) \end{bmatrix} . $$

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[4]Recall that a life table population is a stationary population that would result if the mortality and migration schedules were applied to arbitrary regional radices.

Table 3. integrals of generalized net maternity function

initial region of cohort slovenia

age	slovenia	r.yugos.
0	0.000000	0.000000
5	0.000000	0.000000
10	0.000333	0.000009
15	0.073085	0.005626
20	0.314869	0.030727
25	0.272029	0.035961
30	0.172122	0.025579
35	0.093916	0.014786
40	0.031564	0.007811
45	0.002824	0.001403
50	0.001133	0.000462
55	0.000000	0.000000
60	0.000000	0.000000
65	0.000000	0.000000
70	0.000000	0.000000
75	0.000000	0.000000
80	0.000000	0.000000
85	0.000000	0.000000
total	0.961876	0.122364

initial region of cohort r.yugos.

age	slovenia	r.yugos.
0	0.000000	0.000000
5	0.000000	0.000000
10	0.000001	0.000297
15	0.000324	0.116755
20	0.002626	0.384742
25	0.003247	0.321421
30	0.002429	0.189757
35	0.001464	0.099737
40	0.000525	0.050403
45	0.000049	0.008840
50	0.000020	0.002859
55	0.000000	0.000000
60	0.000000	0.000000
65	0.000000	0.000000
70	0.000000	0.000000
75	0.000000	0.000000
80	0.000000	0.000000
85	0.000000	0.000000
total	0.010687	1.174812

The numerical approximation of (2.7) is

$$\bar{\underset{\sim}{R}}(n) = \sum_{\alpha-5}^{\beta-5} (x + 2.5)^n \bar{\underset{\sim}{\phi}}(x)$$

$$= \sum_{\alpha-5}^{\beta-5} (x + 2.5)^n \underset{\sim}{F}(x) \underset{\sim}{L}(x) \quad . \tag{2.8}$$

Observe that the 0-th moment, $\underset{\sim}{R}(0)$, is identical to $\underset{\sim}{NRR}$.

The 0-th, first and second moments of the GNM function of the two-region system Slovenia-Rest of Yugoslavia are given in Table 4. The column totals of $\underset{\sim}{R}(0)$ represent the total number of offspring per woman born in a certain region, e.g.

$$_iR(0) = \sum_j {}_iR_j(0) \quad . \tag{2.9}$$

The row totals of $\underset{\sim}{R}(0)$ give the total number of children born in a certain region during one generation per woman born in that region. It is the number of daughters by which a girl child in a region is replaced. Noting that $\underset{\sim}{R}(0) = \underset{\sim}{NRR}$, the total number of children born in region j during one generation, is

$$Q_{2j} = \sum_i {}_iR_j(0)\; Q_{1i} \tag{2.10}$$

and the row total of the j-th region is

$$R_j(0) = \frac{Q_{2j}}{Q_{1j}} = \sum_i \frac{Q_{1i}}{Q_{1j}}\; {}_iR_j(0) \quad . \tag{2.11}$$

The value of $R_j(0)$ depends on the radix ratio Q_{1i} / Q_{1j} of the life table population. Since we have assumed unit radices in all regions, the row totals in Table 4, i.e. $R_j(0)$, are the sum of the elements in the row. Other radices would give $R_j(0)$ and the row totals of the subsequent moments other values.

Table 4. moments of integral function

0 moment

	total	slovenia	r.yugos.
slovenia	0.972563	0.961876	0.010687
r.yugos.	1.297176	0.122364	1.174812
total		1.084240	1.185499

1 moment

	total	slovenia	r.yugos.
slovenia	26.813101	26.499439	0.313662
r.yugos.	35.749592	3.587494	32.162098
total		30.086933	32.475761

2 moment

	total	slovenia	r.yugos.
slovenia	777.565430	767.940002	9.625456
r.yugos.	1044.039429	110.833084	933.206299
total		878.773071	942.831726

Table 5 repeats $\underset{\sim}{R}(0)$ or $\underset{\sim}{NRR}$ and gives the dominant eigenvalue and associated eigenvectors of $\underset{\sim}{R}(0)$. The eigenvalue of $\underset{\sim}{R}(0)$, $\lambda_1(\underset{\sim}{R}(0))$, gives an indication of the net reproduction rate of the whole system or country (Rogers and Willekens, 1976c, p.28). A life table radix ratio that yields a global NRR equal to $\lambda_1(\underset{\sim}{R}(0))$ is given by the right eigenvector of $\underset{\sim}{R}(0)$. The global NRR resulting from a radix ratio as specified by the user, 1:1 say, is also given in Table 5. It is equal to 1.224257. The net reproduction allocation ${}_i\rho_j$ denotes the fraction of the offspring of the i-born women, that are born in region j (Rogers, 1975b, p. 2.).[5] For example,

$$ {}_1\rho_2 = \frac{{}_1NRR_2}{{}_1NRR_{\bullet}} = \frac{0.122364}{1.084240} = 0.112857 \quad , $$

i.e. 11.29% of the daughters of Slovenia-born women, are born in the Rest of Yugoslavia.

The moments of the GNM function give rise to other demographically meaningful statistics: the mean and the variance of the GNM function. In the single region case, the mean of the net maternity function is defined as (Keyfitz, 1968, p. 102)

$$ \mu = \frac{\sum_x (x + 2.5)\ F(x)\ L(x)}{\sum_x F(x)\ L(x)} = \frac{\bar{R}(1)}{\bar{R}(0)} \quad . \qquad (2.12) $$

It represents the mean age of childbearing of the life table population (given the observed fertility schedule). The variance of the net maternity function is

$$ \sigma^2 = \frac{\sum_x (x + 2.5 - \mu)^2\ F(x)\ L(x)}{\sum_x F(x)\ L(x)} = \frac{R(2)}{R(0)} - \mu^2 \quad , \qquad (2.13) $$

[5] The arrangements of the elements in Table 5 is the transpose of Table 2 in Rogers (1975b, p. 5).

Table 5.. spatial fertility expectancies
****

net reproduction rate

	total	slovenia	r.yugos.
slovenia	0.972563	0.961876	0.010687
r.yugos.	1.297176	0.122364	1.174812
total		1.084240	1.185499
eigenvalue	1.180786		
eigenvector			
- right		1.000000	20.483938
- left		1.000000	1.789025

net reproduction allocations

	total	slovenia	r.yugos.
slovenia	0.896158	0.887143	0.009015
r.yugos.	1.103842	0.112857	0.990985
total		1.000000	1.000000

global nrr = 1.224257

and represents the variance of the mean age of childbearing. Multiregional generalizations of (2.12) and (2.13) are (Rogers, 1975a, p. 106):

$$ {}_{i}\mu_{j} = \frac{\sum_{x} (x + 2.5)\ F_{j}(x)\ {}_{i0}L_{j}(x)}{\sum_{x} F_{j}(x)\ {}_{i0}L_{j}(x)} = \frac{{}_{i}\bar{R}_{j}(1)}{{}_{i}\bar{R}_{j}(0)} \tag{2.14} $$

and

$$ {}_{i}\sigma_{j}^{2} = \frac{\sum_{x} (x + 2.5 - {}_{i}\mu_{j})^{2}\ F_{j}(x)\ {}_{i0}L_{j}(x)}{\sum_{x} F_{j}(x)\ {}_{i0}L_{j}(x)} = \frac{{}_{i}\bar{R}_{j}(2)}{{}_{i}\bar{R}_{j}(0)} - {}_{i}\mu_{j}^{2}\ , \tag{2.15} $$

respectively.

The matrix of mean ages of childbearing of the life table population is given in Table 6 as Alternative 1. For example, the mean age of childbearing among Slovenia-born women who are living in the Rest of Yugoslavia is 29.32 years. The mean age of the women living in Slovenia is lower, namely 27.55 years. This is consistent with the observation that mothers who have migrated are normally older.

The single-region measures (2.12) and (2.13) may be generalized to a multiregional system in a different way, one which is analogous to the extension of the single-region survivorship proportion to the multiregional survivorship matrix in the life table. The mean age of childbearing matrix in this case is

$$ \begin{aligned} \underset{\sim}{\mu} &= \left[\sum_{x} (x + 2.5)\ \underset{\sim}{F}(x)\ \underset{\sim}{L}(x)\right]\left[\sum_{x} \underset{\sim}{F}(x)\ \underset{\sim}{L}(x)\right]^{-1} \\ &= \left[\underset{\sim}{\bar{R}}(1)\right]\left[\underset{\sim}{\bar{R}}(0)\right]^{-1} \end{aligned} \tag{2.16} $$

and the variance matrix is

$$ \underset{\sim}{\sigma}^{2} = \left[\underset{\sim}{\bar{R}}(2)\right]\left[\underset{\sim}{\bar{R}}(0)\right]^{-1} - \underset{\sim}{\mu}^{2}\ . \tag{2.17} $$

These matrices are given in Table 6 as Alternative 2. The average age at childbearing of a woman who conceived in Slovenia is 27.795 years. Of this total 27.548 have been lived in Slovenia and 0.247 in the Rest of Yugoslavia.

Table 6. matrices of mean ages and variances

** alternative 1 **

means

	total	slovenia	r.yugos.
slovenia	28.449963	27.549740	29.350185
r.yugos.	28.347355	29.318327	27.376379
total		28.434034	28.363283

variances

	slovenia	r.yugos.
slovenia	39.388977	39.246277
r.yugos.	46.204773	44.879150

** alternative 2 **

means

	total	slovenia	r.yugos.
slovenia	27.564051	27.547653	0.016397
r.yugos.	27.621458	0.247328	27.374130
total		27.794981	27.390528

variances

	total	slovenia	r.yugos.
slovenia	39.412544	39.381409	0.031137
r.yugos.	45.476387	0.607306	44.869080
total		39.988716	44.900215

2.2 The Weighted Generalized Net Maternity Function

Thus far we limited ourselves to the fertility analysis of a population, distributed as in the multiregional life table. It is a stationary population that is generated by the observed mortality and migration schedules. The life table population was augmented by the observed fertility schedules to give the GNM function and the derived statistics discussed above. We now replace the life table population by the stable population, given in Table 1c, and perform an analogous analysis. The regional radices, used in the life table, are now replaced by the regional births in the stable population. As before we assume unit birth cohorts.

Computationally, the fertility analysis in the stable population is completely analogous to the one described above. The only difference is that $\underset{\sim}{\ell}(x)$ is replaced by

$$\underset{\sim}{\ell}^{(r)}(x) = e^{-rx}\,\underset{\sim}{\ell}(x) \tag{1.3a}$$

and $\underset{\sim}{L}(x)$ by

$$\underset{\sim}{L}^{(r)}(x) = e^{-r(x+2.5)}\,\underset{\sim}{L}(x) \quad . \tag{1.3b}$$

Define the Weighted Generalized Net Maternity (WGNM) Function as the product

$$\underset{\sim}{\phi}^{(r)}(x) = \underset{\sim}{m}(x)\,\underset{\sim}{\ell}^{(r)}(x) = e^{-rx}\,\underset{\sim}{m}(x)\,\underset{\sim}{\ell}(x) \quad . \tag{2.18}$$

The weight applied is e^{-rx}. Since this may be considered as a discounting to birth, with r being the rate of discount, we may denote the WGNM function as a GNM function with discounting. The usefulness of the notion of discounting for demographic analysis becomes clear in the treatment of the reproductive value (Rogers and Willekens, 1976b). An element ${}_i\phi_j^{(r)}(x)$ denotes the expected number of children to be born in region j to an i-born woman of exact age x who is part of the stable population. It may also be considered as the number of children discounted back to the time of birth of the mother.

The numerical approximation of (2.18) is

$$\bar{\underset{\sim}{\phi}}^{(r)}(x) = \underset{\sim}{F}(x)\ \underset{\sim}{L}^{(r)}(x) \quad , \tag{2.19}$$

and the result is given in Table 7. Table 7 is obtained by multiplying the fertility rates of Table 2 by the age composition of the stable population (Table 1c). For example,

$$\bar{\underset{\sim}{\phi}}^{(r)}(20) = \begin{bmatrix} 0.070652 & 0 \\ 0 & 0.087978 \end{bmatrix} \begin{bmatrix} 3.885117 & 0.032403 \\ 0.304468 & 3.812357 \end{bmatrix}$$

$$= \begin{bmatrix} 0.274491 & 0.002289 \\ 0.026786 & 0.335404 \end{bmatrix}$$

The WGNM function gives the number of offspring by age of a unit birth in the stable population. Summing over all age groups we get

$$\underset{\sim}{\Psi}(r) = \sum_x \bar{\underset{\sim}{\phi}}^{(r)}(x) \quad . \tag{2.20}$$

The matrix $\underset{\sim}{\Psi}(r)$ is the <u>characteristic matrix</u> of the multiregional population system (Rogers, 1975a, p. 93). An element ${}_i\Psi_j(r)$ denotes the total number of children expected to be born in region j to a woman who was born in region i, and who is a member of the stable population. The characteristics matrix is the stable analogue of the $\underset{\sim}{NRR}$ matrix. It gives the regional distribution of the offspring per unit birth in each region of the stable population. For example, Table 7 shows that a woman born in the stable population in Slovenia gives birth to a total of 0.916100 children on the average. Of them, 0.813686 are born in Slovenia and 0.102414 in the Rest of Yugoslavia.

Table 7. integrals of weighted generalized net maternity function

initial region of cohort slovenia

age	slovenia	r.yugos.
0	0.000000	0.000000
5	0.000000	0.000000
10	0.000309	0.000008
15	0.065686	0.005056
20	0.274491	0.026786
25	0.230022	0.030408
30	0.141171	0.020979
35	0.074715	0.011763
40	0.024356	0.006027
45	0.002114	0.001050
50	0.000823	0.000335
55	0.000000	0.000000
60	0.000000	0.000000
65	0.000000	0.000000
70	0.000000	0.000000
75	0.000000	0.000000
80	0.000000	0.000000
85	0.000000	0.000000
total	0.813686	0.102414

initial region of cohort r.yugos.

age	slovenia	r.yugos.
0	0.000000	0.000000
5	0.000000	0.000000
10	0.000001	0.000275
15	0.000292	0.104935
20	0.002289	0.335404
25	0.002746	0.271786
30	0.001992	0.155635
35	0.001165	0.079345
40	0.000405	0.038894
45	0.000037	0.006617
50	0.000015	0.002075
55	0.000000	0.000000
60	0.000000	0.000000
65	0.000000	0.000000
70	0.000000	0.000000
75	0.000000	0.000000
80	0.000000	0.000000
85	0.000000	0.000000
total	0.008942	0.994966

If the stable distribution of births is $\{\underset{\sim}{Q}^s\}$, then the distribution of offspring is also $\{\underset{\sim}{Q}^s\}$ (Rogers, 1975a, p. 93):

$$\{\underset{\sim}{Q}^s\} = \underset{\sim}{\Psi}(r)\{\underset{\sim}{Q}^s\} \quad . \tag{2.21}$$

Equation (2.21) is the multiregional characteristic equation. It can be seen from (2.21) that the relative distribution of births is given by the right eigenvector of $\underset{\sim}{\Psi}(r)$. In our numerical example,

$$\{\underset{\sim}{Q}_1^s\} = \begin{bmatrix} 1 \\ \\ 20.823662 \end{bmatrix} , \tag{2.22}$$

where the subscript denotes "arbitrary norming." Since the eigenvector of a matrix is fixed up to a scalar, we may choose the norming of the eigenvector freely. The result (2.22) implies that 4.58% of the births occur in Slovenia and 95.42% in the Rest of Yugoslavia (in the observed population it was 6.91% and 93.09%, respectively).

As with the GNM function, we define the n-th moment of the WGNM function (2.18) as (Rogers, 1975a, p. 112)

$$\begin{aligned} \underset{\sim}{R}^{(r)}(n) &= \int_\alpha^\beta x^n \, \underset{\sim}{\phi}^{(r)}(x) \, dx \\ &= \int_\alpha^\beta x^n \, e^{-rx} \, \underset{\sim}{\phi}(x) \, dx \quad , \end{aligned} \tag{2.23}$$

and evaluate it numerically as follows:

$$\begin{aligned} \bar{\underset{\sim}{R}}^{(r)}(n) &= \sum_{\alpha-5}^{\beta-5} (x + 2.5)^n \, \bar{\underset{\sim}{\phi}}^{(r)}(x) \\ &= \sum_{\alpha-5}^{\beta-5} (x + 2.5)^n \, e^{-r(x+2.5)} \, \underset{\sim}{F}(x) \, \underset{\sim}{L}(x) \quad . \end{aligned} \tag{2.24}$$

The moments are given in Table 8. Note that the 0-th moment of the WGNM function coincides with $\underset{\sim}{\Psi}(r)$. The column totals of $\underset{\sim}{\Psi}(r)$ represent the total number of offspring in the stable population per woman by her place of birth, e.g.

$$ {}_{i}\Psi(r) = \sum_{j} {}_{i}\Psi_{j}(r) \quad . \tag{2.25} $$

The row totals give the total number of daughters by which a female baby is replaced in her region of birth in the stable population. It depends of course on the stable ratio of births:

$$ \Psi_{j}(r) = \sum_{i} \frac{Q^{s}_{1i}}{Q^{s}_{1j}} \; {}_{i}\Psi_{j}(r) \quad , \tag{2.26} $$

where Q^{s}_{1i} is an element of the right eigenvector of $\underset{\sim}{\Psi}(r)$.

Table 9 repeats the $\underset{\sim}{\Psi}(r)$ matrix. In addition, it shows the net reproduction allocations ${}_{i}\rho^{(r)}_{j}$, with

$$ {}_{i}\rho^{(r)}_{j} = \frac{{}_{i}\Psi_{j}(r)}{{}_{i}\Psi(r)} \quad . \tag{2.27} $$

For example,

$$ {}_{1}\rho^{(r)}_{2} = \frac{{}_{1}\Psi_{2}(r)}{{}_{1}\Psi(r)} = \frac{0.102414}{0.916100} = 0.111793 \quad , $$

i.e. 11.18% of the daughters born to Slovenia-born women, are born in the Rest of Yugoslavia.

The mean and the variance of the WGNM function are given in Table 10. Again, two alternative expressions are distinguished.

Table 8. moments of integral function

0 moment

	total	slovenia	r.yugos.
slovenia	0.822628	0.813686	0.008942
r.yugos.	1.097380	0.102414	0.994966
total		0.916100	1.003908

1 moment

	total	slovenia	r.yugos.
slovenia	22.483915	22.223598	0.260318
r.yugos.	29.944412	2.974082	26.970331
total		25.197681	27.230650

2 moment

	total	slovenia	r.yugos.
slovenia	646.212097	638.288757	7.923365
r.yugos.	865.373718	90.987785	774.385925
total		729.276550	782.309265

Table 9. spatial fertility expectancies

net reproduction rate

	total	slovenia	r.yugos.
slovenia	0.822628	0.813686	0.008942
r.yugos.	1.097380	0.102414	0.994966
total		0.916100	1.003908
eigenvalue[6]	0.999884		
eigenvector			
- right		1.000000	20.823662
- left		1.000000	1.818116

net reproduction allocations

	total	slovenia	r.yugos.
slovenia	0.897113	0.888207	0.008907
r.yugos.	1.102887	0.111793	0.991093
total		1.000000	1.000000

[6] The eigenvalue should be equal to one. Deviation is due to rounding of the intrinsic growth rate r to six decimal places. The growth rate has been computed by projecting the population growth matrix until stability.

Alternative 1 (Rogers, 1975a, p. 113):

The matrix of mean ages of childbearing in the stable population, $\underset{\sim}{A}$, has elements:

$$ {}_iA_j = \frac{\sum_x (x + 2.5)\ e^{-r(x+2.5)}\ F_j(x)\ {}_{i0}L_j(x)}{\sum_x e^{-r(x+2.5)}\ F_j(x)\ {}_{i0}L_j(x)} = \frac{{}_i\bar{R}_j^{(r)}(1)}{{}_i\bar{R}_j^{(r)}(0)} \quad , \tag{2.28} $$

and the variance $\underset{\sim}{\sigma}^2$ with elements

$$ {}_i\sigma_j^{\ 2} = \frac{\sum_x (x + 2.5 - {}_iA_j)^2\ e^{-r(x+2.5)}\ F_j(x)\ {}_{i0}L_j(x)}{\sum_x e^{-r(x+2.5)}\ F_j(x)\ {}_{i0}L_j(x)} $$

$$ = \frac{{}_i\bar{R}_j^{(r)}(2)}{{}_i\bar{R}_j^{(r)}(0)} - {}_iA_j^{\ 2} \quad . \tag{2.29} $$

Alternative 2:

$$ \underset{\sim}{A} = \left[\sum_x (x + 2.5)\ e^{-r(x+2.5)}\ \underset{\sim}{F}(x)\ \underset{\sim}{L}(x)\right]\left[\sum_x e^{-r(x+2.5)}\ \underset{\sim}{F}(x)\ \underset{\sim}{L}(x)\right]^{-1} $$

$$ = \left[\underset{\sim}{\bar{R}}^{(r)}(1)\right]\left[\underset{\sim}{\bar{R}}^{(r)}(0)\right]^{-1} \tag{2.30} $$

$$ \underset{\sim}{\sigma}^2 = \left[\underset{\sim}{\bar{R}}^{(r)}(2)\right]\left[\underset{\sim}{\bar{R}}^{(r)}(0)\right]^{-1} - \underset{\sim}{A}^2 \quad . \tag{2.31} $$

Table 10. matrices of mean ages and variances

**** alternative 1 ****

means

	total	slovenia	r.yugos.
slovenia	28.212582	27.312254	29.112906
r.yugos.	28.073296	29.039808	27.106783
total		28.176031	28.109844

variances

	slovenia	r.yugos.
slovenia	38.481873	38.555481
r.yugos.	45.120789	43.526062

**** alternative 2 ****

means

	total	slovenia	r.yugos.
slovenia	27.326414	27.310213	0.016201
r.yugos.	27.348167	0.243574	27.104593
total		27.553787	27.120794

variances

	total	slovenia	r.yugos.
slovenia	38.507473	38.474243	0.033230
r.yugos.	44.138874	0.622760	43.516113
total		39.097004	43.549343

3. MOBILITY ANALYSIS

There are two alternative approaches to expressing the level of migration in a multiregional system (Rogers, 1975b). The first expresses the migration level in terms of expected durations , i.e. the fraction of an individual's lifetime that is spent in a particular region. The expectation of life at birth by place of residence is computed in the multiregional life table. The life expectancy matrix

$$\underset{\sim}{e}(0) = \begin{bmatrix} {}_1e_1(0) & {}_2e_1(0) \\ {}_1e_2(0) & {}_2e_2(0) \end{bmatrix} \tag{3.1}$$

for the system Slovenia -- Rest of Yugoslavia is given in Table 11. The total life expectancy of a girl born in Slovenia is 72.48 years, of which 64.90 years are expected to be lived in Slovenia (${}_1e_1(0)$) and 7.57 years in the Rest of Yugoslavia (${}_1e_2(0)$).

Expressing these expectancies as fractions of the total lifetime yields the migration levels ${}_i\theta_j$:

$${}_i\theta_j = {}_ie_j(0)/{}_ie(0) \tag{3.2}$$

The second approach adopts a fertility perspective to migration analysis. Unlike death, migration is a recurrent event, analogous to birth. Like fertility, its level can be measured by counting the events, i.e. the number of moves an average person makes during his lifetime[7]. Such indices have been developed by Wilber (1963) and Long (1973) for a population aggregated at the national level. Rogers (1975b) combines Wilber's and Long's ideas of "expected moves" with the approach generalizing the expected number of children (NRR) to a multiregional system ($\underset{\sim}{NRR}$).

---

[7]The number of moves is defined here as the number of times a person is in another region at the end of the unit time interval. Back and forth moves during a unit interval are not counted (a similar assumption has been adopted by Wilber (1963) and Long (1973)).

Table 11. spatial migration expectancies

expectations of life

	total	slovenia	r.yugos.
slovenia	65.712997	64.902672	0.810323
r.yugos.	73.009148	7.573801	65.435349
total		72.476471	66.245674
eigenvalue	67.660629		
eigenvector			
- right		1.000000	3.403525
- left		1.000000	0.364144

migration levels

	total	slovenia	r.yugos.
slovenia	0.907732	0.895500	0.012232
r.yugos.	1.092268	0.104500	0.987768
total		1.000000	1.000000

As before, let $\underset{\sim}{\ell}(x)$ be the distribution of the life table population of exact age x, and let $\underset{\sim}{L}(x)$ be the stationary life table population aged x to x + 4, by place of birth and residence. Define $\underset{\sim}{m}^{O}$ as the diagonal matrix of annual regional outmigration rates of people at exact age x, and $\underset{\sim}{M}^{O}(x)$ as the diagonal matrix of outmigration rates of people in age group x to x + 4, e.g.

$$\underset{\sim}{M}^{O}(x) = \begin{bmatrix} M_1^{O}(x) & 0 \\ & \\ 0 & M_2^{O}(x) \end{bmatrix} \tag{3.3}$$

with $M_i^{O}(x) = \sum_{j \neq i} M_{ij}(x)$, $M_{ij}(x)$ being the age specific migration rate from region i to region j. Integration of the matrices of age-specific outmigration rates over all ages gives the <u>gross migra-production rate matrix</u>:

$$\underset{\sim}{GMR} = \int_0^{\omega} \underset{\sim}{m}^{O}(x)\ dx \doteq \sum_x \underset{\sim}{M}^{O}(x) \quad .$$

The origin-destination migration rates of the two-region system Slovenia - Rest of Yugoslavia are given in Table 3 of Willekens and Rogers (1976a, p. 9). Table 12 shows the age-specific regional total outmigration rates. Since the system under consideration contains only two regions, $M_i^{O}(x) = M_{ij}(x)$ for $i \neq j$. The column totals denote the regional gross migra-production rates.

