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MATHEMATICAL MODELS OF CATASTROPHES. CONTROL OF CATASTROHIC PROCESSES.

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Contents

Summary

The basic notions of Singularity Theory of differentiable mappings and the Bifurcation theory of dynamical systems are described.

They form the foundation of Mathematical Catastrophe Theory, which is essentially a qualitative analysis of complicated systems depending on parameters, such as life supporting systems in real life.

We demonstrate some simple examples of systems wherein the choice of the control parameter makes it possible to avoid abrupt (catastrophic) behavior.

The aim of the chapter is to emphasize that the general methodology of these mathematical theories are important in investigations of the specific complicated systems, even at the stage of creating adequate models.

1. Introduction

In mathematics sudden and abrupt changes in the response of a system to a smooth change in external conditions are termed as *catastrophes*.

The mathematical catastrophe models exhibit certain common traits of the most diverse phenomena of jump change in a system behavior. Mostly catastrophe theory is a collection of the applications and general ideas of Singularity Theory and Bifurcation Theory of dynamical systems.

The main object studied by these theories is a system which continuously depends on its parameters. Any mathematical model of a real system (for instance, physical ecological or economical) determines a set of numerical parameters, whose values reflects important features of the system.

In a relatively simple case (which might be called a *stationary system*) *the state of a system* is described by a point from a certain subset of the space of all possible values of parameters. This subset can have a complicated structure and singularities. Here we may well rely on the singularity theory of differentiable mapping.

In other cases, for *evolution processes*, time is one of the parameters and a *system* is modeled by a dynamical system, governed by certain system of differential equations or by relations of more complicated nature.

Even the simplest general mathematical conclusions on the properties of such systems can often help in investigation of specified complicated modeling problems.

The basic idea of the catastrophe theory according to E.C.Zeeman, is the following:

Assume that the parameters defining the states of the system are separated into two groups: *internal* and *external.* It is assumed that there exists dependence among the parameters. However, the values of the internal parameters are not uniquely determined by the values of the external ones. Geometrically the states of the system are described by the points in the product of the spaces of internal and external parameters. The meaning of the dependence is that this point (the current state of the system) always lies in some subset of the product space. In the simplest case one may assume that this subset is a smooth submanifold in general position in the product space and its dimension is equal to the dimension of the space of external parameters. The projection of this submainfold onto the space of external parameters is not generically one-to-one. In other words, even in this simplest model the internal parameters do not depend smoothly on the external ones: Under small changes of external parameters our system may jump from one visible state (i.e. from a point in the internal parameter space) to another one.

Singularity Theory yields information on the critical points and critical values of such mappings. Analytical, geometrical and topological methods often ensure the appearance of certain types of critical sets without ambiguity.

Since smooth mappings are found everywhere, their singularities must be everywhere also, and since Singularity Theory gives significant information on the singularities of generic mappings, one can try to use this information to study a lot of diverse phenomena and processes in all areas of science.

The mathematical description of the world depends on a delicate interplay between continuous and discontinuous (discrete) phenomena. The latters are perceived first. “Functions, just like living beings are characterized by their singularities”, as P. Montel proclaimed.

The modern singularity theory began in the 1950s by the works of H. Whitney. Also in the early 1930s A.A Andropov started the theory of the bifurcations of dynamical systems. However similar ideas and objects go up to classics: Hamilton, Monge, Cayley, Poincaré. Now there exist thousands of publications on the development of their results and on various applications. The singularity theory is now one of the central areas of mathematics, where the most abstract parts (differential and algebraic geometry and topology, group theory, the theory of complex spaces) come together with the most applied ones (stability of motion of dynamical systems, bifurcation of equilibrium states, optics, optimal control).

Its methods were applied to various branches of knowledge, for example, to heat beats modeling, to geometrical and physical optics, to embryology, linguistics, economics, hydrodynamics, geology, computer vision, elasticity theory, stability of ships, etc.

Certain features of catastrophes in natural, technological, social and other processes might be understood using methods of Singularity Theory and Bifurcation Theory.

In Section 2 the general methods and simple examples of catastrophe theory are outlined.

That is, in Subsection 2.1 we describe the traditional example of Zeeman’s “catastrophe mashine”, which exhibits loss of stability and jump–like dynamics in mechanical systems with rigid and elastic elements. Also the bifurcation of steady state positions of loaded elastic beam is analyzed. These two models provide the simplest examples of the real catastrophe which may occur in the building and construction technology.

In Subsection 2.2 the simplest resource exploitation model (based on natural law of resource growth) is considered. We show that the catastrophic vanishing of the resource due to the intensification of extraction can be avoided by the manipulation of a feedback control parameter. In this simplest model of the “control of the ecological catastrophe” the average production might be maintained at the optimal level.

In the following sections recent developments in these theories and domains of their application are described.

Singularities of functions described in Subsection 3.1 provide mathematical models of typical catastrophes in systems with several