The application of the age-specific outmigration rates to the life table and to the stable populations yields, respectively, the generalized and the weighted generalized net mobility functions.

Table 12. migration analysis
****

observed rates
==============

age	slovenia	r.yugos.
0	0.002832	0.000272
5	0.002294	0.000166
10	0.001485	0.000157
15	0.005158	0.000679
20	0.007170	0.000937
25	0.005534	0.000506
30	0.003756	0.000350
35	0.001765	0.000226
40	0.001013	0.000183
45	0.000543	0.000094
50	0.000663	0.000130
55	0.000629	0.000205
60	0.000884	0.000203
65	0.000949	0.000156
70	0.000876	0.000078
75	0.001111	0.000099
80	0.000704	0.000196
85	0.000000	0.000000
gmr	0.186830	0.023185

3.1 The Generalized Net Mobility Function

The generalized net mobility (GM) function is the product

$$\underset{\sim}{\gamma}(x) = \underset{\sim}{m}^{o}(x)\ \underset{\sim}{\ell}(x), \quad \text{or} \tag{3.4}$$

$$\begin{bmatrix} {}_1\gamma_1(x) & {}_2\gamma_1(x) \\ {}_1\gamma_2(x) & {}_2\gamma_2(x) \end{bmatrix} = \begin{bmatrix} m_1^o(x)\,{}_{10}\ell_1(x) & m_1^o(x)\,{}_{20}\ell_1(x) \\ m_2^o(x)\,{}_{10}\ell_2(x) & m_2^o(x)\,{}_{20}\ell_2(x) \end{bmatrix}$$

An element ${}_i\gamma_j(x)$ denotes the expected number of migrations out of region j, made during a unit time interval following age x, by a woman born in region i. Since the system only consists of two regions, ${}_i\gamma_j(x)$ measures the return migration of the x-year old.

The numerical evaluation of equation (3.4) is

$$\bar{\gamma}(x) = M^o(x) \quad L(x) \quad . \tag{3.5}$$

The values of $\bar{\gamma}(x)$ are given in Table 13. The computational procedure is completely analogous to the one used in the fertility analysis. The only difference is that $F(x)$ of (2.4) is replaced by $M^o(x)$. For example, $\bar{\gamma}(20)$ is

$$\bar{\gamma}(20) = \begin{bmatrix} 0.007170 & 0 \\ 0 & 0.000937 \end{bmatrix} \begin{bmatrix} 4.456621 & 0.037170 \\ 0.349255 & 4.373158 \end{bmatrix}$$

$$= \begin{bmatrix} 0.031954 & 0.000267 \\ 0.000327 & 0.004098 \end{bmatrix}$$

The expected number of migrations an individual makes during his lifetime is given by the summation of $\bar{\gamma}(x)$ over all x. The result is the <u>net migra-production matrix</u> (Rogers, 1975b, p. 8):

$$NMR = \sum_x \bar{\gamma}(x) \tag{3.6}$$

where

$$NMR = \begin{bmatrix} {}_1NMR_1 & {}_2NMR_1 \\ {}_1NMR_2 & {}_2NMR_2 \end{bmatrix}$$

$$\overline{{}_1NMR.} \qquad \overline{{}_2NMR.}$$

Table 13. integrals of generalized net mobility function

initial region of cohort slovenia

age	slovenia	r.yugos.
0	0.013849	0.000009
5	0.010892	0.000015
10	0.006972	0.000021
15	0.023773	0.000144
20	0.031954	0.000327
25	0.023813	0.000245
30	0.015729	0.000202
35	0.007251	0.000142
40	0.004101	0.000119
45	0.002160	0.000061
50	0.002573	0.000084
55	0.002354	0.000130
60	0.003130	0.000123
65	0.003040	0.000087
70	0.002335	0.000037
75	0.002183	0.000036
80	0.000817	0.000047
85	0.000000	0.000000
total	0.156926	0.001830

initial region of cohort r.yugos.

age	slovenia	r.yugos.
0	0.000009	0.001287
5	0.000019	0.000739
10	0.000017	0.000697
15	0.000106	0.002996
20	0.000267	0.004098
25	0.000284	0.002190
30	0.000222	0.001500
35	0.000113	0.000958
40	0.000068	0.000765
45	0.000038	0.000386
50	0.000046	0.000521
55	0.000044	0.000788
60	0.000062	0.000730
65	0.000063	0.000502
70	0.000050	0.000208
75	0.000047	0.000197
80	0.000018	0.000250
85	0.000000	0.000000
total	0.001473	0.018813

The column sum ${}_{i}NMR$ denotes the total expected number of migrations to be made by a person born in region i. Some of these, i.e. ${}_{i}NMR_{j}$ migrations are made out of region j. In other words, ${}_{i}NMR_{j}$ is the number of times a person born in region i is expected to leave region j. The total number of migrations expected to be made by the current birth cohorts out of region j is of course

$$E_j = \sum_i {}_{i}NMR_{j} Q_{1i}$$

or in matrix notation

$$\{\underset{\sim}{E}\} = \underset{\sim}{NMR} \; \{\underset{\sim}{Q}_1\} \tag{3.7}$$

The moments of the GM-function are completely analogous to those of the GNM - function. The n-th moment of the GM-function is defined as:

$$\underset{\sim}{D}(n) = \int_0^{\omega} x^n \; \underset{\sim}{\gamma}(x) \; dx \tag{3.8}$$

where ω is the highest age of the population. The numerical approximation of (3.8) is:

$$\bar{\underset{\sim}{D}}(n) = \sum_{x=0}^{z-5} (x + 2.5)^n \; \bar{\underset{\sim}{\gamma}}(x)$$

$$= \sum_{x=0}^{z-5} (x + 2.5)^n \; \underset{\sim}{M}^{o}(x) \; \underset{\sim}{L}(x) \tag{3.9}$$

with z being the highest age in the discrete case and z-5 the starting age of the highest age group.

The moments of the GM-function are contained in Table 14. The zeroth moment, $\bar{\underset{\sim}{D}}(0)$, is identical to the migra-production matrix, which is given in Table 15 together with the migra-production allocations. The row sums of $\bar{\underset{\sim}{D}}(0)$ represent the elements of $\{\underset{\sim}{E}\}$ for the case of unit regional radices. The net migra-production allocation ${}_i\varepsilon_j$ denotes the fraction of the migrations made by an i born individual, that are out of region j (Rogers, 1975b, p. 8). For example,

$$ {}_1\varepsilon_2 = \frac{{}_1NMR_2}{{}_1NMR.} = \frac{0.001830}{0.158755} = 0.011526. $$

The <u>global NMR</u> or the <u>Wilber-index</u> is computed as:

$$ \sum_x M^n(x)\ L^n(x), $$

where $M^n(x)$ are the average national age-specific migration rates (see Table 4 in Willekens and Rogers, 1975, p. 11) and $L^n(x)$ is the aggregated life table population distribution:

$$ L^n(x) = \frac{1}{\sum_k Q_k} \sum_i Q_i \ [\sum_j {}_iL_j(x)] $$

The mean and the variance of the GM-function are given by formulas (2.14) to (2.17) in which $F_j(x)$ is replaced by $M_j^o(x)$ and $\underset{\sim}{F}(x)$ by $\underset{\sim}{M}^o(x)$. The global NMR is given in Table 15.

Table 16 lists the means and variances of the generalized mobility function.

Table 14. moments of integral function

0 moment

	total	slovenia	r.yugos.
slovenia	0.158398	0.156926	0.001473
r.yugos.	0.020642	0.001830	0.018813
total		0.158755	0.020285

1 moment

	total	slovenia	r.yugos.
slovenia	4.184498	4.130428	0.054071
r.yugos.	0.632692	0.071622	0.561070
total		4.202050	0.615141

2 moment

	total	slovenia	r.yugos.
slovenia	156.705719	154.243912	2.461800
r.yugos.	26.584549	3.429838	23.154711
total		157.673752	25.616510

Table 15. spatial migration expectancies
****

net migraproduction rate

	total	slovenia	r.yugos.
slovenia	0.158398	0.156926	0.001473
r.yugos.	0.020642	0.001830	0.018813
total		0.158755	0.020285
eigenvalue	0.156945		
eigenvector			
- right		1.000000	0.013246
- left		1.000000	0.010662

net migraproduction allocations

	total	slovenia	r.yugos.
slovenia	1.061078	0.988474	0.072604
r.yugos.	0.938922	0.011526	0.927396
total		1.000000	1.000000

global nmr = 0.032255 (wilber index)

Table 16. matrices of mean ages and variances

** alternative 1 **

means

	total	slovenia	r.yugos.
slovenia	31.516935	26.320940	36.712929
r.yugos.	34.483528	39.142857	29.824196
total		32.731899	33.268562

variances

	slovenia	r.yugos.
slovenia	290.119568	323.677246
r.yugos.	342.310913	341.327515

** alternative 2 **

means

	total	slovenia	r.yugos.
slovenia	27.125751	26.311443	0.814309
r.yugos.	29.924438	0.108755	29.815681
total		26.420198	30.629990

variances

	total	slovenia	r.yugos.
slovenia	298.155457	289.901672	8.253784
r.yugos.	342.566650	1.407888	341.158752
total		291.309570	349.412537

3.2 The Weighted Generalized Net Mobility Function

Mobility analysis of the stable population leads to the concept of the weighted generalized net mobility (WGM) function. The WGM-function is estimated by replacing the life table population in (3.4) and (3.5) by the stable population.

$$\underset{\sim}{\gamma}^{(r)}(x) = \underset{\sim}{m}^{o}(x)\ e^{-rx}\ \underset{\sim}{\ell}(x) \tag{3.10}$$

and

$$\underset{\sim}{\bar{\gamma}}^{(r)}(x) = \underset{\sim}{M}^{o}(x)\ e^{-r(x+2.5)}\ \underset{\sim}{L}(x) \quad . \tag{3.11}$$

The weights are e^{-rx} and $e^{-r(x+2.5)}$ respectively. The numerical values of $\underset{\sim}{\bar{\gamma}}^{(r)}(x)$ are given in Table 17. Summation of $\underset{\sim}{\bar{\gamma}}^{(r)}(x)$ over all x yields the characteristic mobility matrix $\underset{\sim}{\Gamma}(r)$:

$$\underset{\sim}{\Gamma}(r) = \sum_{x} \underset{\sim}{\bar{\gamma}}^{(r)}(x) \quad . \tag{3.12}$$

An element ${}_i\Gamma_j(r)$ denotes the average number of migrations out of region j in the stable population that an i born person is expected to make during his lifetime. The right eigenvector of $\underset{\sim}{\Gamma}(r)$ represents the regional distribution of births that would result in an equal distribution of the outmigrants. In other words, if the births are distributed according to the right eigenvector of $\Gamma(r)$, $\{\underset{\sim}{z}\}$ say, then the relative regional distribution of the migrants and the births are the same. This can easily be seen by writing the characteristic equation

$$\lambda[\underset{\sim}{\Gamma}(r)]\ \{\underset{\sim}{z}\} = \underset{\sim}{\Gamma}(r)\ \{\underset{\sim}{z}\} \tag{3.13}$$

$$\lambda[\underset{\sim}{\Gamma}(r)]\ \{\underset{\sim}{z}\} = [\sum_{x} \underset{\sim}{M}^{o}(x)\ e^{-r(x+2.5)}\ \underset{\sim}{L}(x)]\ \{\underset{\sim}{z}\}$$

$$= \sum \underset{\sim}{M}^{o}(x)\ \underset{\sim}{L}^{(r)}(x)\ \{\underset{\sim}{z}\}$$

Table 17. integrals of weighted generalized net mobility function

initial region of cohort slovenia

age	slovenia	r.yugos.
0	0.013639	0.000009
5	0.010405	0.000015
10	0.006460	0.000020
15	0.021367	0.000130
20	0.027856	0.000285
25	0.020136	0.000207
30	0.012900	0.000166
35	0.005768	0.000113
40	0.003164	0.000092
45	0.001617	0.000046
50	0.001868	0.000061
55	0.001658	0.000091
60	0.002138	0.000084
65	0.002014	0.000058
70	0.001500	0.000024
75	0.001361	0.000022
80	0.000494	0.000028
85	0.000000	0.000000
total	0.134346	0.001450

initial region of cohort r.yugos.

age	slovenia	r.yugos.
0	0.000009	0.001267
5	0.000018	0.000706
10	0.000016	0.000645
15	0.000095	0.002693
20	0.000232	0.003572
25	0.000240	0.001852
30	0.000182	0.001230
35	0.000090	0.000762
40	0.000053	0.000591
45	0.000028	0.000289
50	0.000034	0.000378
55	0.000031	0.000555
60	0.000042	0.000499
65	0.000042	0.000333
70	0.000032	0.000134
75	0.000030	0.000123
80	0.000011	0.000151
85	0.000000	0.000000
total	0.001184	0.015780

where $\underset{\sim}{L}^{(r)}(x)$ is the distribution of the age group x to x + 4 in the stable population, by place of residence and by place of birth, and $\lambda[\underset{\sim}{\Gamma}(r)]$ is the dominant eigenvalue of $\underset{\sim}{\Gamma}(r)$. In our numerical example, (3.13), is:

$$0.134360 \cdot \begin{bmatrix} 1.000000 \\ \\ 0.012229 \end{bmatrix} = \begin{bmatrix} 0.134346 & 0.001184 \\ \\ 0.001450 & 0.015780 \end{bmatrix} \begin{bmatrix} 1.000000 \\ \\ 0.012229 \end{bmatrix} .$$

At stability, the migrants have not only the same relative regional distribution as the births, but they also are proportional to the number of births. If the vector of births is $\{\underset{\sim}{Q}^m\}$, with elements proportional to $\{\underset{\sim}{z}\}$, then the vector of migrants $\{\underset{\sim}{Z}\}$ is:

$$\{\underset{\sim}{Z}\} = \underset{\sim}{\Gamma}(r)\{\underset{\sim}{Q}^m\} = \lambda[\underset{\sim}{\Gamma}(r)]\{\underset{\sim}{Q}^m\}$$

For the system Slovenia - Rest of Yugoslavia $\lambda[\underset{\sim}{\Gamma}(r)] = 0.134360$, i.e. the number of migrants is 13 percent of the number of births. In other words, if the births are distributed according to $\{\underset{\sim}{Q}^m\}$, then the number of people leaving Slovenia during one generation (independent on where they are born) is 13% of the births in Slovenia in the beginning of this generation.

The moments of the WGM-function are defined analogously to (2.23):

$$\underset{\sim}{D}^{(r)}(n) = \int_0^{\omega} x^n \underset{\sim}{\gamma}^{(r)}(x)\,dx = \int_0^{\omega} x^n e^{-rx} \underset{\sim}{\gamma}(x)\,dx$$

and

$$\underset{\sim}{\bar{D}}^{(r)}(n) = \sum_0^{z-5} x^n \underset{\sim}{\bar{\gamma}}^{(r)}(x) = \sum_0^{z-5} x^n e^{-r(x+2.5)} \underset{\sim}{\bar{\gamma}}(x) .$$

The moments are given in Table 18. The zeroth moment is of course equal to $\underset{\sim}{\Gamma}(r)$, which is repeated in Table 19. The mean and variance of the WGM-function are derived in an analogous manner as equations (2.28) to (2.31) (Table 20). Finally, the discounted life expectancy matrix is represented in Table 21.

Table 18. moments of integral function

0 moment

	total	slovenia	r.yugos.
slovenia	0.135530	0.134346	0.001184
r.yugos.	0.017230	0.001450	0.015780
total		0.135796	0.016964

1 moment

	total	slovenia	r.yugos.
slovenia	3.353349	3.312093	0.041256
r.yugos.	0.493278	0.053842	0.439436
total		3.365935	0.480692

2 moment

	total	slovenia	r.yugos.
slovenia	118.014839	116.233238	1.781602
r.yugos.	19.543728	2.459748	17.083981
total		118.692986	18.865582

Table 19. spatial migration expectancies

net migraproduction rate

	total	slovenia	r.yugos.
slovenia	0.135530	0.134346	0.001184
r.yugos.	0.017230	0.001450	0.015780
total		0.135796	0.016964
eigenvalue	0.134360		
eigenvector			
- right		1.000000	0.012229
- left		1.000000	0.009986

net migraproduction allocations

	total	slovenia	r.yugos.
slovenia	1.059125	0.989321	0.069804
r.yugos.	0.940874	0.010679	0.930196
total		1.000000	1.000000

Table 20. matrices of mean ages and variances

** alternative 1 **

means

	total	slovenia	r.yugos.
slovenia	29.746502	24.653543	34.839462
r.yugos.	32.488285	37.129063	27.847506
total		30.891304	31.343485

variances

	slovenia	r.yugos.
slovenia	257.384216	290.717163
r.yugos.	317.659180	307.145569

** alternative 2 **

means

	total	slovenia	r.yugos.
slovenia	25.410284	24.645287	0.764997
r.yugos.	27.940247	0.100267	27.839979
total		24.745554	28.604977

variances

	total	slovenia	r.yugos.
slovenia	265.060608	257.196289	7.864311
r.yugos.	308.356445	1.366035	306.990417
total		258.562317	314.854736

Table 21. spatial migration expectancies

expectations of life

	total	slovenia	r.yugos.
slovenia	52.827232	52.227596	0.599635
r.yugos.	57.965794	5.629006	52.336788
total		57.856602	52.936424
eigenvalue	54.120216		
eigenvector			
- right		1.000000	3.156286
- left		1.000000	0.336226

migration levels

	total	slovenia	r.yugos.
slovenia	0.914035	0.902708	0.011327
r.yugos.	1.085965	0.097292	0.988672
total		1.000000	1.000000

4. FERTILITY ANALYSIS: CONTINUED

In this section we approach fertility analysis from a different perspective. Although the starting point is the net reproduction rate matrix ($\underset{\sim}{\text{NRR}}$) and the characteristic matrix ($\underset{\sim}{\Psi}(r)$ or $\underset{\sim}{R}^{(r)}(0)$), the interpretation is different. This allows us to derive additional useful information on the fertility behavior of the population.

Recall that both $\underset{\sim}{\text{NRR}}$ and $\underset{\sim}{\Psi}(r)$ represent the regional distribution of the offspring by place of birth of the mother. The matrix $\underset{\sim}{\text{NRR}}$ refers to a life table population, and $\underset{\sim}{\Psi}(r)$ to a stable population. The intrinsic or stable growth rate is r. In equation (2.18), the rate r may also be considered to be a rate of discount. Introducing the notion of discounting, and hence a time preference to the fact of having children, adds an interesting new dimension to fertility analysis.

The central concept is the reproductive value. It has been developed by Fisher (1929), and studied by Goodman (1967, 1971), Keyfitz (1975) and others. For a reformulation of the concept and a generalization to multiregional demographic systems, see Rogers and Willekens (1976b) and Willekens (1977). In this paper we highlight only a few important elements of the theory of spatial reproductive value (section 1), and focus on the computational algorithms (section 2).

4.1. The Theory of the Spatial Reproductive Value

Fisher (1929) looks at a life as a debt one has incurred at birth, and at the offspring of a child as the repayment of this debt. Let the debt or loan incurred at birth be equal to unity. At stability, the present value of the subsequent repayment must equal the debt:

$$1 = \int_0^\infty e^{-ra}\, m(a)\ell(a)da = \Psi(r) \tag{4.1}$$

where $m(a)\ell(a)da$ is the expected number of children to be born between ages a and a + da to a baby born in a life table population and obeying the observed fertility schedule, and r is the rate of discount. Equation (4.1) is of course identical to the characteristic equation of a single-region population system.

The multiregional counterpart of (4.1) is (Rogers, 1975a, p.93)

$$\{\underset{\sim}{Q}^S\} = \underset{\sim}{\Psi}(r) \{\underset{\sim}{Q}^S\} \quad , \tag{4.2}$$

where $\{\underset{\sim}{Q}^S\}$ is the right eigenvector, associated with the dominant eigenvalue of $\underset{\sim}{\Psi}(r)$. An alternative generalization of (4.1) is

$$\{\underset{\sim}{\nu}(0)\}' = \{\underset{\sim}{\nu}(0)\}' \underset{\sim}{\Psi}(r) \tag{4.3}$$

where $\{\underset{\sim}{\nu}(0)\}'$ is the corresponding left eigenvector of $\underset{\sim}{\Psi}(r)$ and the prime denotes the transpose.

Both formulations (4.2) and (4.3) have their demographic significance. Equation (4.2) has already been considered in section 2 of this paper. The eigenvector $\{\underset{\sim}{Q}^S\}$ gives the regional distribution of births in the stable population. Following the investment approach to life and childbearing, $\{\underset{\sim}{Q}^S\}$ denotes the spatial distribution of the investments (or births) which makes the intrinsic rate of return of each investment equal to r, the equilibrium rate of return. Whereas $\{\underset{\sim}{Q}^S\}$ denotes the <u>number</u> of births, the left eigenvector $\{\underset{\sim}{\nu}(0)\}'$ represents the marginal <u>value</u> of an additional unit birth, or in other words, the <u>reproductive value</u> of a 0-year old girl. The value is measured in terms of contribution to the ultimate population of the demographic system. It reflects the capacity to produce new life. Note that, since the model we consider is linear, the marginal value of one birth is equal to its average value.

We explore now the investment approach to fertility analysis a little further. If the regional distribution of births is $\{\underset{\sim}{Q}^S\}$, then the present value of the offspring must also equal $\{\underset{\sim}{Q}^S\}$ (equation (4.2)). This implies that

$$Q_i^S = \sum_j {}_j\Psi_i(r)\; Q_j^S \quad . \tag{4.4}$$

In each region, the discounted number of offspring must be equal to the current number of births. In other words, each region must pay back the debt it has incurred by receiving Q_i^S births. A part of this debt is paid back by people born in another region. People born in region j, for example, contribute a total of ${}_j\mathrm{NRR}_i\, Q_j^S$ to region i, which has a discounted value of ${}_j\Psi_i Q_j^s$. Recall that in the numerical illustration of Slovenia - Rest of Yugoslavia,

$$\underset{\sim}{NRR} = \begin{bmatrix} 0.961876 & 0.010687 \\ 0.122364 & 1.174812 \end{bmatrix}$$

Equation (4.2) is

$$\begin{bmatrix} 1.000000 \\ 20.823662 \end{bmatrix} = \begin{bmatrix} 0.813686 & 0.008942 \\ 0.102414 & 0.994966 \end{bmatrix} \begin{bmatrix} 1.000000 \\ 20.823662 \end{bmatrix}$$

One baby born in Slovenia is replaced by an average of

$$0.961876 * 1.000000 + 0.010687 * 20.823662 =$$

$$0.961876 + 0.222542 = 1.184418$$

babies in the stable population. An average of 0.961876 babies will be born to mothers who are born in Slovenia themselves, and 0.222542 will be born to mothers born in the Rest of Yugoslavia. The present value of the 0.961876 babies is 0.813686, and of 0.222542 is 0.008942 * 20.823662 = 0.186205. Hence the average present value of a baby born in Slovenia to a Slovenia-born woman is

$$\frac{0.813686}{0.961876} = 0.845936,$$

while that of a baby born in Slovenia to a Rest of Yogoslavia-born woman is

$$\frac{0.994966}{1.174812} = 0.846915$$

The difference is explained by the difference in mean ages at childbearing in the stable population and the stationary population.

Equation (4.2) expresses births in one generation as a function of the number of births in the previous generation. It denotes the number of daughters by which a woman is replaced in the stable population, or, alternatively, the present value of the daughters replacing a woman, the mortality and migration behavior of which is given by the life table. The regional distribution of births is consistent with the given fertility, mortality and migration schedules and with the growth rate or rate of discount r. Since these schedules differ from one region to another while r is unique, a birth in a less fertile region contributes less to the sustainment of the overall r than a birth in a highly fertile area. The value of a birth for sustaining r depends on the capacity of the 0-year old to produce new lives. This capacity is measured by the reproductive value.

The vector $\{\underset{\sim}{\nu}(0)\}'$ denotes the reproductive value of a baby or a 0-year old girl by region. If the reproductive value of a 0-year old in region i is $\nu_i(0)$, then the value of the discounted number of offsprings must also be $\nu_i(0)$, i.e.

$$\nu_i(0) = \nu_i(0)\ {}_i\Psi_i(r) + \nu_j(0)\ {}_i\Psi_j(r),$$

or

$$\nu_i(0) = \sum_x e^{-r(x+2.5)} \nu_i(0)\ F_i(x)\ {}_iL_i(x) + \sum_x e^{-r(x+2.5)}\ \nu_j(0)\ F_j(x)\ {}_iL_j(x) \quad . \qquad (4.5)$$

Equation (4.5) indicates an equivalent formulation: the present worth of the reproductive value of the offspring must equal to the reproductive value of the 0-year old. If $\nu_i(0)$ represents the value (cost) of the life invested in an individual, then he must pay off the value of this investment. Since $\nu_i(0) \neq \nu_j(0)$, $\sum_j {}_i\Psi_j(r) \neq 1$, which means that the discounted number of offspring of an individual does not have to be exactly one.

Consider the Slovenia - Rest of Yugoslavia example. The matrix $\underset{\sim}{\Psi}(r)$ is given in Table 9. The left eigenvector is

$$\{\underset{\sim}{\nu}(0)\} = \begin{bmatrix} 1.000000 \\ \\ 1.818116 \end{bmatrix} , \qquad (4.6)$$

and equation (4.3) becomes

$$[1.000000 \quad 1.818116] = [1.000000 \quad 1.818116] \begin{bmatrix} 0.813686 & 0.008942 \\ \\ 0.102414 & 0.994966 \end{bmatrix}$$

Assuming the reproductive value of a 0-year old in Slovenia to be unity, then the reproductive value of the baby in the Rest of Yugoslavia is 1.818. Any norming may be used since the eigenvector is fixed up to a scalar. Throughout this paper, the regional reproductive values are scaled such that $\nu_1(0) = 1$.

Note that the discounted number of daughters of a Slovenia-born girl is 0.916100, i.e. less than unity. Therefore, she does not replace herself by one child (discounted). The value of the offspring, however, is equal to her reproductive value at birth:

$$\nu_1(0) = 1.000 = 1.000 * 0.814 + 1.818 * 0.102 .$$

4.2 The Computation of the Spatial Reproductive Value

The above interpretation of (4.3) suggests asking what is the productive capacity of a girl aged x. The answer is the expected number of subsequent children discounted back to age x, and weighted for the region of birth. The vector of reproductive values of x-year old women, differentiated by region of residence, is:

$$\{\underset{\sim}{v}(x)\}' = \{\underset{\sim}{v}(0)\}' \int_x^\infty e^{-r(a-x)}\ \underset{\sim}{m}(a)\ \underset{\sim}{\ell}(a)\ da\ [\underset{\sim}{\ell}(x)]^{-1}$$

$$= \{\underset{\sim}{v}(0)\}'\ \underset{\sim}{n}(x),\ \text{say.} \qquad (4.7)$$

The matrix

$$\underset{\sim}{n}(x) = \begin{bmatrix} n_{11}(x) & n_{21}(x) \\ n_{12}(x) & n_{22}(x) \end{bmatrix} \qquad (4.8)$$

represents the expected total number of female offspring per woman at age x, discounted back to age x. The element $n_{ij}(x)$ gives the discounted number of daughters to be born in region j to a woman now x years of age and a resident of region i. There are two approaches to evaluate (4.2) and (4.7) numerically. The first approach evaluates the reproductive values at exact age x:

$$\{\underset{\sim}{v}(x)\}' = \{\underset{\sim}{v}(0)\}' \sum_{a=x}^{\beta-5} [e^{-r(a+2.5-x)}\ \underset{\sim}{M}(a)\underset{\sim}{L}(a)][\underset{\sim}{\ell}(x)]^{-1} \qquad (4.9)$$

$$= \{\underset{\sim}{v}(0)\}'\ \bar{\underset{\sim}{n}}_x\ ,\ \text{say.} \qquad (4.10)$$

Both $\bar{\underset{\sim}{n}}_x$ and $\{\underset{\sim}{v}(x)\}$ refer to exact age x. The values of $\bar{\underset{\sim}{n}}_x$ for Slovenia - Rest of Yugoslavia are given in Table 22. For example, the discounted number of female descendants of a woman living in Slovenia and 10 years old is 1.002. A total of 0.9168 is expected to be born in Slovenia and 0.0852 in the Rest of Yugoslavia. On the other hand, a woman of the same age in the Rest of Yugoslavia has an expected discounted number of daughters of 1.1984. Because of the low

Table 22. results for people at exact age x
****
****

discounted number of offspring per person
****

region of residence slovenia
----

	region of birth of offspring		
	total	slovenia	r.yugos.
0	0.916100	0.813686	0.102414
5	0.971974	0.877292	0.094682
10	1.002009	0.916767	0.085242
15	1.032697	0.953190	0.079507
20	0.981935	0.929147	0.052788
25	0.652048	0.630543	0.021506
30	0.351120	0.344533	0.006587
35	0.153903	0.152558	0.001344
40	0.042952	0.042710	0.000242
45	0.004836	0.004807	0.000028
50	0.001423	0.001417	0.000006
55	0.000000	0.000000	0.000000
60	0.000000	0.000000	0.000000
65	0.000000	0.000000	0.000000
70	0.000000	0.000000	0.000000
75	0.000000	0.000000	0.000000
80	0.000000	0.000000	0.000000

region of residence r.yugos.
----

	region of birth of offspring		
	total	slovenia	r.yugos.
0	1.003908	0.008942	0.994966
5	1.158391	0.009091	1.149300
10	1.198425	0.008654	1.189771
15	1.238313	0.008204	1.230109
20	1.148223	0.005266	1.142957
25	0.743626	0.001747	0.741879
30	0.394423	0.000555	0.393869
35	0.184441	0.000131	0.184310
40	0.072078	0.000023	0.072054
45	0.013779	0.000002	0.013777
50	0.003460	0.000000	0.003460
55	0.000000	0.000000	0.000000
60	0.000000	0.000000	0.000000
65	0.000000	0.000000	0.000000
70	0.000000	0.000000	0.000000
75	0.000000	0.000000	0.000000
80	0.000000	0.000000	0.000000

migration level out of the Rest of Yugoslavia and the relatively low fertility in Slovenia, only an average of 0.0087 daughters will be born in Slovenia to these women.

The reproductive values by age, $\{\underset{\sim}{v}(x)\}$ are represented in Table 23. For instance, the reproductive value of 10-year old girls is

$$\{\underset{\sim}{v}(10)\}' = \{\underset{\sim}{v}(0)\}' \ \bar{\underset{\sim}{n}}_{10} \quad , \quad \text{or}$$

$$[1.071747 \quad 2.171796] = [1.000000 \quad 1.818166] \begin{bmatrix} 0.916767 & 0.008654 \\ 0.085242 & 1.189771 \end{bmatrix}$$

Note that $\bar{\underset{\sim}{n}}_o$ is identical to the characteristic matrix $\underset{\sim}{\Psi}(r)$.

The second approach computes the average reproductive value for each age group x to x + 4. Denote this by $\{{}_5\underset{\sim}{V}_x\}$, then

$$\{{}_5\underset{\sim}{V}_x\}' = \{\underset{\sim}{v}(0)\}' \ \frac{5}{2} \sum_{a=x}^{\beta-5} [e^{-r(a-x)}\underset{\sim}{M}(a) \ \underset{\sim}{L}(a)$$

$$+ e^{-r(a+5-x)}\underset{\sim}{M}(a+5)\underset{\sim}{L}(a+5)] \ [\underset{\sim}{L}(x)]^{-1}$$

$$= \{\underset{\sim}{v}(0)\}' \ \frac{5}{2} \sum_{a=x}^{\beta-5} [\underset{\sim}{M}(a) + e^{-5r} \ \underset{\sim}{M}(a+5)\underset{\sim}{S}(a)]$$

$$\cdot \quad e^{-r(a-x)}\underset{\sim}{L}(a) \ [\underset{\sim}{L}(x)]^{-1} \qquad (4.11)$$

$$= \{\underset{\sim}{v}(0)\}' \ {}_5\underset{\sim}{N}_x \quad , \quad \text{say.} \qquad (4.12)$$

The matrix ${}_5\underset{\sim}{N}_x$ gives the discounted number of offspring per person in age group x to x + 4, and not the number per person at exact age x (Table 24). It has been shown by Willekens

(1977, p. 14) that ${}_5\underset{\sim}{N}_x$ may be expressed in terms of ${}_5\underset{\sim}{N}_{x+5}$:

$${}_5\underset{\sim}{N}_x = \frac{5}{2}\,\underset{\sim}{M}(x) + [\frac{5}{2}\,\underset{\sim}{M}(x+5) + {}_5\underset{\sim}{N}_{x+5}]\; e^{-5r}\,\underset{\sim}{S}(x) \quad . \qquad (4.13)$$

The associated average reproductive values by age group are listed in Table 25.

Table 23. spatial reproductive value per person
****

	slovenia	r.yugos.
0	1.000000	1.818116
5	1.049436	2.098651
10	1.071747	2.171796
15	1.097742	2.244684
20	1.025122	2.083294
25	0.669643	1.350569
30	0.356509	0.716653
35	0.155003	0.335227
40	0.043150	0.131027
45	0.004859	0.025051
50	0.001427	0.006291
55	0.000000	0.000000
60	0.000000	0.000000
65	0.000000	0.000000
70	0.000000	0.000000
75	0.000000	0.000000
80	0.000000	0.000000

Table 24. results for people in age group x

discounted number of offspring per person

region of residence slovenia

region of birth of offspring

x	total	slovenia	r.yugos.
0	0.943847	0.844677	0.099170
5	0.986890	0.896790	0.090100
10	1.017244	0.934792	0.082452
15	1.007583	0.941816	0.065767
20	0.819229	0.785372	0.033857
25	0.503857	0.492104	0.011753
30	0.254235	0.251223	0.003013
35	0.099454	0.098905	0.000549
40	0.024254	0.024192	0.000062
45	0.003166	0.003156	0.000010
50	0.000730	0.000730	0.000000
55	0.000000	0.000000	0.000000
60	0.000000	0.000000	0.000000
65	0.000000	0.000000	0.000000
70	0.000000	0.000000	0.000000
75	0.000000	0.000000	0.000000
80	0.000000	0.000000	0.000000

region of residence r.yugos.

region of birth of offspring

x	total	slovenia	r.yugos.
0	1.076651	0.009032	1.067619
5	1.178234	0.008880	1.169355
10	1.218204	0.008436	1.209768
15	1.194267	0.006720	1.187547
20	0.949871	0.003167	0.946704
25	0.572510	0.000973	0.571537
30	0.291552	0.000260	0.291292
35	0.129433	0.000047	0.129386
40	0.043597	0.000004	0.043593
45	0.008752	0.000001	0.008751
50	0.001785	0.000000	0.001785
55	0.000000	0.000000	0.000000
60	0.000000	0.000000	0.000000
65	0.000000	0.000000	0.000000
70	0.000000	0.000000	0.000000
75	0.000000	0.000000	0.000000
80	0.000000	0.000000	0.000000

Table 25. spatial reproductive value per person
****

	slovenia	r.yugos.
0	1.024979	1.950086
5	1.060603	2.134902
10	1.084699	2.207934
15	1.061388	2.165818
20	0.846928	1.724385
25	0.513472	1.040094
30	0.256700	0.529862
35	0.099904	0.235286
40	0.024305	0.079261
45	0.003175	0.015911
50	0.000730	0.003245
55	0.000000	0.000000
60	0.000000	0.000000
65	0.000000	0.000000
70	0.000000	0.000000
75	0.000000	0.000000
80	0.000000	0.000000

The discounted number of offspring and the reproductive value in (4.12) and (4.13) are expressed per person in age group x to x + 4 of the life table population. To obtain an estimate of the discounted number of offspring and the reproductive value of the total observed population, we multiply ${}_5\tilde{N}_x$ and $\{{}_5\tilde{V}_x\}$ by the observed population distribution and sum over all age groups:

$$\tilde{NK} = \sum_{x=0}^{z-5} {}_5\tilde{N}_x \, \tilde{K}(x) \qquad (4.14)$$

and

$$\{\tilde{V}\} = \{\tilde{\nu}(0)\}' \, {}_5\tilde{N}_x \, \tilde{K}(x) = \{\tilde{\nu}(0)\}' \, \tilde{NK} \quad , \qquad (4.15)$$

where $\tilde{K}(x)$ is the diagonal matrix containing the regional populations aged x to x + 4.

The value of $\tilde{NK}$ is given in Table 26a. Under the 1961 regime of fertility, mortality and migration, the total discounted number of female offspring of Yugoslavia is 5,528,628. Of them, 382,695 or 6.92 % will be born in Slovenia. However, the female residents of Slovenia will account for only 379,094 or 6.68% of the total discounted number of births. Of the ultimate discounted 382,695 female children born in Slovenia, 29,934 can be attributed to women now residing in the Rest of Yugoslavia. On the other hand, of the discounted 379,094 daughters born to the female population of Slovenia, 26,333 will be born in the Rest of Yugoslavia, and 352,761 in Slovenia.

The reproductive value of the total female population by place of residence is obtained by (4.15), i.e. by weighting the discounted number of offspring for the region of birth. If we attach to a birth in Slovenia the reproductive value of unity, then a birth in the Rest of Yugoslavia has a reproductive value of 1.818. Adopting this scaling, the total reproductive value

Table 26a. total discounted number of offspring
****

of observed population in 100,000.
****

	total	slovenia	r.yugos.
slovenia	3.826951	3.527613	0.299338
r.yugos.	51.459335	0.263329	51.196007
total		3.790943	51.495346

Table 26b. reproductive value of the total
****

population in 100,000.
****

slovenia	4.006376
r.yugos.	93.379601
total	97.385979

by region of residence is:

$$[1.000000 \quad 1.818116]\begin{bmatrix} 352,761 & 29,934 \\ \\ 26,333 & 5,119,601 \end{bmatrix} = \begin{bmatrix} 400,638 \\ \\ 9,337,960 \end{bmatrix}$$

The total reproductive value for the whole of Yugoslavia is (Table 26b):

$$V = 400,638 + 9,337,960 = 9,738,598 \quad .$$

Note that the unit in which V is measured is the reproductive value of a birth or a 0-year old in Slovenia. The choice of the unit is arbitrary, since its only function is that of a "numeraire".

5. FURTHER STABLE POPULATION ANALYSIS

In sections 2 and 3 of this paper, we performed some introductory analyses of fertility and migration characteristics of stationary populations. In this section, stable population analysis is advanced by means of the notion of spatial reproductive value, developed in the previous section.

If age-specific birth, death and migration rates remain fixed, then a population exposed to these rates ultimately will evolve into a stable population whose principal characteristics are: unchanging regional age compositions and regional shares; constant regional annual rates of birth, death, and migration; and a fixed multiregional annual rate of growth that also is the annual rate of population growth in each and every region (Rogers and Willekens, 1976c, p. 12). The constant growth rate implies that births and population increase at the same rate and follow an exponential growth path. This trajectory may be expressed in terms of observed population characteristics. This is the topic of the first part of this section. The second part focuses on the calculation of the intrinsic rates of birth, death, out- and inmigration.

5.1 The Ultimate Trajectory of Births and Population

When a multiregional population system has reached stability (steady-state equilibrium), its births grow exponentially and their regional distribution remains constant. The ultimate birth trajectory is (Willekens, 1977, p.29)[8]:

$$\{\underset{\sim}{Q}^{(t)}\} = e^{rt} \frac{V}{\{\underset{\sim}{\nu}(0)\}'\underset{\sim}{\kappa}\{\underset{\sim}{Q}_1\}}\{\underset{\sim}{Q}_1\} , \qquad (5.1)$$

where r is the stable growth rate, V is the total reproductive value of the whole population system, $\{\underset{\sim}{\nu}(0)\}'$ and $\{\underset{\sim}{Q}_1\}$ are respectively the left and right eigenvectors of $\underset{\sim}{\Psi}(r)$, associated with the dominant eigenvalue, and $\underset{\sim}{\kappa}$ is the matrix of mean ages of childbearing in the stable population, defined in (2.30):

$$\underset{\sim}{\kappa} = [\underset{\sim}{R}^{(r)}(1)][\underset{\sim}{R}^{(r)}(0)]^{-1} \qquad (5.2)$$

The expression $\{\underset{\sim}{\nu}(0)\}'\underset{\sim}{\kappa}\{\underset{\sim}{Q}_1\}$ is a normalizing factor. Writing

$$\kappa = \{\underset{\sim}{\nu}(0)\}' \underset{\sim}{\kappa} \{\underset{\sim}{Q}_1\} , \qquad (5.3)$$

yields the simple expression for the ultimate birth trajectory:

$$\{\underset{\sim}{Q}^{(t)}\} = e^{rt} \frac{V}{\kappa} \{\underset{\sim}{Q}_1\} . \qquad (5.4)$$

If $\{\underset{\sim}{Q}_1\}$ is chosen such that its elements sum up to unity, then the ultimate total number of births is proportional to the total reproductive value. The total number of births is then allocated to the different regions according to $\{\underset{\sim}{Q}_1\}$.

---

[8] The superscript of $\{\underset{\sim}{Q}^s\}$ is dropped for convenience

Substituting V in (5.4) and rewriting shows that the stable number of births in each region $\{\underset{\sim}{Q}^{(t)}\}$ also is a linear combination of the discounted number of offspring by region of birth (for details, see Willekens, 1977, pp. 32-33). The stable equivalent of births is:

$$\{\underset{\sim}{Q}^{(0)}\} = \{\underset{\sim}{Q}\} = \frac{V}{\kappa} \{\underset{\sim}{Q}_1\} \quad . \tag{5.5}$$

Recall our numerical illustration. The matrix of mean ages of childbearing is given in Table 11. Since the growth rate r is 0.006099, the normalizing factor, (5.3), is 1054.266 (Table 28). The total reproductive value V has been computed to be 9,738,598; hence the stable equivalent of births is by (5.5):

$$\{\underset{\sim}{Q}\} = \frac{9{,}738{,}598}{1054.266} \begin{bmatrix} 1.000000 \\ 20.823662 \end{bmatrix}$$

$$= \begin{bmatrix} 9{,}237 \\ 192{,}355 \end{bmatrix} \quad . \tag{5.6}$$

The total number of births is 201,592. Of this number of babies, 4.58% will be born in Slovenia and 95.42% in the Rest of Yugoslavia[9].

[9]Compare this with the observed number of births (205,010) and its regional distribution: 6.90% in Slovenia vs. 93.10% in the Rest of Yugoslavia.

The stable equivalent population in each age group x to x + 4 is easily obtained by the formula (1.5):

$$\{\underset{\sim}{K}(x)\} = e^{-r(x+2.5)} \underset{\sim}{L}(x) \{\underset{\sim}{Q}\} \quad . \tag{1.5}$$

The stable equivalent of the total population is:

$$\{\underset{\sim}{Y}\} = \sum_x \{\underset{\sim}{K}(x)\} = [\sum_x e^{-r(x+2.5)} \underset{\sim}{L}(x)] \{\underset{\sim}{Q}\} \quad . \tag{5.7}$$

Defining

$$\sum_x e^{-r(x+2.5)} \underset{\sim}{L}(x) = \underset{\sim}{e}^{(r)}(x) \tag{5.8}$$

as the matrix of discounted life expectancies at birth, equation (5.7) becomes

$$\{\underset{\sim}{Y}\} = \underset{\sim}{e}^{(r)}(0) \{\underset{\sim}{Q}\} \quad . \tag{5.9}$$

The numerical values of the stable equivalent population are given in Table 27. Note that those values are very close to the ones given by Willekens and Rogers (1976, p. 52), which were computed by projecting the observed population[10].

Equations (1.5) and (5.7) demonstrate that for population analysis it is more convenient to express the relative age composition of the population in unit births instead of in fractions or percentages of the total population. The values of

$$e^{-r(x+2.5)} \underset{\sim}{L}(x)$$

are given in Table 1c.

---

[10]Minor deviations are due to rounding error.

Table 27. stable equivalent of total population

	total	slovenia	r.yugos.
0	941654.	45086.	896568.
5	862431.	43386.	819045.
10	834174.	42248.	791926.
15	806463.	41802.	764661.
20	778260.	42121.	736138.
25	749753.	41965.	707788.
30	721359.	41049.	680310.
35	693192.	39990.	653202.
40	664290.	38859.	625432.
45	633669.	37448.	596221.
50	599231.	35742.	563489.
55	558765.	33875.	524890.
60	507986.	31559.	476427.
65	441715.	28066.	413649.
70	356051.	22843.	333208.
75	257354.	16436.	240917.
80	159177.	9514.	149663.
85	151501.	5796.	145705.
total	10717026.	597786.	10119240.

percentage distribution

	total	slovenia	r.yugos.
0	0.087865	0.075422	0.088600
5	0.080473	0.072577	0.080939
10	0.077836	0.070674	0.078259
15	0.075251	0.069928	0.075565
20	0.072619	0.070462	0.072746
25	0.069959	0.070201	0.069945
30	0.067310	0.068669	0.067229
35	0.064681	0.066898	0.064550
40	0.061985	0.065004	0.061806
45	0.059127	0.062644	0.058920
50	0.055914	0.059791	0.055685
55	0.052138	0.056667	0.051871
60	0.047400	0.052793	0.047081
65	0.041216	0.046950	0.040877
70	0.033223	0.038213	0.032928
75	0.024014	0.027495	0.023808
80	0.014853	0.015916	0.014790
85	0.014136	0.009696	0.014399

5.2 Stable Equivalents and Intrinsic Rates

The fertility, mortality and migration characteristics of a stable population may be described by a small number of parameters, namely the intrinsic rates. (Rogers, 1975a, pp. 109 - 115). The intrinsic rates are directly related to the stable equivalents of births, deaths, and migrants. Therefore, we treat both statistics simultaneously.

Applying the fixed age-specific schedules of fertility, mortality and migration to the stable equivalent of the population gives the stable equivalent of births, deaths and migrants. The stable equivalent of births has already been computed. Applying the fertility schedule to the population distribution of (1.5) and summing over all age groups yield of course the characteristic equation:

$$\{\underset{\sim}{Q}\} = \sum_x \underset{\sim}{F}(x) \{\underset{\sim}{K}(x)\}$$

$$= [\sum_x \underset{\sim}{F}(x)\, e^{-r(x+2.5)} \underset{\sim}{L}(x)] \{\underset{\sim}{Q}\} = \underset{\sim}{\bar{\Psi}}\{\underset{\sim}{Q}\} \quad .$$

The intrinsic birth rate of region i is the ratio between Q_i and the stable equivalent population Y_i, which may be written as (Rogers, 1975a, p. 115):

$$b_i = \frac{Q_i}{Y_i} = \frac{Q_i}{\sum_x e^{-r(x+2.5)} \sum_j {}_jL_i(x)\, Q_j}$$

$$= \frac{1}{\sum_x e^{-r(x+2.5)} \sum_j \frac{Q_j}{Q_i}\, {}_jL_i(x)} \quad .$$

The vector of intrinsic birth rates is:

$$\{\underset{\sim}{b}\} = \underset{\sim}{Y}^{-1} \{\underset{\sim}{Q}\} \quad , \qquad (5.10)$$

where $\underset{\sim}{Y}$ is the diagonal matrix of stable equivalents of total populations, i.e.

$$\underset{\sim}{Y} \{\underset{\sim}{1}\} = \{\underset{\sim}{Y}\} .$$

The vector $\{\underset{\sim}{b}\}$ also may be expressed as

$$\{\underset{\sim}{b}\} = \sum_x \underset{\sim}{F}(x) \{\underset{\sim}{C}(x)\} , \qquad (5.11)$$

where $\{\underset{\sim}{C}(x)\}$ denotes the age composition of the population as fractions of the total, i.e.

$$\{\underset{\sim}{C}(x)\} = \underset{\sim}{Y}^{-1} \{\underset{\sim}{K}(x)\} . \qquad (5.12)$$

The proportion of the regional population, which is aged x to x + 4, also may be written as follows:

$$\{\underset{\sim}{C}(x)\} = \underset{\sim}{Q}^{-1} \underset{\sim}{b} e^{-r(x+2.5)} \underset{\sim}{L}(x) \{\underset{\sim}{Q}\} , \qquad (5.13)$$

since by (5.10) $\underset{\sim}{Y}^{-1}$ is equal to $\underset{\sim}{Q}^{-1}\underset{\sim}{b}$, where both $\underset{\sim}{Q}$ and $\underset{\sim}{b}$ are diagonal matrices. Defining $\underset{\sim}{C}(x)$ as (Rogers, 1975a, p. 115):

$$\underset{\sim}{C}(x) = \underset{\sim}{b} e^{-r(x+2.5)} \underset{\sim}{L}(x) , \qquad (5.14)$$

gives

$$\{\underset{\sim}{C}(x)\} = \underset{\sim}{Q}^{-1} \underset{\sim}{C}(x) \{\underset{\sim}{Q}\} . \qquad (5.15)$$

To compute the stable equivalents of deaths, outmigrants and inmigrants, we must reconsider the age-specific death and migration rates (11). The deaths and outmigrants in age group x to x + 4 in a life table population are given by (Rogers and Ledent, 1976, p. 289).

$$\underset{\sim}{\ell}(x) - \underset{\sim}{\ell}(x+5) = \underset{\sim}{M}(x)\ \underset{\sim}{L}(x), \qquad (5.16)$$

where $\underset{\sim}{\ell}(x)$ represents the distribution of the life table population at exact age x by place of birth and place of residence,

$\underset{\sim}{L}(x)$ is given in (1.1) and represents the distribution of the life table population aged x to x + 4 by place of birth and place of residence, and

$\underset{\sim}{M}(x)$ is the matrix

$$\underset{\sim}{M}(x) = \begin{bmatrix} \left[M_{1\delta}(x) + \sum\limits_{j\neq 1} M_{1j}(x)\right] & -M_{21}(x) & \cdots\cdots & -M_{n1}(x) \\ -M_{12}(x) & \left[M_{2\delta}(x) + \sum\limits_{j\neq 2} M_{2j}(x)\right] & \cdots\cdots & -M_{n2}(x) \\ \vdots & \vdots & & \vdots \\ -M_{1n}(x) & -M_{2n}(x) & \cdots\cdots & \left[M_{n\delta}(x) + \sum\limits_{j\neq n} M_{nj}(x)\right] \end{bmatrix} \qquad (5.17)$$

---

(11) The authors are grateful to Jacques Ledent for pointing out the argument, which is analagous to the one proposed by Keyfitz (1968, pp. 19-20).

with $M_{i\delta}(x)$ and $M_{ij}(x)$ the age-specific life table death rate and migration rate, respectively.

Equation (5.16) is the discrete approximation of the continuous relation

$$\underset{\sim}{\ell}(x) - \underset{\sim}{\ell}(x + 5) = \int_0^5 \underset{\sim}{\mu}(x + t)\underset{\sim}{\ell}(x + t)dt, \qquad (5.18)$$

where $\underset{\sim}{\mu}(x)$ is a matrix of the format $\underset{\sim}{M}(x)$. Its elements are the age-specific forces of mortality $\mu_{i\delta}(x)$ and of migration $\mu_{ij}(x)$, i.e.

$$\underset{\sim}{\mu}(x) = -\frac{1}{dx}[d\underset{\sim}{\ell}(x)][\underset{\sim}{\ell}(x)]^{-1}$$

$$= -\frac{d\ \ell n\ \underset{\approx}{\ell}(x)}{dx}$$

Equation (5.18) represents the decrements by death and outmigration in a stationary population. To derive the decrements in a population growing at rate r, we write

$$\underset{\sim}{\ell}^{(r)}(x) - \underset{\sim}{\ell}^{(r)}(x + 5) = \int_0^5 \underset{\sim}{\mu}(x + t)\ \underset{\sim}{\ell}^{(r)}(x + t)dt,$$

with $\underset{\sim}{\ell}^{(r)}(x) = e^{-rx}\underset{\sim}{\ell}(x)$. Hence

$$\underset{\sim}{\ell}^{(r)}(x) - \underset{\sim}{\ell}^{(r)}(x + 5) = \int_0^5 \underset{\sim}{\mu}(x + t)\ e^{-r(x + t)}\underset{\sim}{\ell}(x + t)dt,$$

$$= \int_0^5 e^{-r(x + t)}\ d\ \underset{\sim}{\ell}(x + t)\ . \qquad (5.19)$$

Integration by parts yields

$$\underset{\sim}{\ell}^{(r)}(x) - \underset{\sim}{\ell}^{(r)}(x+5) = e^{-rx}\underset{\sim}{\ell}(x) - e^{-r(x+5)}\underset{\sim}{\ell}(x+5)$$

$$- r\int_0^5 e^{-r(x+t)}\, \underset{\sim}{\ell}(x+t)dt \;.$$

$$= e^{-rx}\underset{\sim}{\ell}(x) - e^{-r(x+5)}\underset{\sim}{\ell}(x+5)$$

$$- r\, \underset{\sim}{L}^{(r)}(x) \;.$$

The age-specific death and outmigration rates in the stable population are given by the matrix

$$\underset{\sim}{M}^{(r)}(x) = [\underset{\sim}{\ell}^{(r)}(x) - \underset{\sim}{\ell}^{(r)}(x+5)]\; [\underset{\sim}{L}^{(r)}(x)]^{-1}$$

$$= [e^{-rx}\underset{\sim}{\ell}(x) - e^{-r(x+5)} - r\underset{\sim}{L}^{(r)}(x)]\; [\underset{\sim}{L}^{(r)}(x)]^{-1}$$

$$= [e^{-rx}\underset{\sim}{\ell}(x) - e^{-r(x+5)}\; \underset{\sim}{\ell}(x+5)][e^{-r(x+2.5)}\underset{\sim}{L}(x)]^{-1} - r\underset{\sim}{I}$$

which after substitution yields

$$\underset{\sim}{M}^{(r)}(x) = \frac{2}{5}\, e^{2.5r}[\underset{\sim}{I} - e^{-5r}\underset{\sim}{P}(x)]\; [\underset{\sim}{I} + \underset{\sim}{P}(x)]^{-1} - r\underset{\sim}{I} \qquad (5.20)$$

For the last age group z, the rates are:

$$\underset{\sim}{M}^{(r)}(z) = \underset{\sim}{\ell}^{(r)}(z)\; [\underset{\sim}{L}^{(r)}(z)]^{-1} - r\underset{\sim}{I}$$

$$= e^{2.5r}\underset{\sim}{\ell}(z)[\underset{\sim}{L}(z)]^{-1} - r\underset{\sim}{I}$$

$$= e^{2.5r}\underset{\sim}{M}(z) - r\underset{\sim}{I} \;. \qquad (5.21)$$

The outmigration rates $M_{ij}^{(r)}(x)$ are contained in the off-diagonal elements of $\underset{\sim}{M}^{(r)}(x)$. The death rates $M_{i\delta}^{(r)}(x)$ are equal to the diagonal elements minus the outmigration rates, i.e. plus the off-diagonal elements in the same column.

To facilitate futher analysis, define the diagonal matrix ${}^{\delta}\underset{\sim}{M}^{(r)}(x)$ of regional death rates, and the diagonal matrix ${}^{o}\underset{\sim}{M}^{(r)}(x)$ of <u>total</u> regional outmigration rates, i.e.

$$ {}^{o}M_{ii}^{(r)}(x) = \sum_{j \neq i} M_{ij}^{(r)}(x) . \tag{5.22}$$

Let ${}^{oo}\underset{\sim}{M}^{(r)}(x)$ be the matrix of outmigration rates, i.e.

$$ {}^{oo}\underset{\sim}{M}^{(r)} = \begin{bmatrix} 0 & M_{21}^{(r)}(x) & \cdots\cdots & M_{n1}^{(r)}(x) \\ M_{12}^{(r)} & 0 & \cdots\cdots & M_{n2}^{(r)}(x) \\ M_{1n}^{(r)} & M_{2n}^{(r)} & \cdots\cdots & 0 \end{bmatrix} \tag{5.23}$$

Once consistent age-specific death and migration rates are derived, we may proceed with the computation of the stable equivalents of deaths and out- and inmigrants and of the associated intrinsic rates. The stable equivalent of deaths is:

$$ \{\underset{\sim}{D}\} = \sum_{x} {}^{\delta}\underset{\sim}{M}^{(r)}(x) \; \{\underset{\sim}{K}(x)\} $$

$$ = [\sum_{x} {}^{\delta}\underset{\sim}{M}^{(r)}(x) \; e^{-r(x + 2.5)} \underset{\sim}{L}(x)]\{\underset{\sim}{Q}\} \tag{5.24}$$

The intrinsic death rates follow immediately:

$$\{\underset{\sim}{d}\} = \underset{\sim}{Y}^{-1} \{\underset{\sim}{D}\} \tag{5.25}$$

or

$$\{\underset{\sim}{d}\} = \sum {}^{\delta}\underset{\sim}{M}^{(r)}(x) \{\underset{\sim}{C}(x)\} . \tag{5.26}$$

The stable equivalent of the outmigrants from region i to region j is:

$$O_{ij} = \sum_x M_{ij}^{(r)}(x) \ K_i(x) \tag{5.27}$$

where $M_{ij}^{(r)}(x)$ is the age-specific migration rate and $K_i(x)$ is the stable population of region i aged x to x + 4. In general, we may write the origin destination flow of stable equivalent migrations as

$$\underset{\sim}{O} = \sum_x {}^{oo}\underset{\sim}{M}^{(r)}(x) \ \underset{\sim}{K}(x) \tag{5.28}$$

where ${}^{oo}\underset{\sim}{M}^{(r)}(x)$ is defined in (5.23) and $\underset{\sim}{K}(x)$ is a diagonal matrix of stable regional populations of ages x to x + 4. The outmigration rates are simply:

$$\underset{\sim}{o} = \underset{\sim}{O} \ \underset{\sim}{Y}^{-1} \tag{5.29}$$

or

$$\underset{\sim}{o} = \sum_x {}^{oo}\underset{\sim}{M}^{(r)}(x) \ \underset{\sim}{C}(x) \tag{5.30}$$

where $\underset{\sim}{C}(x) = \underset{\sim}{K}(x) \ \underset{\sim}{Y}^{-1}$, i.e. $\underset{\sim}{C}(x)\{\underset{\sim}{1}\} = \{\underset{\sim}{C}(x)\}$.

The stable equivalent of the total number of outmigrants is:

$$\{\underset{\sim}{O}\}' = \{\underset{\sim}{1}\}' \ \underset{\sim}{O} \tag{5.31}$$

and the total outmigration rates are:

$$\{\underset{\sim}{o}\}' = \{\underset{\sim}{1}\}' \underset{\sim}{o} = \{\underset{\sim}{1}\}' \underset{\sim}{O} \underset{\sim}{Y}^{-1} = \{\underset{\sim}{O}\}' \underset{\sim}{Y}^{-1} \ . \tag{5.32}$$

An equivalent expression for (5.31) is:

$$\begin{aligned} \{O\}' &= \sum_x \{1\}'\, {}^{oo}\underset{\sim}{M}^{(r)}(x)\ \underset{\sim}{K}(x) \\ &= \sum_x \{{}^{o}\underset{\sim}{M}^{(r)}(x)\}' \underset{\sim}{K}(x), \end{aligned} \tag{5.33}$$

where $\{{}^{o}\underset{\sim}{M}^{(r)}(x)\}$ is the vector of total outmigration rates, defined in (5.22).

The stable equivalent of the total number of inmigrants by region is:

$$\{\underset{\sim}{I}\} = \underset{\sim}{O}\{\underset{\sim}{1}\} \ . \tag{5.34}$$

The inmigration rates are:

$$\begin{aligned} \{\underset{\sim}{i}\} &= \underset{\sim}{Y}^{-1} \{\underset{\sim}{I}\} \\ &= \underset{\sim}{Y}^{-1} \underset{\sim}{O} \{\underset{\sim}{1}\} \ . \end{aligned} \tag{5.35}$$

The matrix $\underset{\sim}{h} = \underset{\sim}{Y}^{-1} \underset{\sim}{O}$ contains inmigration rates by region of origin and region os destination. An element h_{ij} denotes the migrants from region i to j as a fraction of the population in j.

There exists a unique relationship between inmigration rates and outmigration rates. Since by (5.29)

$$\underset{\sim}{O} = \underset{\sim}{o}\ \underset{\sim}{Y},$$

we have

$$\underset{\sim}{i} = \underset{\sim}{Y}^{-1} \underset{\sim}{o}\ \underset{\sim}{Y}, \tag{5.36}$$

and the total inmigration rates

$$\{\underset{\sim}{i}\} = \underset{\sim}{i}\ \{\underset{\sim}{1}\} = \underset{\sim}{Y}^{-1} \underset{\sim}{o}\ \underset{\sim}{Y}\ \{\underset{\sim}{1}\}. \tag{5.37}$$

The stable equivalents of births, deaths and outmigrants and inmigrants are given in Table 28, together with the intrinsic rates. Note that the intrinsic rates obey the following relationship:

$$r = b_i - d_i - o_i + i_i. \tag{5.38}$$

Equation (5.38) provides an independent check of the results.

Table 28

stable equivalents and intrinsic rates
****

	births		deaths		outmigration		inmigration	
	number	rate	number	rate	number	rate	number	rate
slovenia	9237.	0.015453	7172.	0.011998	1469.	0.002457	3049.	0.005101
r.yugos.	192355.	0.019009	129052.	0.012753	3049.	0.000301	1469.	0.000145
total	201592.	0.018810	136224.	0.012711	4518.	0.000422	4518.	0.000422

stable growth rate 0.006099

normalizing factor 1054.2656

6. SPATIAL ZERO POPULATION GROWTH

The demographic system we have considered thus far is one that is characterized by constant fertility, mortality and migration schedules. The ultimate population evolving under these conditions is a stable population, with the following features: fixed age and regional structure, unchanging regional birth, death and migration rates, and a unique and constant growth rate.

The growing public concern about rapid population increase has generated a vast literature on the social and economic impacts of high fertility and has focused attention on fertility decline as a means for relieving socio-economic problems. An immediate drop of fertility to replacement level would not stop population growth however. In a growing population, children outnumber parents. Consequently, the number of potential parents in the next generation will be larger than at present. This built-in tendency to continued growth causes the number of people to increase for some time before the population becomes stationary (stable, but with zero growth). The ratio by which the ultimate stationary population exceeds a current population is the momentum of that population. The momentum of a population undisturbed by migration has been studied recently by Keyfitz (1971).

Although population growth is an important concern, the question where people choose to live in the future presents issues and problems that are potentially as serious as those posed by the number of children they choose to have. A drop in fertility, for example, not only causes the population to continue to grow for a while, but, together with the built-in migration forces, also affects the regional distribution of this population. The spatial impact of fertility reduction has been studied by Rogers and Willekens (1976a, 1976b).

The spatial momentum of zero population growth may be computed numerically and, if the initial population is stable, analytically. In the first section, the numerical approach is discussed. The analytical approach is examined in the following section.

6.1. The Numerical Approach

The numerical approach to spatial zero population growth (ZPG) analysis is to substitute fertility schedules representing fertility at replacement level for observed schedules. All the computations for population projection and stable population analysis are done over, and the results are compared with the results obtained using the original fertility schedules.

Many alternative fertility reduction schemes are possible. Some age groups may have a proportionally greater decline than others, due to differences in birth control practiced, or to shifting of the marriage and divorce functions. Or the decline may be age-independent, i.e. the proportional decline of the age-specific fertility rate is the same at all ages. Keyfitz (1971) considers a fertility drop that is age-independent. Most demographers have followed this practice and it is also adopted in this paper.

Regional differences in fertility decline are introduced through two alternative schemes:

Alternative 1: the cohort replacement alternative: in every region, fertility of each female cohort is reduced to bare replacement level, i.e. to a level of one daughter (net) per woman born there.

Alternative 2: the proportional reduction alternative: each regional fertility schedule is reduced by the same proportion at all ages.

To derive mathematical expressions for both alternatives, recall (2.6), which may be written as:

$$\{\underset{\sim}{Q}_2\} = \underset{\sim}{R}(0)\{\underset{\sim}{Q}_1\} = \left[\sum_x \underset{\sim}{F}(x)\ \underset{\sim}{L}(x)\right]\{\underset{\sim}{Q}_1\} \quad , \tag{6.1}$$

where $\{\underset{\sim}{Q}_1\}$ is the vector of births and $\{\underset{\sim}{Q}_2\}$ the vector of their offspring, i.e. births in the next generation.

Equation (6.1) expresses the births in one generation as a function of the births in the previous generation. A multiregional population system that is growing exhibits a net reproduction matrix $\underset{\sim}{R}(0)$ with a dominant characteristic root $\lambda_1[\underset{\sim}{R}(0)]$ that is greater than unity. The total number of offspring per woman born in a certain region is given by the column totals of $\underset{\sim}{R}(0)$, i.e.:

$$ {}_iR(0) = \sum_j {}_iR_j(0) \quad . \tag{2.9}$$

If fertility is reduced according to the cohort replacement alternative, then

$$ {}_i\hat{R}(0) = 1 \qquad \text{for all } i \quad ; $$

or, in matrix form,

$$ \underset{\sim}{\hat{R}}(0)' \{\underset{\sim}{1}\} = \{\underset{\sim}{1}\} \quad . \tag{6.2}$$

This means that every woman would have a net reproduction rate of unity. The problem is now to determine by how much the observed age-specific fertility rates must be altered for each woman to have a net reproduction rate of unity. Let γ_i be the required fertility adjustment factor for region i, i.e.

$$ {}_i\hat{R}(0) = 1 = \gamma_i \, {}_iR(0) \quad . $$

In general, we have

$$ \underset{\sim}{\hat{R}}(0) = \underset{\sim}{\gamma}\underset{\sim}{R}(0) \quad , \tag{6.3}$$

where $\underset{\sim}{\gamma}$ is a diagonal matrix of regional fertility adjustment factors. Substituting (6.3) into (6.2) gives

$$ \underset{\sim}{R}(0)'\underset{\sim}{\gamma}\{\underset{\sim}{1}\} = \{\underset{\sim}{1}\} \quad , $$

whence

$$ \{\underset{\sim}{\gamma}\} = [\underset{\sim}{R}(0)']^{-1}\{\underset{\sim}{1}\} \quad . \tag{6.4}$$

Therefore, the cohort replacement alternative yields the replacement fertility rates $\hat{\underset{\sim}{F}}(x)$:

$$\hat{\underset{\sim}{F}}(x) = \underset{\sim}{\gamma}\underset{\sim}{F}(x) \quad , \tag{6.5}$$

where $\underset{\sim}{F}(x)$ is the diagonal matrix of observed regional fertility rates of age group x to x + 4, and $\underset{\sim}{\gamma}$ is the diagonal matrix with the elements of $\{\underset{\sim}{\gamma}\}$ in the diagonal.

Recall our numerical illustration: the two-region system of Slovenia and the rest of Yugoslavia. The matrix of fertility adjustment factors is given in Table 29. Since originally the women of both regions had a net reproduction rate greater than unity, the fertility adjustment factors are less than one, causing a fertility drop in both regions. In Slovenia, the fertility rates drop to 93.24% of their original values, while in the rest of Yugoslavia they decline to a level of 84.27%. The difference is caused by differences in the initial fertility levels. The new fertility rates $\hat{\underset{\sim}{F}}(x)$ are also given in Table 29. Note that the gross rates of reproduction drop in the same proportion as the age-specific fertility rates.

With these new rates, fertility analysis is performed as before (see sections 2, 4 and 5). The results are listed in Tables 30 to 40. A comparison of these results with Tables 3 to 10 reveals the impact of the fertility drop to replacement level.

In the proportional reduction alternative, the age-specific fertility rates of each region are reduced in the same proportion. The fertility adjustment factor is identical for each region and is equal to

$$\gamma_i = \gamma_j = \gamma = \frac{1}{\lambda_1[\underset{\sim}{R}(0)]} \quad . \tag{6.6}$$

The matrix of fertility adjustment factors is given in Table 41, together with the new fertility rates $\hat{\hat{\underset{\sim}{F}}}(x)$

$$\hat{\hat{\underset{\sim}{F}}}(x) = \underset{\sim}{\gamma}\underset{\sim}{F}(x) \quad , \tag{6.7}$$

where $\underset{\sim}{\gamma} = \gamma\underset{\sim}{I}$.

Table 29. zero population growth ********************** alternative 1 *************

matrix of fertility adjustment factors

	total	slovenia	r.yugos.
slovenia	0.932430	0.932430	0.000000
r.yugos.	0.842718	0.000000	0.842718
total		0.932430	0.842718

fertility analysis

observed rates

age	slovenia	r.yugos.
0	0.000000	0.000000
5	0.000000	0.000000
10	0.000066	0.000056
15	0.014786	0.022297
20	0.065878	0.074141
25	0.058946	0.062580
30	0.038326	0.037324
35	0.021317	0.019831
40	0.007270	0.010156
45	0.000662	0.001813
50	0.000272	0.000602
55	0.000000	0.000000
60	0.000000	0.000000
65	0.000000	0.000000
70	0.000000	0.000000
75	0.000000	0.000000
80	0.000000	0.000000
85	0.000000	0.000000
grr	1.037617	1.143994

Table 30. integrals of generalized net maternity function

initial region of cohort slovenia

age	slovenia	r.yugos.
0	0.000000	0.000000
5	0.000000	0.000000
10	0.000311	0.000008
15	0.068147	0.004741
20	0.293593	0.025894
25	0.253648	0.030305
30	0.160491	0.021556
35	0.087570	0.012460
40	0.029431	0.006582
45	0.002634	0.001182
50	0.001057	0.000389
55	0.000000	0.000000
60	0.000000	0.000000
65	0.000000	0.000000
70	0.000000	0.000000
75	0.000000	0.000000
80	0.000000	0.000000
85	0.000000	0.000000
total	0.896882	0.103118

initial region of cohort r.yugos.

age	slovenia	r.yugos.
0	0.000000	0.000000
5	0.000000	0.000000
10	0.000001	0.000251
15	0.000303	0.098391
20	0.002449	0.324229
25	0.003028	0.270867
30	0.002265	0.159912
35	0.001365	0.084050
40	0.000490	0.042476
45	0.000046	0.007450
50	0.000019	0.002409
55	0.000000	0.000000
60	0.000000	0.000000
65	0.000000	0.000000
70	0.000000	0.000000
75	0.000000	0.000000
80	0.000000	0.000000
85	0.000000	0.000000
total	0.009965	0.990035

Table 31. moments of integral function

0 moment

	total	slovenia	r.yugos.
slovenia	0.906847	0.896882	0.009965
r.yugos.	1.093153	0.103118	0.990035
total		1.000000	1.000000

1 moment

	total	slovenia	r.yugos.
slovenia	25.001335	24.708868	0.292468
r.yugos.	30.126825	3.023245	27.103580
total		27.732113	27.396048

2 moment

	total	slovenia	r.yugos.
slovenia	725.025208	716.050171	8.975063
r.yugos.	879.830750	93.401039	786.429687
total		809.451233	795.404724

Table 32. spatial fertility expectancies

net reproduction rate

	total	slovenia	r.yugos.
slovenia	0.906847	0.896882	0.009965
r.yugos.	1.093153	0.103118	0.990035
total		1.000000	1.000000
eigenvalue	1.000000		
eigenvector			
- right		1.000000	10.348231
- left		1.000000	1.000000

net reproduction allocations

	total	slovenia	r.yugos.
slovenia	0.906847	0.896882	0.009965
r.yugos.	1.093153	0.103118	0.990035
total		1.000000	1.000000

Table 33. matrices of mean ages and variances

** alternative 1 **

means

	total	slovenia	r.yugos.
slovenia	28.449961	27.549738	29.350182
r.yugos.	28.347355	29.318325	27.376385
total		28.434032	28.363283

variances

	slovenia	r.yugos.
slovenia	39.389099	39.246460
r.yugos.	46.204895	44.878845

** alternative 2 **

means

	total	slovenia	r.yugos.
slovenia	27.565796	27.547653	0.018143
r.yugos.	27.597666	0.223531	27.374134
total		27.771185	27.392277

variances

	total	slovenia	r.yugos.
slovenia	39.415859	39.381409	0.034452
r.yugos.	45.417957	0.548938	44.869019
total		39.930347	44.903469

Table 34. The spatial reproductive value:

results for people at exact age x

discounted number of offspring per person

region of residence slovenia

	region of birth of offspring total	slovenia	r.yugos.
0	1.000000	0.896882	0.103118
5	1.030595	0.937944	0.092650
10	1.031792	0.950707	0.081085
15	1.032309	0.958818	0.073492
20	0.958979	0.911241	0.047738
25	0.631086	0.611872	0.019214
30	0.337589	0.331761	0.005828
35	0.146983	0.145803	0.001180
40	0.040820	0.040610	0.000210
45	0.004615	0.004591	0.000025
50	0.001346	0.001342	0.000005
55	0.000000	0.000000	0.000000
60	0.000000	0.000000	0.000000
65	0.000000	0.000000	0.000000
70	0.000000	0.000000	0.000000
75	0.000000	0.000000	0.000000
80	0.000000	0.000000	0.000000

region of residence r.yugos.

	region of birth of offspring total	slovenia	r.yugos.
0	1.000000	0.009965	0.990035
5	1.119092	0.009841	1.109251
10	1.122916	0.009097	1.113819
15	1.125393	0.008375	1.117018
20	1.018806	0.005253	1.013553
25	0.653396	0.001723	0.651673
30	0.344486	0.000541	0.343945
35	0.160095	0.000127	0.159968
40	0.062111	0.000022	0.062089
45	0.011878	0.000002	0.011876
50	0.002961	0.000000	0.002960
55	0.000000	0.000000	0.000000
60	0.000000	0.000000	0.000000
65	0.000000	0.000000	0.000000
70	0.000000	0.000000	0.000000
75	0.000000	0.000000	0.000000
80	0.000000	0.000000	0.000000

Table 35. spatial reproductive value per person
****

	slovenia	r.yugos.
0	1.000000	1.000000
5	1.030595	1.119093
10	1.031792	1.122917
15	1.032310	1.125394
20	0.958979	1.018807
25	0.631086	0.653396
30	0.337589	0.344486
35	0.146983	0.160095
40	0.040820	0.062112
45	0.004615	0.011878
50	0.001346	0.002961
55	0.000000	0.000000
60	0.000000	0.000000
65	0.000000	0.000000
70	0.000000	0.000000
75	0.000000	0.000000
80	0.000000	0.000000

Table 36. results for people in age group x

****

****

discounted number of offspring per person

****

region of residence slovenia

----

	region of birth of offspring		
	total	slovenia	r.yugos.
0	1.015376	0.916952	0.098424
5	1.031201	0.944281	0.086921
10	1.032055	0.954744	0.077311
15	0.995478	0.935348	0.060130
20	0.795003	0.764489	0.030514
25	0.484480	0.473984	0.010495
30	0.242553	0.239886	0.002667
35	0.094082	0.093599	0.000483
40	0.022787	0.022732	0.000055
45	0.002992	0.002983	0.000009
50	0.000681	0.000681	0.000000
55	0.000000	0.000000	0.000000
60	0.000000	0.000000	0.000000
65	0.000000	0.000000	0.000000
70	0.000000	0.000000	0.000000
75	0.000000	0.000000	0.000000
80	0.000000	0.000000	0.000000

region of residence r.yugos.

----

	region of birth of offspring		
	total	slovenia	r.yugos.
0	1.056170	0.009920	1.046250
5	1.121000	0.009472	1.111528
10	1.124153	0.008738	1.115415
15	1.072242	0.006781	1.065462
20	0.836701	0.003148	0.833553
25	0.499562	0.000958	0.498603
30	0.252691	0.000254	0.252438
35	0.111355	0.000045	0.111309
40	0.037178	0.000004	0.037174
45	0.007463	0.000001	0.007463
50	0.001504	0.000000	0.001504
55	0.000000	0.000000	0.000000
60	0.000000	0.000000	0.000000
65	0.000000	0.000000	0.000000
70	0.000000	0.000000	0.000000
75	0.000000	0.000000	0.000000
80	0.000000	0.000000	0.000000

Table 37. spatial reproductive value per person
****

	slovenia	r.yugos.
0	1.015376	1.056171
5	1.031201	1.121001
10	1.032055	1.124153
15	0.995478	1.072243
20	0.795003	0.836702
25	0.484480	0.499562
30	0.242553	0.252691
35	0.094082	0.111355
40	0.022787	0.037178
45	0.002992	0.007463
50	0.000681	0.001504
55	0.000000	0.000000
60	0.000000	0.000000
65	0.000000	0.000000
70	0.000000	0.000000
75	0.000000	0.000000
80	0.000000	0.000000

Table 38. total discounted number of offspring
****

of observed population in 100,000.
****

	total	slovenia	r.yugos.
slovenia	3.899955	3.585212	0.314742
r.yugos.	47.410378	0.250228	47.160149
total		3.835440	47.474892

reproductive value of the total population in 100,000.
****

slovenia	3.835440
r.yugos.[12]	47.474907
total	51.310349

---

[12] The small deviation from the total discounted number of offspring of the observed population is due to rounding error.

Table 39. stable equivalent of total population
****

	total	slovenia	r.yugos.
0	888828.	81159.	807670.
5	840853.	79659.	761194.
10	838527.	79376.	759152.
15	835824.	79476.	756348.
20	831644.	79814.	751830.
25	826109.	79703.	746406.
30	819573.	79119.	740455.
35	812088.	78653.	733436.
40	802469.	78237.	724233.
45	789281.	77366.	711914.
50	769609.	75847.	693762.
55	740058.	73708.	666351.
60	694006.	70341.	623664.
65	622513.	64142.	558371.
70	517445.	53632.	463813.
75	385494.	39683.	345811.
80	245110.	23585.	221525.
85	237110.	14741.	222369.
total	12496541.	1208239.	11288302.

percentage distribution
----

	total	slovenia	r.yugos.
0	0.071126	0.067171	0.071549
5	0.067287	0.065929	0.067432
10	0.067101	0.065695	0.067251
15	0.066884	0.065778	0.067003
20	0.066550	0.066058	0.066603
25	0.066107	0.065967	0.066122
30	0.065584	0.065483	0.065595
35	0.064985	0.065097	0.064973
40	0.064215	0.064753	0.064158
45	0.063160	0.064032	0.063067
50	0.061586	0.062775	0.061459
55	0.059221	0.061004	0.059030
60	0.055536	0.058218	0.055249
65	0.049815	0.053087	0.049465
70	0.041407	0.044388	0.041088
75	0.030848	0.032844	0.030634
80	0.019614	0.019520	0.019624
85	0.018974	0.012200	0.019699

Table 40

stable equivalents and intrinsic rates
****

	births		deaths		outmigration		inmigration	
	number	rate	number	rate	number	rate	number	rate
slovenia	16486.	0.013645	16888.	0.013978	2838.	0.002349	3240.	0.002681
r.yugos.	170603.	0.015113	170200.	0.015078	3240.	0.000287	2838.	0.000251
total	187089.	0.014971	187088.	0.014971	6078.	0.000486	6078.	0.000486

stable growth rate 0.000000
normalizing factor 0.311 .2329

This reduction scheme results in a different stationary population. A baby girl born in Slovenia is replaced by only 0.918 daughters on the average, while a girl born in the Rest of Yugoslavia replaces herself with 1.004 daughters. Further results of this replacement alternative are given in Tables 41 to 52.

Table 41. zero population growth alternative 2

matrix of fertility adjustment factors

	total	slovenia	r.yugos.
slovenia	0.846894	0.846894	0.000000
r.yugos.	0.846894	0.000000	0.846894
total		0.846894	0.846894

fertility analysis

observed rates

age	slovenia	r.yugos.
0	0.000000	0.000000
5	0.000000	0.000000
10	0.000060	0.000057
15	0.013429	0.022407
20	0.059835	0.074508
25	0.053539	0.062890
30	0.034810	0.037509
35	0.019362	0.019929
40	0.006603	0.010206
45	0.000601	0.001822
50	0.000247	0.000605
55	0.000000	0.000000
60	0.000000	0.000000
65	0.000000	0.000000
70	0.000000	0.000000
75	0.000000	0.000000
80	0.000000	0.000000
85	0.000000	0.000000
grr	0.942432	1.149663

Table 42. integrals of generalized net maternity function

initial region of cohort slovenia

age	slovenia	r.yugos.
0	0.000000	0.000000
5	0.000000	0.000000
10	0.000282	0.000008
15	0.061896	0.004765
20	0.266661	0.026022
25	0.230380	0.030455
30	0.145769	0.021663
35	0.079537	0.012522
40	0.026731	0.006615
45	0.002392	0.001188
50	0.000960	0.000391
55	0.000000	0.000000
60	0.000000	0.000000
65	0.000000	0.000000
70	0.000000	0.000000
75	0.000000	0.000000
80	0.000000	0.000000
85	0.000000	0.000000
total	0.814607	0.103629

initial region of cohort r.yugos.

age	slovenia	r.yugos.
0	0.000000	0.000000
5	0.000000	0.000000
10	0.000001	0.000252
15	0.000275	0.098879
20	0.002224	0.325835
25	0.002750	0.272209
30	0.002057	0.160704
35	0.001240	0.084467
40	0.000445	0.042686
45	0.000042	0.007487
50	0.000017	0.002421
55	0.000000	0.000000
60	0.000000	0.000000
65	0.000000	0.000000
70	0.000000	0.000000
75	0.000000	0.000000
80	0.000000	0.000000
85	0.000000	0.000000
total	0.009051	0.994941

Table 43. moments of integral function

0 moment

	total	slovenia	r.yugos.
slovenia	0.823658	0.814607	0.009051
r.yugos.	1.098570	0.103629	0.994941
total		0.918236	1.003992

1 moment

	total	slovenia	r.yugos.
slovenia	22.707848	22.442211	0.265638
r.yugos.	30.276110	3.038226	27.237883
total		25.480438	27.503521

2 moment

	total	slovenia	r.yugos.
slovenia	658.515381	650.363647	8.151740
r.yugos.	884.190491	93.863853	790.326660
total		744.227478	798.478394

Table 44. spatial fertility expectancies
****

net reproduction rate
----

	total	slovenia	r.yugos.
slovenia	0.823658	0.814607	0.009051
r.yugos.	1.098570	0.103629	0.994941
total		0.918236	1.003992
eigenvalue	1.000000		
eigenvector			
- right		1.000000	20.483936
- left		1.000000	1.788998

net reproduction allocations
----

	total	slovenia	r.yugos.
slovenia	0.896158	0.887143	0.009015
r.yugos.	1.103842	0.112857	0.990985
total		1.000000	1.000000

Table 45. matrices of mean ages and variances

** alternative 1 **

means

	total	slovenia	r.yugos.
slovenia	28.449965	27.549742	29.350185
r.yugos.	28.347355	29.318327	27.376381
total		28.434034	28.363283

variances

	slovenia	r.yugos.
slovenia	39.388916	39.246338
r.yugos.	46.204712	44.879028

** alternative 2 **

means

	total	slovenia	r.yugos.
slovenia	27.564054	27.547657	0.016397
r.yugos.	27.621460	0.247328	27.374132
total		27.794985	27.390530

variances

	total	slovenia	r.yugos.
slovenia	39.412422	39.381287	0.031137
r.yugos.	45.476391	0.607310	44.869080
total		39.988598	44.900215

Table 46. The Spatial Reproductive Value

results for people at exact age x

discounted number of offspring per person

region of residence slovenia

	region of birth of offspring		
	total	slovenia	r.yugos.
0	0.918236	0.814607	0.103629
5	0.945012	0.851902	0.093110
10	0.944981	0.863494	0.081487
15	0.944717	0.870861	0.073856
20	0.875623	0.827649	0.047974
25	0.575051	0.555742	0.019309
30	0.307184	0.301327	0.005857
35	0.133614	0.132428	0.001186
40	0.037096	0.036884	0.000211
45	0.004194	0.004170	0.000025
50	0.001223	0.001219	0.000005
55	0.000000	0.000000	0.000000
60	0.000000	0.000000	0.000000
65	0.000000	0.000000	0.000000
70	0.000000	0.000000	0.000000
75	0.000000	0.000000	0.000000
80	0.000000	0.000000	0.000000

region of residence r.yugos.

	region of birth of offspring		
	total	slovenia	r.yugos.
0	1.003992	0.009051	0.994941
5	1.123686	0.008938	1.114748
10	1.127601	0.008262	1.119339
15	1.130160	0.007607	1.122553
20	1.023347	0.004771	1.018576
25	0.656467	0.001565	0.654902
30	0.346140	0.000491	0.345649
35	0.160876	0.000115	0.160761
40	0.062417	0.000020	0.062397
45	0.011937	0.000002	0.011935
50	0.002975	0.000000	0.002975
55	0.000000	0.000000	0.000000
60	0.000000	0.000000	0.000000
65	0.000000	0.000000	0.000000
70	0.000000	0.000000	0.000000
75	0.000000	0.000000	0.000000
80	0.000000	0.000000	0.000000

Table 47. spatial reproductive value per person
****

	slovenia	r.yugos.
0	1.000000	1.788998
5	1.018475	2.003219
10	1.009274	2.010757
15	1.002989	2.015852
20	0.913474	1.827001
25	0.590286	1.173183
30	0.311805	0.618857
35	0.134549	0.287715
40	0.037262	0.111648
45	0.004214	0.021353
50	0.001227	0.005323
55	0.000000	0.000000
60	0.000000	0.000000
65	0.000000	0.000000
70	0.000000	0.000000
75	0.000000	0.000000
80	0.000000	0.000000

Table 48. results for people in age group x

discounted number of offspring per person

region of residence slovenia

	region of birth of offspring total	slovenia	r.yugos.
0	0.931747	0.832836	0.098912
5	0.945009	0.857657	0.087352
10	0.944855	0.867161	0.077694
15	0.909972	0.849544	0.060428
20	0.725024	0.694359	0.030665
25	0.441051	0.430504	0.010547
30	0.220561	0.217880	0.002681
35	0.085498	0.085013	0.000485
40	0.020702	0.020647	0.000055
45	0.002718	0.002709	0.000009
50	0.000618	0.000618	0.000000
55	0.000000	0.000000	0.000000
60	0.000000	0.000000	0.000000
65	0.000000	0.000000	0.000000
70	0.000000	0.000000	0.000000
75	0.000000	0.000000	0.000000
80	0.000000	0.000000	0.000000

region of residence r.yugos.

	region of birth of offspring total	slovenia	r.yugos.
0	1.060445	0.009010	1.051435
5	1.125639	0.008603	1.117036
10	1.128878	0.007937	1.120942
15	1.076900	0.006159	1.070741
20	0.840543	0.002860	0.837683
25	0.501945	0.000870	0.501074
30	0.253919	0.000230	0.253688
35	0.111902	0.000041	0.111861
40	0.037362	0.000004	0.037359
45	0.007500	0.000001	0.007500
50	0.001512	0.000000	0.001512
55	0.000000	0.000000	0.000000
60	0.000000	0.000000	0.000000
65	0.000000	0.000000	0.000000
70	0.000000	0.000000	0.000000
75	0.000000	0.000000	0.000000
80	0.000000	0.000000	0.000000

Table 49. spatial reproductive value per person
****

	slovenia	r.yugos.
0	1.009788	1.890025
5	1.013929	2.006978
10	1.006155	2.013299
15	0.957649	1.921713
20	0.749219	1.501474
25	0.449373	0.897291
30	0.222676	0.454078
35	0.085881	0.200160
40	0.020745	0.066838
45	0.002725	0.013417
50	0.000618	0.002704
55	0.000000	0.000000
60	0.000000	0.000000
65	0.000000	0.000000
70	0.000000	0.000000
75	0.000000	0.000000
80	0.000000	0.000000

Table 50. total discounted number of
offspring of observed population in 100,000.

	total	slovenia	r.yugos.
slovenia	3.542193	3.256324	0.285870
r.yugos.	47.645302	0.251468	47.393837
total		3.507792	47.679707

reproductive value of the total population in 100,000.

slovenia	3.706200
r.yugos.	85.073341
total	88.779541

Table 51. stable equivalent of total population
****

	total	slovenia	r.yugos.
0	877045.	42646.	834399.
5	828160.	42294.	785866.
10	825832.	42450.	783382.
15	823123.	43278.	779845.
20	818936.	44912.	774024.
25	813373.	46093.	767281.
30	806806.	46462.	760344.
35	799313.	46651.	752662.
40	789710.	46725.	742984.
45	776636.	46417.	730219.
50	757174.	45670.	711504.
55	727909.	44617.	683292.
60	682258.	42847.	639411.
65	611628.	39279.	572350.
70	508282.	32956.	475326.
75	378761.	24445.	354316.
80	241512.	14587.	226925.
85	236926.	9160.	227766.
total	12303383.	701488.	11601895.

percentage distribution
----

	total	slovenia	r.yugos.
0	0.071285	0.060794	0.071919
5	0.067312	0.060292	0.067736
10	0.067122	0.060514	0.067522
15	0.066902	0.061694	0.067217
20	0.066562	0.064024	0.066715
25	0.066110	0.065707	0.066134
30	0.065576	0.066233	0.065536
35	0.064967	0.066503	0.064874
40	0.064186	0.066609	0.064040
45	0.063124	0.066169	0.062940
50	0.061542	0.065105	0.061327
55	0.059163	0.063604	0.058895
60	0.055453	0.061080	0.055113
65	0.049712	0.055993	0.049332
70	0.041312	0.046980	0.040970
75	0.030785	0.034848	0.030539
80	0.019630	0.020794	0.019559
85	0.019257	0.013058	0.019632

Table 52

stable equivalents and intrinsic rates
****

	births		deaths		outmigration		inmigration	
	number	rate	number	rate	number	rate	number	rate
slovenia	8607.	0.012270	10330.	0.014726	1610.	0.002296	3333.	0.004751
r.yugos.	176307.	0.015196	174583.	0.015048	3333.	0.000287	1610.	0.000139
total	184914.	0.015030	184914.	0.015029	4943.	0.000402	4943.	0.000402

stable growth rate	0.000000
normalizing factor	1031.471

6.2 The Analytical Approach

If the initial population is stable, the momentum of spatial zero population growth may be expressed as a simple analytical formula. The ultimate number of stationary equivalent births is by (5.1):

$$\{\hat{\mathbf{Q}}\} = \frac{1}{\{\hat{\mathbf{v}}(0)\}'\hat{\boldsymbol{\kappa}}\{\hat{\mathbf{Q}}_1\}} \int_0^{\omega} \{\hat{\mathbf{v}}(x)\}'\{\mathbf{k}(x)\}dx\ \{\hat{\mathbf{Q}}_1\} \quad , \qquad (5.1)$$

where the caret indentifies a stationary population, and the total reproductive value $\hat{V}$ is

$$\hat{V} = \int_0^{\omega} \{\hat{\mathbf{v}}(x)\}'\{\mathbf{k}(x)\}\ dx \quad ,$$

with $\{\mathbf{k}(x)\}$ being the vector defining regional distribution of people at exact age x. If the distribution $\{\mathbf{k}(x)\}$ is stable, then by (1.6)

$$\{\mathbf{k}(x)\} = e^{-rx}\ \boldsymbol{\ell}(x)\ \{\mathbf{Q}\} \quad , \qquad (6.8)$$

where $\{\mathbf{Q}\}$ represents the regional distribution of births before the drop in fertility. Substituting $\{\mathbf{k}(x)\}$ in (5.1) into (6.8) and reworking yields (Rogers and Willekens, 1976b, p. 22):

$$\{\hat{\mathbf{Q}}\} = \frac{1}{\mu r}\ [\{\hat{\mathbf{v}}(0)\}'\boldsymbol{\gamma}[\mathbf{R}(0) - \boldsymbol{\Psi}(r)]\ \{\mathbf{Q}\}]\ \{\hat{\mathbf{Q}}_1\} \quad , \qquad (6.9)$$

where $\mu = \{\hat{\mathbf{v}}(0)\}'\hat{\boldsymbol{\kappa}}\ \{\hat{\mathbf{Q}}_1\}$, with $\hat{\boldsymbol{\kappa}} = \boldsymbol{\mu} = \boldsymbol{\gamma}\ \mathbf{R}(1)\ \mathbf{R}^{-1}(0)\ \boldsymbol{\gamma}^{-1}$ being the matrix of mean ages of childbearing in the stationary population (after decline in fertility). The matrices $\mathbf{R}(0)$ and $\boldsymbol{\Psi}(r)$ and the vector of stable equivalent of births refer to the stable population before the drop in fertility. The matrix of fertility adjustment factors is $\boldsymbol{\gamma}$.

It can be shown that equation (6.9) is equivalent to

$$\{\hat{\underset{\sim}{Q}}\} = \frac{1}{r} \underset{\sim}{R}(1)^{-1} [\underset{\sim}{R}(0) - \underset{\sim}{\Psi}(r)] \{\underset{\sim}{Q}\} \tag{6.10}$$

$$= \underset{\sim}{S}^{o} \{\underset{\sim}{Q}\} , \quad \text{say.}$$

The stationary births are therefore a linear combination of the stable births, before the drop in fertility. The conversion matrix is $\underset{\sim}{S}^{o}$. The numerical evaluation is given in Table 53. The ultimate stationary population is

$$\{\hat{\underset{\sim}{Y}}\} = \left[\int_0^{\omega} \underset{\sim}{\ell}(x)dx \right] \{\hat{\underset{\sim}{Q}}\} = \underset{\sim}{e}(0) \{\hat{\underset{\sim}{Q}}\} \tag{6.11}$$

and the total reproductive value is

$$\hat{V} = \{\hat{\underset{\sim}{\nu}}(0)\} \frac{1}{r} \underset{\sim}{\gamma}[\underset{\sim}{R}(0) - \underset{\sim}{\Psi}(r)] \{\underset{\sim}{Q}\} . \tag{6.12}$$

Let $\underset{\sim}{Y}$ be the diagonal matrix of the total observed population, then

$$\underset{\sim}{Y} \{\underset{\sim}{1}\} = \left[\int_0^{\omega} e^{-rx} \underset{\sim}{\ell}(x)dx \right] \{\underset{\sim}{Q}\}$$

$$= \underset{\sim}{e}^{(r)}(0) \{\underset{\sim}{Q}\} , \tag{6.13}$$

where $\underset{\sim}{e}^{(r)}(0)$ has been labeled the matrix of discounted life expectancies. Recalling the characteristic equation, (6.13) also may be written as

$$\{\underset{\sim}{Y}\} = \underset{\sim}{e}^{(r)}(0) [\underset{\sim}{\Psi}(r)]^{-1} \{\underset{\sim}{Q}\} , \tag{6.14}$$

and therefore

$$\{\underset{\sim}{Q}\} = \underset{\sim}{\Psi}(r) [\underset{\sim}{e}^{(r)}(0)]^{-1} \{\underset{\sim}{Y}\} = \underset{\sim}{b} \{\underset{\sim}{Y}\} \tag{6.15}$$

The spatial momentum of zero population growth is

$$\underset{\sim}{Y}^{-1}\ \{\hat{\underset{\sim}{Y}}\} = \frac{1}{\mu r}\ [\{\underset{\sim}{\hat{Q}}(0)\}'\underset{\sim}{\gamma}[\underset{\sim}{R}(0) - \underset{\sim}{\Psi}(r)]\ \{\underset{\sim}{Q}\}]\ \underset{\sim}{Y}^{-1}\ \underset{\sim}{e}(0)\ \{\hat{\underset{\sim}{Q}}_1\} \quad (6.16)$$

$$= \frac{1}{\mu r}\ [\{\underset{\sim}{\hat{Q}}(0)\}'\underset{\sim}{\gamma}[\underset{\sim}{R}(0) - \underset{\sim}{\Psi}(r)]\ \{\underset{\sim}{b}\}]\ \underset{\sim}{e}(0)\ \{\hat{\underset{\sim}{Q}}_1\}\ , \quad (6.17)$$

where $\{\underset{\sim}{b}\}$ is the vector of regional intrinsic birth rates before the drop in fertility. Applying (6.10) the momentum becomes

$$\underset{\sim}{Y}^{-1}\ \{\hat{\underset{\sim}{Y}}\} = \underset{\sim}{e}(0)\frac{1}{r}\ \underset{\sim}{R}(1)^{-1}\ [\underset{\sim}{R}(0) - \underset{\sim}{\Psi}(r)]\{\underset{\sim}{b}\}\ . \quad (6.18)$$

Introducing (6.15) into (6.16) gives still another expression for the momentum

$$\underset{\sim}{Y}^{-1}\ \{\hat{\underset{\sim}{Y}}\} = \frac{1}{\mu r}\ [\{\hat{Q}(0)'\ \underset{\sim}{\gamma}[\underset{\sim}{R}(0) - \underset{\sim}{\Psi}(r)]\ \underset{\sim}{b}\ \{\underset{\sim}{1}\}]\ \underset{\sim}{e}(0)\{\hat{\underset{\sim}{Q}}_1\}\ . \quad (6.19)$$

The analytical approach is illustrated in Table 53. It is assumed that the initial population coincides with the stable equivalent population of Slovenia and the Rest of Yugoslavia. Hence the regional births are contained in the vector

$$\{\underset{\sim}{Q}\} = \begin{bmatrix} 9,237 \\ \\ 192,355 \end{bmatrix},$$

and the population by age-group and region are given in Table 27. Table 53 reveals that, given a population of 597,786 in Slovenia and 10,119,240 in the Rest of Yugoslavia, an immediate drop of fertility to replacement level would result in an ultimate population increase of 15.74% in Slovenia and of 14.66% in the Rest of Yugoslavia. This momentum is a consequence of the growth potential in the initial age and regional distribution of the population[13].

---

[13] Note that the stationary population distribution in unit births is given in Table 1 b.

Table 53. spatial momentum of zero population growth
****
****

matrix converting stable to stationary births

	total	slovenia	r.yugos.
slovenia	0.916789	0.916844	-0.000054
r.yugos.	0.916212	-0.000574	0.916786
total		0.916270	0.916732

stable and stationary equivalent

	births		population		population
	stable	stationary	stable	stationary	momentum
slovenia	9237.	8459.	597786.	691891.	1.1574
r.yugos.	192355.	176343.	10119240.	11603140.	1.1466
total	201592.	184802.	10717026.	12295031.	1.1472

7. PROGRAM DESCRIPTION

The concept underlying the programs is that of a modular system. It consists of a set of subroutines each of which performs a specific task, such as matrix inversion, calculating the dominant eigenvalue and associated eigenvectors, computing the integral functions and their moments, and so on. The main program is kept very short. It coordinates the computations through CALL statements and transmits information from one subroutine to another through labeled COMMON statements. No major computations are performed in the main program.

The subroutines consist of the frequently used general purpose subroutines and special purpose subroutines:

i. General purpose subroutines:

MULTIP: matrix multiplication

INVERT: matrix inversion

EIGEN: computation of dominant eigenvalue and associated right and left eigenvectors.

ii. Special purpose subroutines:

READ2: reads in the data.

AGEDIS: generates the population distribution by age and region:

- observed population
- life table population
- stable population.

STABCH: computes the integral functions, i.e. the (weighted) generalized maternity and mobility functions, and their zero-th, first and second moments. In addition, it calculates the matrices of mean ages at childbearing and mobility and the matrices of the variances of the ages at childbearing, and mobility.

ZERO: replaces the observed regional fertility schedules with fertility schedules at replacement level. Two alternative fertility reduction schemes are possible.

RVALUE: computes the discounted number of offspring and computes the spatial reproductive values by age and region.

RINTR: computes the stable equivalents of births, deaths, outmigrants and inmigrants, and the intrinsic rates.

MOMENT: computes the spatial momentum of zero population growth.

The purpose of separating each major task into subroutines is to keep the whole structure of the programs very clear and to enable the user to change parts of the programs according to his needs. Clarity and flexibility are major objectives which we tried to keep in mind while writing the programs. Computational efficiency was of secondary importance. In a rapidly growing field such as multiregional demographic analysis, computer programs must be flexible and easy to adapt to new theoretical or methodological developments. The computer programs published here are not final fixed products; they are working tools to produce useful numerical demographic results. The user is urged to adapt them to fit his own needs in order to get the most out of them.

The relationships between the various subroutines is illustrated in Figure 1. Two rather separate blocks can be distinguished. One deals with the integral functions and the derived statistics such as the means and variance of the age distributions. The numerical approach of ZPG analysis also belongs in this block. The second block focuses on the theory and applications of the spatial reproductive value. It contains further stable population analysis and the analytical approach to the ZPG study.

7.1. The General Purpose Subroutines

a. MULTIP:

SUBROUTINE MULTIP (N,K,L)

Task: multiplication of two matrices $\underset{\sim}{A1}$ and $\underset{\sim}{B}$.

$$\underset{\sim}{C} = \underset{\sim}{A1} * \underset{\sim}{B}$$

Parameters:

N: number of rows of $\underset{\sim}{A1}$

K: number of columns of $\underset{\sim}{A1}$ (and consequently, number of rows of $\underset{\sim}{B}$)

L: number of columns of $\underset{\sim}{B}$.

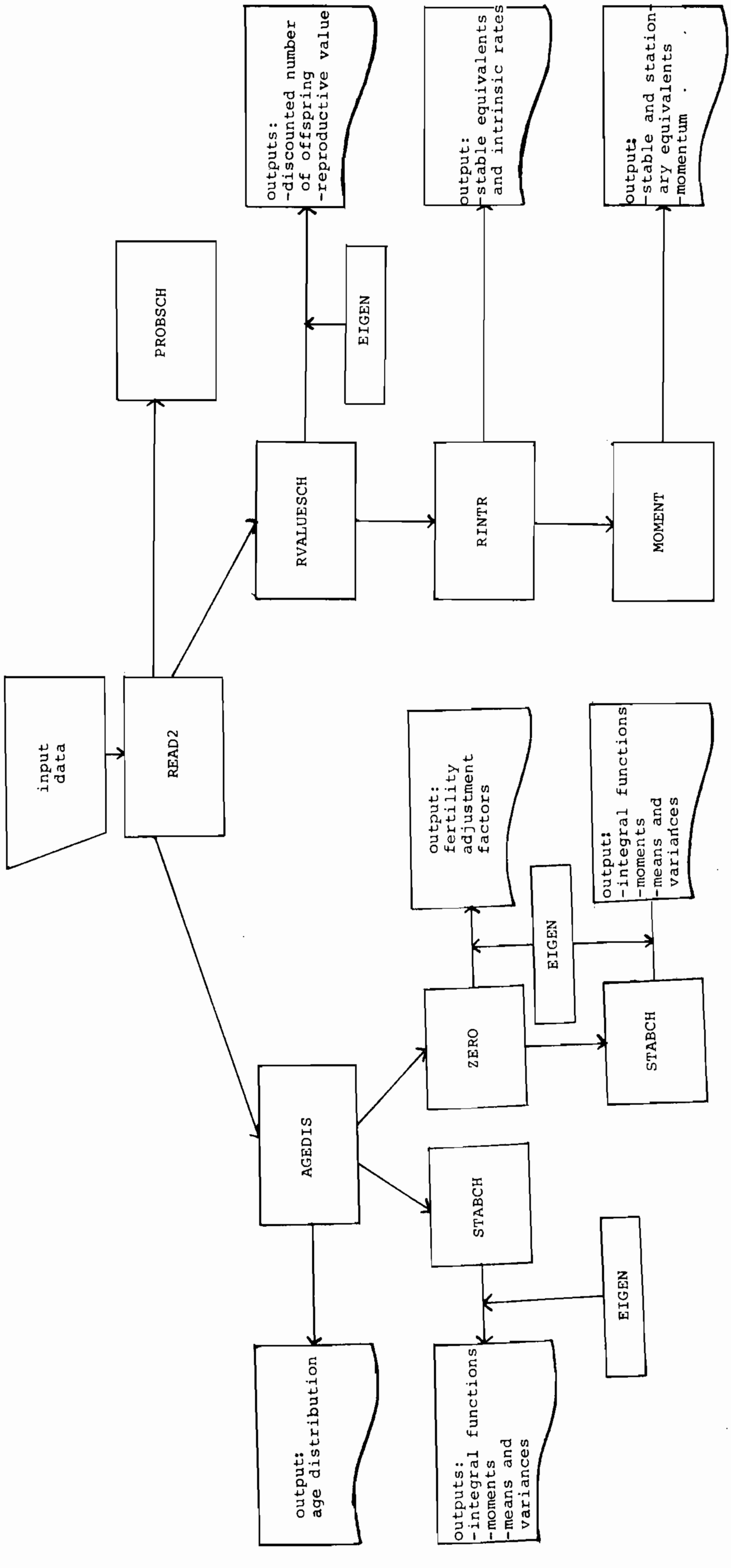
input data
READ2
PROBSCH
RVALUESCH
EIGEN
outputs:
-discounted number of offspring
-reproductive value
RINTR
output:
-stable equivalents and intrinsic rates
MOMENT
output:
-stable and stationary equivalents
-momentum
AGEDIS
output:
age distribution
ZERO
output:
fertility adjustment factors
EIGEN
STABCH
output:
-integral functions
-moments
-means and variances
STABCH
EIGEN
outputs:
-integral functions
-moments
-means and variances

Figure 1

Input:	-parameters in the CALL-statement.
	-matrices A1 and B in a labeled COMMON[14]:
	COMMON /CMUL/ A1(4,4), B(4,4), C(4,4)
Output:	the result of the matrix multiplication is stored in the N x L matrix C.
Printing:	none.

b. INVERT:

SUBROUTINE INVERT (CC,N)

Task:	inversion of the matrix CC
Parameters:	N=rank of CC.
Input:	the parameter N and the matrix CC are transmitted through the CALL statement. The subroutine assumes that CC is nonsingular, and that all the diagonal elements are nonzero.
Output:	the original matrix CC is replaced by the inverted matrix.
Printing:	none.

c. EIGEN

SUBROUTINE EIGEN (N,NP, NEIG)

Task: calculation of the dominant eigenvalue of the matrix CE and of the associated right and left eigenvectors. EIGEN may also be used to compute row and column totals and to print a matrix

Parameters: N: dimension of the matrix.

NEIG: parameter related to the computation.

NEIG = 1: the complete computation procedure is performed: row and column totals, dominant eigenvalue and associated eigenvectors.

NEIG = 0: only the row and column sums are computed and printed. By using this option, EIGEN may be used to print a matrix.

---

[14] In the programs, the dimensions are specified for maximum 4 regions. To create more regions the dimension and common statements have to be changed. Other than that the programs can handle systems of up to 10 regions.

NP: parameter related to printing.

NP = 1: EIGEN prints the original matrix, its row and column sums. The dominant eigenvalue and its right and left eigenvector are printed if NEIG = 1. The row totals denote the sum of the row elements, weighted by the radices (i.e. RADIX(I)).

NP = 0: nothing is printed

NP = 2: the row and column totals are <u>averages</u>, not totals. This option is only used for the matrix of mean ages, Alternative 1.

Input: -parameters in CALL statement.
-the matrix $\underset{\sim}{CE}$ in labeled COMMON:
COMMON /CEIGEN/ CE(4,4), ROOT, VECT(4), VECTL(4)

Output: the dominant eigenvalue ROOT, the right eigenvector VECT(I) and the left eigenvector VECTL(I) are stored in labeled COMMON.

Printing: according to the specification of the parameter NP.

Algorithm: Let the original matrix be $\underset{\sim}{A}$:

$$\underset{\sim}{A} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}$$

The dominant eigenvalue of $\underset{\sim}{A}$ is obtained by the power method (for details see Rogers 1971, Chapter 7):

$$\lambda^{(n)} = \frac{a_{11}^{(n+1)}}{a_{11}^{(n)}} ,$$

where the superscript denotes the iteration and $a_{11}^{(n)}$ is the first element of the matrix $\underset{\sim}{A}^n$. As n gets large, $\lambda^{(n)}$ converges to the true eigenvalue. The iteration terminates when

$$-\varepsilon < \left[\frac{a_{12}^{(n+1)}}{a_{11}^{(n+1)}} - \frac{a_{12}^{(n)}}{a_{11}^{(n)}} \right] < \varepsilon$$

with $\varepsilon = 0.000001$.

The right eigenvector $\{\underset{\sim}{\xi}\}$ associated with λ is proportional to any column of $\underset{\sim}{A}^n$ for n large. In the program, $\{\underset{\sim}{\xi}\}$ is taken to be the first column of $\underset{\sim}{A}^n$,--scaled such that $\xi_1 = 1$. The scaling selected is arbitrary, since an eigenvector is fixed up to a scalar. For convenience, we have retained the scaling $\xi_1 = 1$, i.e. the first element of $\{\underset{\sim}{\xi}\}$ is unity.

The left eigenvector is the solution to the system

$$\{\underset{\sim}{\nu}\}' \left[\underset{\sim}{A} - \lambda \underset{\sim}{I}\right] = \{\underset{\sim}{0}\}' \quad .$$

As before, we take ν_1 to be unity. Therefore

$$\begin{bmatrix} 1 \\ \nu_2 \\ \vdots \\ \nu_n \end{bmatrix}' \begin{bmatrix} a_{11}-\lambda & a_{21} \cdots\cdots & a_{n1} \\ a_{12} & a_{22}-\lambda \cdots & a_{n2} \\ \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} \cdots & a_{nn}-\lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}'$$

and

$$\begin{bmatrix} \nu_2 \\ \vdots \\ \nu_n \end{bmatrix}' = \begin{bmatrix} a_{21} \\ \vdots \\ a_{n1} \end{bmatrix}' \begin{bmatrix} a_{22}-\lambda & & a_{n2} \\ \vdots & & \vdots \\ a_{2n} & \cdots\cdots & a_{nn}-\lambda \end{bmatrix}^{-1}$$

7.2 The Special Purpose Subroutines

a. READ2

SUBROUTINE READ2 (NA,NR,NY,NGRO,NSTADC,ZLAMDK)

Task: reads in the data. The input data required by the programs treated in this paper are produced by the life table program and the population projection program listed in Willekens and Rogers (1976) and contained in the two files:

OUTLIF (life table output)

OUTPROJ (projection output).

These files must be defined in a control card. The following convention is used: the channel (IBM) or unit number (CDC) of OUTLIF is 7, and that of OUTPROJ is 8.

Parameters: NA: number of age groups

NR: number of regions

NY: age group interval (usually five years).

NGRO: zero population growth (ZPG) option:

NGRO = 0: no ZPG analysis

NGRO = 1: ZPG analysis

NSTADC: selection of discrete or continuous analysis of population growth.

NSTADC = 1: discrete analysis

NSTADC = 2: continuous analysis.

(NSTADC has been set equal to 2).

ZLAMDK: dominant eigenvalue of the multiregional growth operator $\underset{\sim}{G}$ (generalized Leslie matrix). ZLAMDK is computed by PROJECT (Willekens and Rogers, 1976).

Input: files OUTLIF and OUTPROJ, created by the programs for life table computation and population projection.

Output: the data are stored in labeled COMMON for easy transmittance from one subroutine to another. The variable names are explained in Appendix 1.

b. AGEDIS

SUBROUTINE AGEDIS (NA,NY,ZFNY,NR,NSTADC,ZLAMDK)

Task: computation and printing of the three types of population distribution: observed population, life table population, and stable population.

Parameters: NA,NY,NR,NSTADC,ZLAMDK: see above

ZFNY: age group interval as a real variable, i.e. ZFNY = FLOAT(NY).

Input: the input data consist of the distribution of the observed and life table population. The data are read in by READ2 and are contained in the following arrays:

- POP(X,I): observed population in age group X and region I.
- L(X,I,J): the life table population of age group X, who are born in region I and residing in region J. The life table population is expressed in unit radices.
- CLLT(X,I): the life table population of age group X, born in region I.

Output: this subroutine produced the tables of Section 1 of the paper. Note that the life table and stable populations are expressed per unit born in each region.

Algorithm: the observed and life table populations are printed as read in. The stable population is computed by (1.3b):

$$\underset{\sim}{L}^{(r)}(x) = EX(x)\ \underset{\sim}{L}(x),$$

where

$$EX(x) = e^{-r(x+NY*0.5)} \quad . \tag{7.1}$$

c. STABCH

SUBROUTINE STABCH (NA,NY,ZFNY,NR,NSTADC,NOPMOB,
NEIG)

Task: STABCH computes and prints the integral functions (generalized maternity and mobility function and the weighted functions), and their moments. From this information, it derives the matrices of mean ages at childbearing and mobility and the variances of the ages at childbearing and mobility.

Parameters: NA,NY,ZFNY,NR,NSTADC, NEIG: see above

NOPMOB: parameter indicating the type of population analysed.

NOPMOB = 1: life table population

NOPMOB = 2: ZPG population

NOPMOB = 3: stable population

Input: transmitted by labeled COMMON. This subroutine is called four times by the main program. Each time, there is a different population distribution - fertility schedule combination:

1. Life table population combined with observed fertility schedule (NOPMOB = 1)
2. Life table population combined with a fertility schedule at replacement level (two alternatives) (NOPMOB = 2)
3. Stable population combined with observed fertility schedule (NOPMOB = 3).

Output: tables of sections 2,3 and 6.1 of this paper.

Algorithm: The integral functions are computed by (2.4),(2.19), (3.5) and (3.11). In the program, it reduces to a single expression:

PATFUN(X,J,I) = EX(X)*L(X,I,J)*ZGRAL(X,J),

where EX(X) is the weighting factor:

EX(X) = 1 in the life table and ZPG population, and is equal to (7.1) in the stable population,

L(X,I,J,) is the number of people in region J aged X to X+NY, who are born in region I,

ZGRAL(X,J) contains the age and region-specific rates applied to the population distribution to give the integral function. In the case of the maternity function,

ZGRAL(X,J) = RATF(X,J), i.e. the regional age-specific fertility rates. In the mobility analysis,

ZGRAL(X,J) = RATOT(X,J), i.e. the regional age-specific total outmigration rates.

The p-th moment of the integral function is given by

$$ZMOM(J,I) = \sum_X Z*PATFUN(X,J,I),$$

where $Z = [(X-1)*NY+NY*0.5]^p$.

The zero-th moment of the generalized net maternity function, i.e. the net reproduction rate matrix is saved in the array ZNRR(I,J) for later use (ZPG-analysis).

d. ZERO

SUBROUTINE ZERO (NA,NR,NZERO)

Task: ZERO replaces the observed regional fertility schedules with fertility schedules at replacement level. The new fertility rates are computed according to two alternative fertility reduction schemes described in section 6.

Parameters: NA,NR: see above.

NZERO: denotes the alternative fertility reduction scheme.

NZERO = 1: the cohort-replacement alternative.

NZERO = 2: the proportional reduction alternative.

Input: transmitted by labeled COMMON.

Output:
- diagonal matrix of fertility adjustment factors. The diagonal elements are stored in the vector VI(I).
- new fertility rates at replacement level: RATF(X,I).

Algorithm: see section 6.

e. RVALUE

SUBROUTINE RVALUE (NA,NY,ZFNY,NR,NSTADC,R,ZVT)

Task: computation and printing of the discounted number of offspring and the reproductive values by age and region.

Parameters: NA,NY,ZFNY,NR,NSTADC: see above.

R: the ultimate growth rate of the population under consideration:

R = 0 in the ZPG population;

R = [ℓn ZLAMDK]/NY in the stable population.

ZVT: total reproductive value of whole system (all regions).

Input: labeled COMMON

Output: tables of section 4 of this paper.

Algorithm: see Section 4.
Note that RVALUE calls the subroutine PROBSCH to compute the number of people at exact age X (equation 4.9). This subroutine has already been published (Willekens and Rogers, 1976, p. 119).

f. RINTR

SUBROUTINE RINTR (NA,NY,ZFNY,NR,NSTADC,ZLAMDA,R,ZVT)

Task: it computes and prints the stable equivalents and the intrinsic rates.

Parameters: NA,NY,ZFNY,NR,NSTADC,R,ZVT: see above.
ZLAMDA: the ultimate growth ratio of the population under consideration.
ZLAMDA = 1 in the life table and ZPG population analysis,
ZLAMDA = ZLAMDK in the stable population analysis.

Input: labeled COMMON

Output: tables of Section 5 of this paper. The stable equivalents of births and total population are saved for later use and stored in the arrays QQ(I) and YY(I) respectively. The stable equivalent population by age and region is contained in POPST(X,I).

Algorithm: see Section 5.

g. MOMENT

SUBROUTINE MOMENT (NA,NY,ZFNY,NR,R)

Task: spatial ZPG - analysis following the analytical approach.

Parameters: NA,NY,ZFNY,NR,R: see above.

Input: labeled COMMON.
It is assumed that the initial population is stable and that the births are given by the vector QQ(I).

Output: tables of Section 6.2 of this paper.

Algorithm: see Section 6.2.

7.3. The Main Program

The main program is kept very short. Its function is to coordinate the calculations. It therefore consists merely of CALL-statements. The zero-population-growth analysis is an option:

NGR0 = 0 no ZPG analysis (only life table and stable population analysis)

NGRO = 1 ZPG analysis included.

NGRO = 2 no life table and ZPG analysis (only stable population analysis).

7.4. The Input Data

The data required by the programs are read in by READ2. All are created by the life table program and the population projection program described in Willekens and Rogers (1976). They are contained in two files. The first, OUTLIF, is created by the life table program and contains the parameters, the age-specific rates, the observed and the life table population distribution, and the survivorship proportions. This file is read in using unit number (channel) 7. The second file, OUTPROJ, is created by the population projection program. It contains the first row of the generalized LESLIE matrix, the population distribution at stability and the dominant eigenvalue of the generalized LESLIE matrix. OUTPROJ is read in through unit number 8.

APPENDIX 1

Glossary of mathematical symbols and FORTRAN names of demographic variables.

Symbol	Page in Rogers (1975)	FORTRAN Name	Description
		Subscripts	
		X	age group (1,2,3,...)
		I	region of residence (or birth)
		J	region of destination (in case of migration)
		Observations	
$K_i(x)$	82	POP(X,I)	population by age and region
		BIRTH (X,I)	births by age and region
$D_i(x)$	82	DEATH(X,I)	deaths " " " "
$K_{ij}(x)$	82	OMIG(X,J,I)	migrants from I to J by age and region
$\ell_i(0)$	73	RADIX(I)	radix of region I
		NAGE(X)	first age of age group X
		REG(I)	name of Region I

		Computed rates and total population	
$M_{i\delta}(x)$	82	RATD(X,I)	age-specific death rates for region I
$F_i(x)$		RATF(X,I)	age-specific fertility rates for region I
$M_{ij}(x)$	83	RATM(X,J,I)	age-specific migration rates from I to J
		RATOT(X,I)	age-specific total out-migration rates of I
		RATDT(X)	age-specific death rates for whole system
		RATFT(X)	age-specific fertility rates for whole system
		RATMT(X)	age-specific migration rates for whole system
		POPC(X)	age distribution of population for total system
		Life Table	
$q_i(x)$	60	Q(X,I)	probability of dying in I
$p_{ij}(x)$	60	P(X,J,I)	probability of being in J at age X + h, while in I at X
		PMIGT(X,I)	probability of being in another region at age X + h, while in I at X
${}_{i0}L_j(x)$	61	L(X,I,J)	number of years lived in J between ages X and X + h by an individual born in I
		CLLT(X,I)	number of years lived between ages X and X + h by an individual born in I
		CLLTOT(X)	average number of years lived between ages X and X + h per unit radix

$s_{ij}(x)$	79	SU(X,I,J)	proportion of people aged X to X + h in region I, surviving to be in region J and X + h to X + 2h years old h years later
		SSU(X,I)	proportion of people aged X to X + h in region I, surviving to be X + h to X + 2h years old h years later

Population Projection

$s_{ij}(x)$	118	SU,(X,I,J)	see life table
$b_{ij}(x)$	118	BR(X,J,I)	average number of babies born during the unit time interval and alive in region J at end of that interval, per X to (X + h) year old residents of region I at beginning of that interval
$P_{ij}(0)$	60	PP1(J,I)	probability that an individual born in I survives to be in J h years later (at beginning of next interval)
$K_i^{(t)}(x)$	118	POPR(X,I)	projected population in age group X in region I
		POPPR(I)	total projected population in region I
		POPPRT	total projected population in whole system

Integral function analysis

		PATFUN(X,J,I)	value of the integral function for I-born people living in region J and X to X + 4 years old
		ZMOM(P,J,I)	P-th moment of the integral function

NRR or $\tilde{R}(0)$	106	ZNRR(J,I)	zero-th moment of the generalized net maternity function
		RONRR	dominant eigenvalue of $\tilde{R}(0)$
		VRNRR(J)	right eigenvector of R(0), associated with RONRR
		VLNRR(I)	left eigenvector of R(0), associated with RONRR
		EX(X)	weighting factor for generalized net maternity and mobility functions EX(X) = 1 for stationary populations EX(X) = $e^{-r(x+NY*0.5)}$ for stable populations
		ZGRAL(X,J)	the age- and region-specific rates entering the integral function
		AGEM(J,I)	matrix of mean ages in stable population (Alternative 2).

Spatial reproductive value analysis

$\tilde{\overline{\psi}}(r)$		PSI(J,I)	zero-th moment of the weighted generalized net maternity function
		ROOTPSI	dominant eigenvalue of $\tilde{\overline{\psi}}(r)$
$\{\tilde{Q}_1^S\}$	93	VRPSI(J)	right eigenvector of $\tilde{\overline{\psi}}(r)$, associated with ROOTPSI
$\{\tilde{v}(0)\}$		VLPSI(I)	left eigenvector of $\tilde{\overline{\psi}}(r)$, associated with ROOTPSI
$\{\tilde{Q}\}$	105	QQ(I)	stable equivalent of regional births
$\{\tilde{Y}\}$	112	YY(I)	stable equivalent of the regional total population
$\{\tilde{K}(x)\}$		POPST(X,I)	stable equivalent of the population by age and region

$\underset{\sim}{\bar{n}}_x$		V(X,J,I)	discounted number of offspring in region J per person residing in region I at age X, respectively in age group X to X + 4

	Momentum of spatial zero population growth		
$\underset{\sim}{e}(o)$	74	EO(J,I)	matrix of life expectancies at birth
$\underset{\sim}{R}(0)$	106	RO(J,I)	$\underset{\sim}{NRR}$ - matrix
$\underset{\sim}{R}(1)$	106	R1(J,I)	first moment of the generalized net maternity function
$\{\underset{\sim}{\hat{Q}}\}$		QQZP(I)	regional allocation of births in ZPG-population
$\{\underset{\sim}{\hat{Y}}\}$		YYZP(I)	regional allocation of the total population in ZPG-population

Appendix 2

Listing of Computer Programs

MAIN PROGRAM

```
MAIN PROGRAM FOR STATIONARY AND STABLE POPULATION ANALYSIS (MAINSTAB.FTN)
      DIMENSION RATFZE(18,4),HU(18)
      COMMON /CSU/ SU(18,4,4),SSU(18,4)
      COMMON /CGROW/ BR(18,4,4),PP1(4,4),POPR(18,4),POPPR(4),POPPRT
      COMMON /CL/ L(18,4,4),CLLT(18,4),CLLTOT(18)
      COMMON /CRAD/ RADIX(4),RADIXT
      COMMON /CRATE/ RATD(18,4),RATM(18,4,4),RATF(18,4)
      COMMON /CRATET/ RATOT(18,4)
      COMMON /CREG/ REG(4)
      COMMON /C1/ POP(18,4),BIRTH(18,4),DEATH(18,4),OMIG(18,4,4)
      COMMON /C4/ NAGE(18)
      COMMON /CMAT/ PATFUN(18,4,4)
      COMMON /CRATS/ BRATE(4),DRATE(4),ORATE(4),OIRATE(4)
      COMMON /CQUE/ QUE(4),QUET,SHA(4)
      COMMON /CTOTRAT/ POPC(18),RATDT(18),RATFT(18),RATMT(18)
      COMMON /CEIGEN/ CE(4,4),ROOT,VECT(4),VECTL(4)
      COMMON /CNRR/ ZNRR(4,4),VRNRR(4),VLNRR(4),RONRR
      COMMON /CPSI/ PSI(4,4),VRPSI(4),VLPSI(4),ROPSI
      COMMON /CEX/ EX(18)
      DOUBLE PRECISION REG
      INTEGER X,XX
      REAL L
      CALL READ2 (NA,NR,NY,NGRO,NSTADC,ZLAMDK)
      NSTADC=2
      ZFNY=FLOAT(NY)
      DO 7 X=1,NA
 7    NAGE(X)=NY*(X-1)
      CALL AGEDIS (NA,NY,ZFNY,NR,NSTADC,ZLAMDK)
      DO 44 X=1,NA
44    HU(X)=EX(X)
      IF (NGRO.EQ.2) GO TO 105
      PRINT 22
22    FORMAT (1H1,1X)
      PRINT 23
23    FORMAT (10X,41(1H*)/10X,41(1H*)/10X,3H* *,35X,3H* *)
      PRINT 43
43    FORMAT (10X,3H* *,1X,33HANALYSIS OF LIFE TABLE POPULATION,
     11X,3H* *)
      PRINT 25
25    FORMAT (10X,3H* *,35X,3H* */10X,41(1H*)/10X,41(1H*)/)
      DO 47 X=1,NA
47    EX(X)=1.
      ZLAMDA=1.
      R=0.
      NEIG=0
      CALL STABCH (NA,NY,ZFNY,NR,2,1,NEIG)
      IF (NGRO.EQ.0) GO TO 105
      PRINT 22
      PRINT 23
      PRINT 24
24    FORMAT (10X,3H* *,1X,33HANALYSIS OF STATIONARY POPULATION,
     11X,3H* *)
      PRINT 25
      DO 30 I=1,NR
      DO 30 X=1,NA
30    RATFZE(X,I)=RATF(X,I)
      NNZERO=2
      DO 33 NZERO=1,NNZERO
      CALL ZERO (NA,NR,NZERO)
      CALL STABCH (NA,NY,ZFNY,NR,2,2,NEIG)
```

```
      CALL RVALUE (NA,NY,ZFNY,NR,NSTADC,R,ZVT)
      CALL RINTR (NA,NY,ZFNY,NR,NSTADC,ZLAMDA,R,ZVT)
      DO 31 I=1,NR
      DO 31 X=1,NA
 31   RATF(X,I)=RATFZE(X,I)
 33   CONTINUE
105   CONTINUE
      PRINT 22
      PRINT 23
      PRINT 45
 45   FORMAT (10X,3H* *,3X,29HANALYSIS OF STABLE POPULATION,3X,
     13H* *)
      PRINT 25
      DO 46 X=1,NA
 46   EX(X)=HU(X)
      R=ALOG(ZLAMDK)/ZFNY
      CALL STABCH (NA,NY,ZFNY,NR,NSTADC,3,NEIG)
       CALL RVALUE (NA,NY,ZFNY,NR,NSTADC,R,ZVT)
      CALL RINTR (NA,NY,ZFNY,NR,NSTADC,ZLAMDK,R,ZVT)
      CALL MOMENT (NA,NY,ZFNY,NR,R)
104   CONTINUE
      IF (IRUNT.EQ.1) IRUN=2
500   CONTINUE
600   CONTINUE
      STOP
      END
```

READ2

```
      SUBROUTINE READ2 (NA,NR,NY,NGRO,NSTADC,ZLAMDK)
      COMMON /CSU/ SU(18,4,4),SSU(18,4)
      COMMON /CGROW/ BR(18,4,4),PP1(4,4),POPR(18,4),POPPR(4),POPPRT
      COMMON /CL/ L(18,4,4),CLLT(18,4),CLLTOT(18)
      COMMON /CRAD/ RADIX(4),RADIXT
      COMMON /CRATE/ RATD(18,4),RATM(18,4,4),RATF(18,4)
      COMMON /CRATET/ RATOT(18,4)
      COMMON /CREG/ REG(4)
      COMMON /C1/ POP(18,4),BIRTH(18,4),DEATH(18,4),OMIG(18,4,4)
      COMMON /CTOTRAT/ POPC(18),RATDT(18),RATFT(18),RATMT(18)
      DOUBLE PRECISION REG
      INTEGER X,XX
      REAL L
      IOPTG=1
100   CONTINUE
      READ (7,20) NA,NR,NY,NZB,NZD,NZO,IRUNT,IOPTG,NGRO,KA,KC,LM,
     1NPAR1,NPAR2,NPAR3,NPAR4,INIT,NPAR5,NPAR6,NPAR7,NPAR8
 20   FORMAT (16I2,5I4)
      NSTADC=NPAR2
      IF (NA.EQ.0) STOP
      READ (7,22) (REG(J),J=1,NR)
 22   FORMAT (9A8)
      READ (7,23) RADIXT,(RADIX(J),J=1,NR)
 23   FORMAT (10F8.0)
      DO 50 I=1,NR
      READ (7,24) (PP1(J,I),J=1,NR)
      DO 50 X=1,NA
      READ (7,24) (SU(X,I,J),J=1,NR)
      READ (7,24) (L(X,I,J),J=1,NR)
 24   FORMAT (9F8.6)
 50   CONTINUE
      DO 51 I=1,NR
      READ (7,25) (POP(X,I),X=1,NA)
 25   FORMAT (8F10.0/8F10.0/2F10.0)
      READ (7,26) (RATF(X,I),X=1,NA)
 26   FORMAT (10F8.6/8F8.6)
      READ (7,26) (RATD(X,I),X=1,NA)
      READ (7,26) (RATOT(X,I),X=1,NA)
      DO 54 J=1,NR
 54   READ (7,26) (RATM(X,J,I),X=1,NA)
      READ (7,24) (CLLT(X,I),X=1,NA)
 51   CONTINUE
      READ (7,26) (RATDT(X),X=1,NA)
      READ (7,26) (RATFT(X),X=1,NA)
      READ (7,26) (RATMT(X),X=1,NA)
      READ (7,24) (CLLTOT(X),X=1,NA)
      DO 59 I=1,NR
      DO 59 J=1,NR
 59   READ (8,26) (BR(X,J,I),X=1,NA)
      DO 52 I=1,NR
 52   READ (8,25) (POPR(X,I),X=1,NA)
      READ (8,25) (POPPR(J),J=1,NR),POPPRT
      READ (8,55) ZLAMDK
 55   FORMAT (E14.7)
      RETURN
      END
```

MULTIP

```
      SUBROUTINE MULTIP (N,K,L)
      COMMON /CMUL/ A1(4,4),B(4,4),C(4,4)
      DO 3 I=1,N
      DO 3 J=1,L
      C(I,J)=0.
      DO 3 JJ=1,K
      C(I,J)=C(I,J)+A1(I,JJ)*B(JJ,J)
 3    CONTINUE
       RETURN
      END
```

INVERT

```
      SUBROUTINE INVERT (CC,NR)
      DIMENSION PIVOT(4),CC(4,4)
      DO 606 I=1,NR
      PIVOT(I)=CC(I,I)
      CC(I,I)=1.0
      DO 607 J=1,NR
      CC(I,J)=CC(I,J)/PIVOT(I)
607   CONTINUE
      IF (NR.EQ.1) GO TO 10
      DO 608 K=1,NR
      IF (K.EQ.I) GO TO 608
      H=CC(K,I)
      CC(K,I)=0.
      DO 609 L=1,NR
      CC(K,L)=CC(K,L)-CC(I,L)*H
609   CONTINUE
608   CONTINUE
606   CONTINUE
10    CONTINUE
      RETURN
       END
```

EIGEN

```
      SUBROUTINE EIGEN (NR,NP,NEIG)
      DIMENSION ZMOMT(4),CC(4,4),HU(4)
      COMMON /CMUL/ A1(4,4),B(4,4),C(4,4)
      COMMON /CRAD/ RADIX(4),RADIXT
      COMMON /CREG/ REG(4)
      COMMON /CEIGEN/ CE(4,4),ROOT(4),VECT(4),VECTL(4)
       DOUBLE PRECISION REG
      IF (NEIG.EQ.0) GO TO 820
      DO 21 I=1,NR
      DO 21 J=1,NR
      A1(J,I)=CE(J,I)
      B(J,I)=CE(J,I)
   21 CONTINUE
      CALL MULTIP (NR,NR,NR)
      Z4=10000.
   23 CONTINUE
      Z5=Z4
       Z=C(1,1)
      DO 22 I=1,NR
      DO 22 J=1,NR
      C(J,I)=C(J,I)/Z
      A1(J,I)=C(J,I)
   22 B(J,I)=C(J,I)
      CALL MULTIP (NR,NR,NR)
      Z4=C(2,1)/C(1,1)
      Z3=Z4-Z5
      TOLEIG=0.000001
      T2=-TOLEIG
      IF ((Z3.LT.T2).OR.(Z3.GT.TOLEIG)) GO TO 23
      DO 24 I=1,NR
      DO 24 J=1,NR
      A1(J,I)=C(J,I)
   24 B(J,I)=CE(J,I)
      CALL MULTIP (NR,NR,NR)
      ROOT(1)=C(1,1)/A1(1,1)
      DO 25 J=1,NR
   25 VECT(J)=C(J,1)/C(1,1)
   26 CONTINUE
      IF (NP.EQ.0) GO TO 30
  820 CONTINUE
      PRINT 62, (REG(J),J=1,NR)
   62 FORMAT (/17X,5HTOTAL,10(3X,A8))
      PRINT 64
   64 FORMAT (1X)
      DO 5 I=1,NR
      ZZTOT=0.
      DO 90 J=1,NR
   90 ZZTOT=ZZTOT+CE(I,J)*RADIX(J)/RADIX(I)
      IF (NP.EQ.2) ZZTOT=ZZTOT/FLOAT(NR)
    5 PRINT 91, REG(I),ZZTOT,(CE(I,J),J=1,NR)
   91 FORMAT (1X,A8,2X,11F11.6)
      DO 6 J=1,NR
      ZMOMT(J)=0.
      DO 8 I=1,NR
    8 ZMOMT(J)=ZMOMT(J)+CE(I,J)
      IF (NP.EQ.2) ZMOMT(J)=ZMOMT(J)/FLOAT(NR)
    6 CONTINUE
      PRINT 7, (ZMOMT(J),J=1,NR)
    7 FORMAT (/4X,5HTOTAL,13X,11F11.6)
C   LEFT EIGENVECTOR
```

```
      PRINT 64
      IF (NEIG.EQ.Ø) GO TO 3Ø
      NR1=NR-1
      DO 11 I=1,NR1
      DO 11 J=1,NR1
      I1=I+1
      J1=J+1
      IF (I.EQ.J) CC(I,J)=CE(J1,I1)-ROOT(1)
      IF (I.NE.J) CC(I,J)=CE(J1,I1)
11    CONTINUE
      CALL INVERT (CC,NR1)
      DO 12 I=1,NR1
      I1=I+1
      B(I,1)=-CE(1,I1)
      DO 12 J=1,NR1
12    A1(I,J)=CC(I,J)
      CALL MULTIP (NR1,NR1,1)
      VECTL(1)=1.
      DO 13 I=1,NR1
      I1=I+1
13    VECTL(I1)=C(I,1)
      DO 92 I=1,1
      PRINT 93, ROOT(I)
93    FORMAT (1X,1ØHEIGENVALUE,F11.6)
      PRINT 94
94    FORMAT (1X,11HEIGENVECTOR)
      PRINT 95, (VECT(J),J=1,NR)
95    FORMAT (5X,7H- RIGHT ,11X,1ØF11.6)
      PRINT 96, (VECTL(J),J=1,NR)
96    FORMAT (5X,7H-  LEFT,11X,1ØF11.6)
92    CONTINUE
3Ø    CONTINUE
      RETURN
      END
```

AGEDIS

```
      SUBROUTINE AGEDIS (NA,NY,ZFNY,NR,NSTADC,ZLAMDK)
      DIMENSION HULP(4),HU1(4)
      COMMON /CL/ L(18,4,4),CLLT(18,4),CLLTOT(18)
      COMMON /CREG/ REG(4)
      COMMON /C1/ POP(18,4),BIRTH(18,4),DEATH(18,4),OMIG(18,4,4)
      COMMON /C4/ NAGE(18)
      COMMON /CEX/ EX(18)
      DOUBLE PRECISION REG
      INTEGER X
      REAL L
      R=ALOG(ZLAMDK)/ZFNY
      DO 3 X=1,NA
      IF (NSTADC.EQ.1) GO TO 4
      Z=(FLOAT(X)-1.)*ZFNY+ZFNY*0.5
      Z=-Z*R
      EX(X)=EXP(Z)
      GO TO 3
    4 IX=X*NY
      Z=-FLOAT(X)
      EX(X)=ZLAMDK**Z
    3 CONTINUE
      PRINT 5
    5 FORMAT (1H1,9X,41HPOPULATION DISTRIBUTION BY AGE AND REGION/
     110X,41(1H*)/)
      PRINT 6
    6 FORMAT (/10X,43HOBSERVED POPULATION (BY PLACE OF RESIDENCE)/
     110X,43(1H=))
      PRINT 7, (REG(J),J=1,NR)
    7 FORMAT (/6X,10(3X,A8))
      PRINT 64
   64 FORMAT (1X)
      DO 25 J=1,NR
      HULP(J)=0.
      DO 25 X=1,NA
   25 HULP(J)=HULP(J)+POP(X,J)
      DO 8 X=1,NA
    8 PRINT 9, NAGE(X),(POP(X,J),J=1,NR)
    9 FORMAT (1X,I3,2X,10F11.0)
      PRINT 15, (HULP(J),J=1,NR)
   15 FORMAT (/1X,5HTOTAL,10F11.0)
      PRINT 10
   10 FORMAT (1H1,9X,21HLIFE TABLE POPULATION/10X,21(1H=))
      DO 19 I=1,NR
      PRINT 11, REG(I)
   11 FORMAT (//10X,24HINITIAL REGION OF COHORT,2X,A8/
     110X,34(1H-)/)
      PRINT 14, (REG(J),J=1,NR)
   14 FORMAT (12X,5HTOTAL,10(3X,A8))
      PRINT 64
      DO 12 X=1,NA
   12 PRINT 13, NAGE(X),CLLT(X,I),(L(X,I,J),J=1,NR)
   13 FORMAT (1X,I3,2X,11F11.6)
      DO 16 J=1,NR
      HULP(J)=0.
      DO 16 X=1,NA
   16 HULP(J)=HULP(J)+L(X,I,J)
      ZH=0.
      DO 17 J=1,NR
   17 ZH=ZH+HULP(J)
      PRINT 18, ZH,(HULP(J),J=1,NR)
```

```
18   FORMAT (/1X,5HTOTAL,11F11.6)
19   CONTINUE
     PRINT 20, R
20   FORMAT (1H1,9X,33HSTABLE POPULATION  (GROWTH RATE =,F10.6,
    12H )/10X,45(1H*))
     DO 21 I=1,NR
     PRINT 11, REG(I)
     PRINT 14, (REG(J),J=1,NR)
     PRINT 64
      ZH=0.
     DO 24 J=1,NR
24    HULP(J)=0.
     DO 22 X=1,NA
     Z=EX(X)*CLIT(X,I)
     DO 23 J=1,NR
     HU1(J)=EX(X)*L(X,I,J)
23   HULP(J)=HULP(J)+HU1(J)
22   PRINT 13, NAGE(X),Z,(HU1(J),J=1,NR)
     DO 26 J=1,NR
26   ZH=ZH+HULP(J)
     PRINT 18, ZH,(HULP(J),J=1,NR)
21   CONTINUE
     RETURN
     END
```

STABCH

```
      SUBROUTINE STABCH (NA,NY,ZFNY,NR,NSTADC,NOPMOB,NEIG)
      DIMENSION ZMOMT(4)
      DIMENSION HULP(4,4),HULP2(4,4),CC(4,4),ZGRR(4),ZMOM(3,4,4)
      DIMENSION HULP4(4)
      DIMENSION HULP5(4),ZGRAL(18,4)
      COMMON /CL/ L(18,4,4),CLLT(18,4),CLLTOT(18)
      COMMON /CMUL/ A1(4,4),B(4,4),C(4,4)
      COMMON /CRAD/ RADIX(4),RADIXT
      COMMON /CRATE/ RATD(18,4),RATM(18,4,4),RATF(18,4)
      COMMON /CRATET/ RATOT(18,4)
      COMMON /CREG/ REG(4)
      COMMON /C4/ NAGE(18)
      COMMON /CMAT/ PATFUN(18,4,4)
      COMMON /CTOTRAT/ POPC(18),RATDT(18),RATFT(18),RATMT(18)
      COMMON /CEIGEN/ CE(4,4),ROOT,VECT(4),VECTL(4)
      COMMON /CNRR/ ZNRR(4,4),VRNRR(4),VLNRR(4),RONRR
      COMMON /CPSI/ PSI(4,4),VRPSI(4),VLPSI(4),ROPSI
      COMMON /CAGEM/ AGEM(4,4)
      COMMON /CEX/ EX(18)
      DOUBLE PRECISION REG
      INTEGER X
      REAL L
   64 FORMAT (1X)
      DO 580 INTEGR=1,2
       IF ((INTEGR.EQ.2).AND.(NOPMOB.EQ.2)) GO TO 580
      DO 121 I=1,NR
      DO 121 X=1,NA
      IF (INTEGR.EQ.2) ZGRAL(X,I)=RATOT(X,I)
      IF ((INTEGR.EQ.1).AND.(NSTADC.EQ.2)) ZGRAL(X,I)=RATF(X,I)
  121 CONTINUE
      IF (INTEGR.EQ.1) PRINT 36
   36 FORMAT (1H0,9X,18HFERTILITY ANALYSIS/10X,18(1H*))
      IF (INTEGR.EQ.2) PRINT 37
   37 FORMAT (1H1,9X,18HMIGRATION ANALYSIS/10X,18(1H*))
C    PRINT RATES
      PRINT 5
    5 FORMAT (//10X,14HOBSERVED RATES/10X,14(1H=))
      PRINT 7, (REG(J),J=1,NR)
    7 FORMAT (/1X,3HAGE,2X,10(3X,A8))
      PRINT 64
      DO 687 X=1,NA
  687 PRINT 686, NAGE(X),(ZGRAL(X,J),J=1,NR)
  686 FORMAT (1X,I3,2X,10F11.6)
      DO 684 J=1,NR
      ZGRR(J)=0.
      DO 684 X=1,NA
  684 ZGRR(J)=ZGRR(J)+ZGRAL(X,J)*ZFNY
      IF (INTEGR.EQ.1) PRINT 685, (ZGRR(J),J=1,NR)
  685 FORMAT (/1X,3HGRR,2X,10F11.6)
      IF (INTEGR.EQ.2) PRINT 688, (ZGRR(J),J=1,NR)
  688 FORMAT (/1X,3HGMR,2X,10F11.6)
      IF ((INTEGR.EQ.1).AND.(NOPMOB.LE.2)) PRINT 120
  120 FORMAT (1H1,9X,47HINTEGRALS OF GENERALIZED NET MATERNITY FUNCTION/
     110X,47(1H=))
      IF ((INTEGR.EQ.1).AND.(NOPMOB.EQ.3)) PRINT 124
  124 FORMAT (1H1,9X,38HINTEGRALS OF WEIGHTED GENERALIZED NET ,
     118HMATERNITY FUNCTION/10X,56(1H=))
      IF ((INTEGR.EQ.2).AND.(NOPMOB.LE.2)) PRINT 122
  122 FORMAT (1H1,9X,46HINTEGRALS OF GENERALIZED NET MOBILITY FUNCTION/
     110X,46(1H=))
```

```
      IF ((INTEGR.EQ.2).AND.(NOPMOB.EQ.3)) PRINT 123
 123  FORMAT (1H1,9X,38HINTEGRALS OF WEIGHTED GENERALIZED NET ,
     117HMOBILITY FUNCTION/10X,55(1H=))
      DO 2 I=1,NR
      DO 2 J=1,NR
      DO 2 X=1,NA
   2  PATFUN(X,J,I)=EX(X)*ZGRAL(X,J)*L(X,I,J)
      DO 110 I=1,NR
      DO 110 J=1,NR
 110   HULP(J,I)=0.
      DO 3 I=1,NR
      PRINT 4, REG(I)
   4  FORMAT (//10X,24HINITIAL REGION OF COHORT,2X,A8/
     110X,34(1H-)/)
      PRINT 7, (REG(J),J=1,NR)
      DO 8 X=1,NA
       DO 109 J=1,NR
 109  HULP(J,I)=HULP(J,I)+PATFUN(X,J,I)
   8  PRINT 9, NAGE(X),(PATFUN(X,J,I),J=1,NR)
   9  FORMAT (1X,I3,2X,11F11.6)
      PRINT 108, (HULP(J,I),J=1,NR)
 108  FORMAT (/1X,5HTOTAL,10F11.6)
   3  CONTINUE
C  COMPUTE MOMENTS OF INTEGRAL FUNCTION
      PRINT 33
  33  FORMAT (1H1,9X,28HMOMENTS OF INTEGRAL FUNCTION/10X,28(1H=))
      NMOMEN=2
      NMOM=NMOMEN+1
      DO 13 IMOM=1,NMOM
      IN8=IMOM-1
      DO 12 I=1,NR
      DO 12 J=1,NR
      ZMOM(IMOM,J,I)=0.
      DO 12 X=1,NA
      Z=1.
      IF (IN8.EQ.0) GO TO 12
      IF (NSTADC.EQ.2) Z=(FLOAT(X)-1.)*ZFNY+ZFNY*0.5
      Z=Z**IN8
      IF (NSTADC.EQ.1) Z=FLOAT(X)
  12  ZMOM(IMOM,J,I)=ZMOM(IMOM,J,I)+Z*PATFUN(X,J,I)
      PRINT 61, IN8
  61  FORMAT (//9X,I2,1X,6HMOMENT/9X,9(1H-))
      DO 90 I=1,NR
      DO 90 J=1,NR
  90  CE(J,I)=ZMOM(IMOM,J,I)
      CALL EIGEN(NR,1,NEIG)
  13  CONTINUE
      PRINT 167
 167  FORMAT (1H1,9X,35HMATRICES OF MEAN AGES AND VARIANCES/10X,35(1H=))
      PRINT 125
 125  FORMAT (//1X)
      IK=1
      PRINT 723, IK
 723  FORMAT (3X,2H**,1X,11HALTERNATIVE,I2,1X,2H**/3X,19(1H*))
      PRINT 67
  67  FORMAT (/9X,5HMEANS/9X,5(1H=))
      DO 19 I=1,NR
      DO 19 J=1,NR
  19  CE(J,I)=ZMOM(2,J,I)/ZMOM(1,J,I)
      CALL EIGEN(NR,2,NEIG)
```

```
      PRINT 68
 68   FORMAT (//9X,9HVARIANCES/9X,9(1H-))
 62   FORMAT (11X,10(3X,A8))
 63   FORMAT (1X,A8,2X,10F11.6)
      PRINT 62, (REG(J),J=1,NR)
      PRINT 64
      DO 21 I=1,NR
      DO 21 J=1,NR
 21   HULP(J,I)=ZMOM(3,J,I)/ZMOM(1,J,I)-CE(J,I)*CE(J,I)
      DO 22 I=1,NR
 22   PRINT 63, REG(I),(HULP(I,J),J=1,NR)
      PRINT 125
      PRINT 64
      IK=2
      PRINT 723, IK
      PRINT 67
      DO 14 I=1,NR
      DO 14 J=1,NR
 14   CC(J,I)=ZMOM(1,J,I)
      CALL INVERT (CC,NR)
      DO 17 I=1,NR
      DO 17 J=1,NR
      A1(J,I)=ZMOM(2,J,I)
 17   B(J,I)=CC(J,I)
      CALL MULTIP (NR,NR,NR)
      DO 91 I=1,NR
      DO 91 J=1,NR
      IF (INTEGR.EQ.1) AGEM(J,I)=C(J,I)
 91   CE(J,I)=C(J,I)
      CALL EIGEN (NR,1,NEIG)
      PRINT 68
       DO 93 I=1,NR
      DO 93 J=1,NR
      A1(J,I)=ZMOM(3,J,I)
 93   B(J,I)=CC(J,I)
      CALL MULTIP (NR,NR,NR)
      DO 94 I=1,NR
      DO 94 J=1,NR
      HULP(J,I)=C(J,I)
      A1(J,I)=CE(J,I)
 94   B(J,I)=CE(J,I)
      CALL MULTIP (NR,NR,NR)
      DO 95 I=1,NR
      DO 95 J=1,NR
 95   CE(J,I)=HULP(J,I)-C(J,I)
      CALL EIGEN (NR,1,0)
      IF (INTEGR.EQ.1) GO TO 579
        PRINT 777
777   FORMAT (1H1,9X,30HSPATIAL MIGRATION EXPECTANCIES/10X,30(1H*)/)
      DO 771 I=1,NR
      DO 771 J=1,NR
      CE(J,I)=0.
      DO 771 X=1,NA
771   CE(J,I)=CE(J,I)+EX(X)*L(X,I,J)
        PRINT 772
772   FORMAT (//10X,20HEXPECTATIONS OF LIFE/10X,20(1H-)/)
       CALL EIGEN (NR,1,1)
      PRINT 774
774   FORMAT (//10X,16HMIGRATION LEVELS/10X,16(1H-)/)
      DO 775 I=1,NR
```

```
      Z=0.
      DO 776 J=1,NR
776   Z=Z+CE(J,I)
      DO 775 J=1,NR
775   CE(J,I)=CE(J,I)/Z
      CALL EIGEN (NR,1,0)
579   CONTINUE
      IF (INTEGR.EQ.2) PRINT 777
      IF (INTEGR.EQ.1) PRINT 882
882   FORMAT (1H1,9X,30HSPATIAL FERTILITY EXPECTANCIES/10X,30(1H*)/)
      IF (INTEGR.EQ.2) PRINT 886
886   FORMAT (//10X,24HNET MIGRAPRODUCTION RATE  /10X,24(1H-)/)
      IF (INTEGR.EQ.1) PRINT 887
887   FORMAT (//10X,21HNET REPRODUCTION RATE/10X,21(1H-)/)
      DO 888 I=1,NR
      DO 888 J=1,NR
888   CE(J,I)=ZMOM(1,J,I)
      CALL EIGEN (NR,1,1)
      IF (INTEGR.NE.1) GO TO 41
      IF (NOPMOB.NE.1) GO TO 43
      RONRR=ROOT
      DO 42 I=1,NR
      VRNRR(I)=VECT(I)
      VLNRR(I)=VECTL(I)
      DO 42 J=1,NR
 42   ZNRR(J,I)=CE(J,I)
      GO TO 41
 43   ROPSI=ROOT
      DO 44 I=1,NR
      VRPSI(I)=VECT(I)
      VLPSI(I)=VECTL(I)
      DO 44 J=1,NR
 44   PSI(J,I)=CE(J,I)
 41   CONTINUE
      PRINT 64
      IF (INTEGR.EQ.2) PRINT 891
891   FORMAT (//10X,31HNET MIGRAPRODUCTION ALLOCATIONS/10X,31(1H-)/)
      IF (INTEGR.EQ.1) PRINT 892
892   FORMAT (//10X,28HNET REPRODUCTION ALLOCATIONS/10X,28(1H-)/)
      DO 889 I=1,NR
      Z=0.
      DO 890 J=1,NR
890   Z=Z+CE(J,I)
      DO 889 J=1,NR
889   CE(J,I)=CE(J,I)/Z
      CALL EIGEN (NR,1,0)
      IF (INTEGR.EQ.1) GO TO 795
      Z=0.
      DO 793 X=1,NA
793   Z=Z+CLLTOT(X)*RATMT(X)
      IF (NOPMOB.EQ.1) PRINT 794,Z
794   FORMAT (///10X,13HGLOBAL NMR =  ,F14.6,5X,14H(WILBER INDEX) )
      GO TO 796
795   CONTINUE
      Z=0.
      DO 798 X=1,NA
798   Z=Z+CLLTOT(X)*RATFT(X)
      IF (NOPMOB.EQ.1) PRINT 797, Z
797   FORMAT (///10X,13HGLOBAL NRR =  ,F14.6)
796   CONTINUE
580   CONTINUE
      RETURN
      END
```

ZERO

```
      SUBROUTINE ZERO (NA,NR,NZERO)
      DIMENSION VI(4),CC(4,4),HULP(4)
      COMMON /CGROW/ BR(18,4,4),PP1(4,4),POPR(18,4),POPPR(4),POPPRT
      COMMON /CRAD/ RADIX(4),RADIXT
      COMMON /CRATE/ RATD(18,4),RATM(18,4,4),RATF(18,4)
      COMMON /CRATET/ RATOT(18,4)
       COMMON /CREG/ REG(4)
      COMMON /C4/ NAGE(18)
       COMMON /CEIGEN/ CE(4,4),ROOT(4),VECT(4),VECTL(4)
     COMMON /CNRR/ ZNRR(4,4),VRNRR(4),VLNRR(4),RONRR
      DOUBLE PRECISION REG
      REAL L
      INTEGER X
 64  FORMAT (1X)
     IF (NZERO.NE.1) PRINT 20
 20  FORMAT (1H1,1X)
        NY=NAGE(2)-NAGE(1)
      ZFNY=FLOAT(NY)
      PRINT 30, NZERO
 30  FORMAT (1H0,10X,22HZERO POPULATION GROWTH,5X,11HALTERNATIVE
    1,I2/11X,22(1H*),5X,13(1H*)/11X,22(1H*),5X,13(1H*)/)
     IF (NZERO.NE.1) GO TO 41
      PRINT 31
 31  FORMAT (10X,42HTRANSPOSE OF NET REPRODUCTION MATRIX R(0)' /
    110X,44(1H-))
     DO 32 I=1,NR
     DO 32 J=1,NR
     CC(J,I)=ZNRR(I,J)
 32  CE(J,I)=ZNRR(I,J)
     CALL EIGEN (NR,5,0)
     PRINT 64
     PRINT 64
     PRINT 34
 34  FORMAT (10X,17HINVERSE OF R(0)' /10X,17(1H-))
     CALL INVERT (CC,NR)
     DO 33 I=1,NR
     DO 33 J=1,NR
 33  CE(J,I)=CC(J,I)
     CALL EIGEN (NR,5,0)
 41  CONTINUE
     PRINT 64
     PRINT 64
     PRINT 35
 35  FORMAT (10X,39HMATRIX OF FERTILITY ADJUSTMENT FACTORS /
    110X,39(1H-))
       DO 38 I=1,NR
       VI(I)=0.
     IF (NZERO.EQ.2) VI(I)=1./RONRR
     IF (NZERO.NE.1) GO TO 38
       DO 39 J=1,NR
 39  VI(I)=VI(I)+CC(I,J)
 38  CONTINUE
     DO 36 I=1,NR
     DO 36 J=1,NR
     CE(J,I)=0.
 36  IF (I.EQ.J) CE(J,I)=VI(I)
     CALL EIGEN (NR,5,0)
     DO 40 I=1,NR
     DO 40 X=1,NA
 40  RATF(X,I)=RATF(X,I)*VI(I)
     PRINT 164
164  FORMAT (1H1,1X)
     RETURN
     END
```

PROBSCH

```
      SUBROUTINE PROBSCH (NA,NY,NR)
      DIMENSION RM(4,4),CC(4,4)
      COMMON /CRATE/ RATD(18,4),RATM(18,4,4),RATF(18,4)
      COMMON /CPQ/ Q(18,4),P(18,4,4),PMIGT(18,4)
      COMMON /CMUL/ A1(4,4),B(4,4),C(4,4)
      INTEGER X
C   COMPUTE RM(X)
      NAA=NA-1
      ZZZ=FLOAT(NY)/2.
      DO 100 X=1,NAA
      IF (NR.EQ.1) GO TO 50
      DO 5 I=1,NR
      Z=RATD(X,I)
      DO 4 J=1,NR
      IF (I.EQ.J) GO TO 4
      Z=Z+RATM(X,J,I)
    4 CONTINUE
      RM(I,I)=(-1.)*Z
      DO 6 J=1,NR
      IF (J.EQ.I) GO TO 6
      RM(J,I)=RATM(X,J,I)
    6 CONTINUE
    5 CONTINUE
      DO 7 I=1,NR
      DO 7 J=1,NR
      IF (I.EQ.J) CC(J,I)=1.-ZZZ*RM(J,I)
    7 IF (I.NE.J) CC(J,I)=-ZZZ*RM(J,I)
      CALL INVERT (CC,NR)
      DO 8 I=1,NR
      DO 8 J=1,NR
       A1(J,I)=CC(J,I)
      IF (J.EQ.I) B(J,I)=1.+ZZZ*RM(J,I)
    8 IF (J.NE.I) B(J,I)=ZZZ*RM(J,I)
      CALL MULTIP (NR,NR,NR)
      DO 9 I=1,NR
      DO 9 J=1,NR
    9 P(X,J,I)=C(J,I)
      GO TO 100
   50 DO 51 X=1,NAA
   51 P(X,1,1)=(1.-ZZZ*RATD(X,1))/(1.+ZZZ*RATD(X,1))
  100 CONTINUE
      DO 10 I=1,NR
      DO 10 J=1,NR
   10 P(NA,J,I)=0.
      DO 12 X=1,NA
      DO 12 I=1,NR
      PMIGT(X,I)=0.
      DO 11 J=1,NR
      IF (I.EQ.J) GO TO 11
      PMIGT(X,I)=PMIGT(X,I)+P(X,J,I)
   11 CONTINUE
      Q(X,I)=1.-P(X,I,I)-PMIGT(X,I)
   12 CONTINUE
       RETURN
      END
```

RVALUE

```
      SUBROUTINE RVALUE (NA,NY,ZFNY,NR,NSTADC,R,ZVT)
      DIMENSION V(18,4,4)
      DIMENSION CC(4,4)
      COMMON /CSU/ SU(18,4,4),SSU(18,4)
      COMMON /CL/ L(18,4,4),CLLT(18,4),CLLTOT(18)
      COMMON /CMUL/ A1(4,4),B(4,4),C(4,4)
      COMMON /CRATE/ RATD(18,4),RATM(18,4,4),RATF(18,4)
      COMMON /CREG/ REG(4)
      COMMON /C1/ POP(18,4),BIRTH(18,4),DEATH(18,4),OMIG(18,4,4)
      COMMON /C4/ NAGE(18)
      COMMON /CEIGEN/ CE(4,4),ROOT(4),VECT(4),VECTL(4)
      COMMON /CPSI/ PSI(4,4),VRPSI(4),VLPSI(4),ROPSI
      COMMON /CEX/ EX(18)
      COMMON /CPQ/ Q(18,4),P(18,4,4),PMIGT(18,4)
      DOUBLE PRECISION REG
      INTEGER X,X1,XX
      REAL L
      NAA=NA-1
 64   FORMAT (1X)
 78   FORMAT (1H1,1X)
      DO 251 I251=1,2
      IF (I251.EQ.1) PRINT 250
250   FORMAT (1H1,1X,32HTHE SPATIAL REPRODUCTIVE VALUE :,1X,
     133HRESULTS FOR PEOPLE AT EXACT AGE X /2X,30(1H*),3X,
     133(1H*)/2X,30(1H*),3X,33(1H*))
      IF (I251.EQ.2) PRINT 258
258   FORMAT (1H1,34X,33HRESULTS FOR PEOPLE IN AGE GROUP X /
     135X,33(1H*)/35X,33(1H*))
      PRINT 51
 51   FORMAT (//10X,41HDISCOUNTED NUMBER OF OFFSPRING PER PERSON,
     1/10X,41(1H*)/)
      IF (I251.EQ.2) GO TO 257
      CALL PROBSCH (NA,NY,NR)
      Z=-ZFNY*0.5*R
      ZZ=EXP(Z)
      ZZZ=ZFNY*0.5*ZZ
      DO 252 I=1,NR
      DO 252 J=1,NR
252   V(NA,J,I)=RATF(NA,J)*ZZZ
      DO 255 X=1,NAA
      XX=NA-X
      X1=XX+1
      DO 254 I=1,NR
      DO 254 J=1,NR
      IF (I.EQ.J) A1(J,I)=ZZZ*RATF(XX,J)+ZZ*ZZ*V(X1,J,I)
      IF (I.NE.J) A1(J,I)=ZZ*ZZ* V(X1,J,I)
254   B(J,I)=P(XX,J,I)
      CALL MULTIP (NR,NR,NR)
      DO 255 I=1,NR
      DO 255 J=1,NR
      IF (I.EQ.J) V(XX,J,I)=ZZZ*RATF(XX,J)+C(J,I)
      IF (I.NE.J) V(XX,J,I)=C(J,I)
255   CONTINUE
      DO 268 I=1,NR
      DO 268 J=1,NR
268   V(1,J,I)=PSI(J,I)
      GO TO 730
257   CONTINUE
      DO 53 I=1,NR
      DO 53 J=1,NR
```

```
 53   V(NA,J,I)=RATF(NA,J)*ZFNY*0.5
      Z=-ZFNY*R
      ZZ=EXP(Z)
      DO 52 X=1,NAA
      XX=NA-X
      X1=XX+1
      DO 54 I=1,NR
      DO 54 J=1,NR
      IF (I.EQ.J) A1(J,I)=ZFNY*0.5*RATF(X1,J)+V(X1,J,I)
      IF (I.NE.J) A1(J,I)=V(X1,J,I)
 54   B(J,I)=SU(XX,I,J)*ZZ
      CALL MULTIP (NR,NR,NR)
      DO 55 I=1,NR
      DO 55 J=1,NR
      IF (I.EQ.J) V(XX,J,I)=ZFNY*0.5*RATF(XX,J)+C(J,I)
  55   IF (I.NE.J) V(XX,J,I)=C(J,I)
 52   CONTINUE
730   CONTINUE
      DO 58 I=1,NR
      PRINT 56, REG(I)
 56   FORMAT (//10X,19HREGION OF RESIDENCE,2X,A8/10X,29(1H-)/)
      PRINT 57, (REG(J),J=1,NR)
 57   FORMAT (10X,28HREGION OF BIRTH OF OFFSPRING/
     110X,5HTOTAL,2X,10(2X,A8))
      PRINT 64
      DO 58 X=1,NAA
      Z=0.
      DO 60 J=1,NR
 60   Z=Z+V(X,J,I)
 58   PRINT 59, NAGE(X),Z,(V(X,J,I),J=1,NR)
 59   FORMAT (1X,I3,1X,F10.6,2X,10F10.6)
      PRINT 61
 61   FORMAT (1H1,10X,37HSPATIAL REPRODUCTIVE VALUE PER PERSON /
     111X,37(1H*)/)
      PRINT 65, (REG(J),J=1,NR)
 65   FORMAT (15X,10(2X,A8))
      PRINT 64
      DO 62 X=1,NAA
      DO 63 J=1,NR
      A1(1,J)=VLPSI(J)
      DO 63 I=1,NR
 63   B(J,I)=V(X,J,I)
      CALL MULTIP (1,NR,NR)
      IF (I251.NE.1) GO TO 264
      IF (X.NE.1) GO TO 264
      DO 263 J=1,NR
263   C(1,J)=VLPSI(J)
264   CONTINUE
      PRINT 66, NAGE(X),(C(1,J),J=1,NR)
 66   FORMAT (1X,I3,1X,10X,11F10.6)
 62   CONTINUE
251   CONTINUE
      PRINT 71
 71   FORMAT (1H1,36HTOTAL DISCOUNTED NUMBER OF OFFSPRING ,
     123H OF OBSERVED POPULATION ,12H IN 100,000.  /2X,70(1H*)/)
      DO 72 I=1,NR
      DO 72 J=1,NR
      CE(J,I)=0.
      DO 72 X=1,NAA
 72   CE(J,I)=CE(J,I)+V(X,J,I)*POP(X,I)*0.00001
```

```
      CALL EIGEN (NR,1,0)
      PRINT   265
265   FORMAT (//1X,42HREPRODUCTIVE VALUE OF THE TOTAL POPULATION
     112H IN 100,000./1X,54(1H*)/)
      DO 260 J=1,NR
      A1(1,J)=VLPSI(J)
      DO 260 I=1,NR
260   B(J,I)=CE(J,I)
      CALL MULTIP (1,NR,NR)
      DO 261 I=1,NR
261   PRINT 262, REG(I),C(1,I)
262   FORMAT (1X,A8,2X,F11.6)
      ZVT=0.
      DO 92 I=1,NR
 92   ZVT=ZVT+C(1,I)
      PRINT 93, ZVT
 93   FORMAT (/4X,5HTOTAL,2X,F11.6)
      ZVT=ZVT*100000.
      RETURN
      END
```

RINTR

```
      SUBROUTINE RINTR (NA,NY,ZFNY,NR,NSTADC,ZLAMDA,R,ZVT)
      DIMENSION DISV(18,4),HULP(5,8),HZ(18),HM(4,4)
      DIMENSION CC(4,4)
      DIMENSION RATO(18,4)
      COMMON /CL/ L(18,4,4),CLLT(18,4),CLLTOT(18)
      COMMON /CMUL/ A1(4,4),B(4,4),C(4,4)
      COMMON /CRATE/ RATD(18,4),RATM(18,4,4),RATF(18,4)
      COMMON /CREG/ REG(4)
      COMMON /C4/ NAGE(18)
      COMMON /CEIGEN/ CE(4,4),ROOT(4),VECT(4),VECTL(4)
      COMMON /CPSI/ PSI(4,4),VRPSI(4),VLPSI(4),ROPSI
      COMMON /CAGEM/ AGEM(4,4)
      COMMON /CEX/ EX(18)
      COMMON /CQQ/ QQ(4),POPST(18,4),YY(4)
      COMMON /CPQ/ Q(18,4),P(18,4,4),PMIGT(18,4)
      DOUBLE PRECISION REG
      INTEGER X,X1,XX
      REAL L
      NAA=NA-1
C  COMPUTE NORMALIZING FACTOR
      DO 85 I=1,NR
      B(I,1)=VRPSI(I)
      DO 85 J=1,NR
   85 A1(J,I)=AGEM(J,I)
      CALL MULTIP (NR,NR,1)
      ZNORM=0.
      DO 86 I=1,NR
   86 ZNORM=ZNORM+C(I,1)*VLPSI(I)
      VKNORM=ZVT/ZNORM
C STABLE EQUIVALENTS OF BIRTHS
       DO 94 I=1,NR
      QQ(1)=VRPSI(I)*VKNORM
   94 CONTINUE
   64 FORMAT (1X)
      PRINT 96
   96 FORMAT (1H1,10X,37HSTABLE EQUIVALENT OF TOTAL POPULATION/
     111X,37(1H*)/)
      DO 121 X=1,NA
      DO 121 J=1,NR
      HULP(J,1)=0.
      DO 120 I=1,NR
  120 HULP(J,1)=HULP(J,1)+L(X,I,J)*QQ(I)
  121 POPST(X,J)=EX(X)*HULP(J,1)
      YT=0.
      DO 122 I=1,NR
      YY(I)=0.
      DO 134 X=1,NA
  134 YY(I)=YY(I)+POPST(X,I)
  122 YT=YT+YY(I)
      PRINT 133, (REG(J),J=1,NR)
  133 FORMAT (11X,5HTOTAL,10(2X,A8))
      PRINT 64
      DO 123 X=1,NA
      HZ(X)=0.
      DO 132 J=1,NR
  132 HZ(X)=HZ(X)+POPST(X,J)
  123 PRINT 124, NAGE(X),HZ(X),(POPST(X,J),J=1,NR)
  124 FORMAT (1X,I3,2X,11F10.0)
      PRINT 125, YT, (YY(J),J=1,NR)
  125 FORMAT (/1X,5HTOTAL,11F10.0)
```

```
      PRINT 31
 31   FORMAT (//10X,23HPERCENTAGE DISTRIBUTION /10X,23(1H-))
      PRINT 133, (REG(J),J=1,NR)
      PRINT 64
      DO 126 X=1,NA
      HZ(X)=HZ(X)/YT
      DO 127 J=1,NR
127   DISV(X,J)=POPST(X,J)/YY(J)
126   PRINT 135, NAGE(X),HZ(X),(DISV(X,J),J=1,NR)
135   FORMAT (1X,I3,2X,11F10.6)
      DO 43 I=1,NR
      HULP(I,1)=QQ(I)
 43   CONTINUE
      PRINT 131
131   FORMAT (1H1,10X,38HSTABLE EQUIVALENTS AND INTRINSIC RATES/
     111X,38(1H*)/)
      PRINT 49
 49   FORMAT (1H0,18X,10X,6HBIRTHS,21X,6HDEATHS,18X,12HOUTMIGRATION,
     116X,11HINMIGRATION/)
      PRINT 115
 115 FORMAT (13X,4(3X,6X,6HNUMBER,8X,4HRATE)/)
      DO 25 X=1,NAA
      DO 21 I=1,NR
      DO 21 J=1,NR
      IF (I.EQ.J) CC(J,I)=P(X,J,I)+1.
 21   IF (I.NE.J) CC(J,I)=P(X,J,I)
      CALL INVERT (CC,NR)
      Z2=0.5*ZFNY
      Z1=Z2*R
      Z=-ZFNY*R
      Z8=EXP(Z)
      Z9=EXP(Z1)/Z2
      DO 22 I=1,NR
      DO 22 J=1,NR
       B(J,I)=CC(J,I)
      IF (I.EQ.J) A1(J,I)=1.-P(X,J,I)*Z8
 22   IF (I.NE.J) A1(J,I)=-P(X,J,I)*Z8
      CALL MULTIP (NR,NR,NR)
      DO 23 I=1,NR
      DO 23 J=1,NR
      IF (I.EQ.J) HM(J,I)=Z9*C(J,I)-R
 23   IF (I.NE.J) HM(J,I)=Z9*C(J,I)
      DO 25 I=1,NR
      Z7=0.
      DO 24 J=1,NR
      IF (I.EQ.J) GO TO 24
      Z7=Z7+HM(J,I)
 24   CONTINUE
      RATD(X,I)=HM(I,I)+Z7
      DO 25 J=1,NR
      RATM(X,J,I)=-HM(J,I)
 25   CONTINUE
      DO 27 I=1,NR
      DO 26 J=1,NR
 26    RATM(NA,J,I)=0.
 27   RATD(NA,I)=EXP(Z1)*RATD(NA,I)-R
      DO 111 I=1,NR
      DO 118 J=2,8
118   HULP(I,J)=0.
      DO 111 X=1,NA
```

```
      RATO(X,I)=0.
      DO 117 J=1,NR
      IF (I.EQ.J) GO TO 117
      RATO(X,I)=RATO(X,I)+RATM(X,J,I)
117   CONTINUE
      HULP(I,3)=HULP(I,3)+POPST(X,I)*RATD(X,I)
      HULP(I,5)=HULP(I,5)+POPST(X,I)*RATO(X,I)
      DO 119 J=1,NR
      IF (I.EQ.J) GO TO 119
      HULP(I,7)=HULP(I,7)+POPST(X,J)*RATM(X,I,J)
119   CONTINUE
111   CONTINUE
      NR1=NR+1
      DO 20 J=1,8
 20   HULP(NR1,J)=0.
      DO 113 I=1,NR
      DO 113 J=1,4
      JK=J*2-1
      JJ=JK+1
      HULP(I,JJ)=HULP(I,JK)/YY(I)
      HULP(NR1,JK)=HULP(NR1,JK)+HULP(I,JK)
113   CONTINUE
      DO 177 I=1,NR
177   PRINT 114, REG(I),(HULP(I,J),J=1,8)
114   FORMAT (5X,A8,4(3X,F12.0,F12.6))
      DO 130 J=1,4
      JJ=J*2
130   HULP(NR1,JJ)=HULP(NR1,JJ-1)/YT
      PRINT 116, (HULP(NR1,J),J=1,8)
116   FORMAT (/8X,5HTOTAL,4(3X,F12.0,F12.6))
      PRINT 178, R
178   FORMAT (//10X,18HSTABLE GROWTH RATE,4X,F10.6)
      PRINT 179, ZNORM
179   FORMAT (/10X,18HNORMALIZING FACTOR,2X,F12.4)
      RETURN
      END
```

MOMENT

```
      SUBROUTINE MOMENT (NA,NY,ZFNY,NR,R)
      DIMENSION RO(4,4),R1(4,4),HU(4,4),QQZP(4),EO(4,4),YYZP(4)
      DIMENSION CC(4,4)
      COMMON /CL/ L(18,4,4),CLLT(18,4),CLLTOT(18)
      COMMON /CMUL/ A1(4,4),B(4,4),C(4,4)
      COMMON /CRATE/ RATD(18,4),RATM(18,4,4),RATF(18,4)
      COMMON /CREG/ REG(4)
      COMMON /C4/ NAGE(18)
      COMMON /CPSI/ PSI(4,4),VRPSI(4),VLPSI(4),ROOTPSI
      COMMON /CEIGEN/ CE(4,4),ROOT(4),VECT(4),VECTL(4)
      COMMON /CQQ/ QQ(4),POPST(18,4),YY(4)
      DOUBLE PRECISION REG
      INTEGER X
      REAL L
      PRINT 50
   50 FORMAT (1H1,10X,42HSPATIAL MOMENTUM OF ZERO POPULATION GROWTH/
     111X,42(1H*)/11X,42(1H*)/)
      DO 3 I=1,NR
      DO 3 J=1,NR
      EO(J,I)=0.
      RO(J,I)=0.
      R1(J,I)=0.
      DO 3 X=1,NA
      EO(J,I)=EO(J,I)+L(X,I,J)
      IZ=NAGE(X)
      Z1=FLOAT(IZ)+ZFNY*0.5
      Z=RATF(X,J)*L(X,I,J)
      RO(J,I)=RO(J,I)+Z
    3 R1(J,I)=R1(J,I)+Z1*Z
      DO 4 I=1,NR
      DO 4 J=1,NR
      HU(J,I)=RO(J,I)-PSI(J,I)
    4 CC(J,I)=R1(J,I)
      CALL INVERT (CC,NR)
       DO 5 I=1,NR
      DO 5 J=1,NR
      A1(J,I)=CC(J,I)
    5 B(J,I)=HU(J,I)
      CALL MULTIP (NR,NR,NR)
      DO 6 I=1,NR
      B(I,1)=QQ(I)
      DO 6 J=1,NR
    6 A1(I,J)=C(I,J)/R
      CALL MULTIP (NR,NR,1)
      DO 7 I=1,NR
    7 QQZP(I)=C(I,1)
      PRINT 11
   11 FORMAT (//10X,45HMATRIX CONVERTING STABLE TO STATIONARY BIRTHS/
     110X,45(1H-))
      DO 10 I=1,NR
      DO 10 J=1,NR
   10 CE(J,I)=A1(J,I)
      CALL EIGEN (NR,1,0)
      DO 8 I=1,NR
      B(I,1)=QQZP(I)
      DO 8 J=1,NR
    8 A1(J,I)=EO(J,I)
      CALL MULTIP (NR,NR,1)
      DO 9 I=1,NR
    9 YYZP(I)=C(I,1)
```

```
      PRINT 12
12    FORMAT (//10X,32HSTABLE AND STATIONARY EQUIVALENT/
     110X,32(1H-))
      PRINT 13
13    FORMAT (/21X,6HBIRTHS,16X,10HPOPULATION,9X,10HPOPULATION)
      PRINT 14
14    FORMAT (/13X,2(5X,6HSTABLE,1X,10HSTATIONARY,2X),3X,8HMOMENTUM)
      PRINT 64
64    FORMAT (1X)
      DO 15 I=1,NR
      IF (YY(I).NE.0.) Z=YYZP(I)/YY(I)
15    PRINT 16, REG(I),QQ(I),QQZP(I),YY(I),YYZP(I),Z
16    FORMAT (3X,A8,2X,2(2F11.0,2X),F11.4)
      DO 17 I=1,NR
17    HU(I,1)=0.
      DO 18 I=1,NR
      HU(1,1)=HU(1,1)+QQ(I)
      HU(2,1)=HU(2,1)+QQZP(I)
      HU(3,1)=HU(3,1)+YY(I)
18    HU(4,1)=HU(4,1)+YYZP(I)
      Z=HU(4,1)/HU(3,1)
      PRINT 19, (HU(I,1),I=1,4),Z
19    FORMAT (/6X,5HTOTAL,2X,2(2F11.0,2X),F11.4)
      RETURN
      END
```

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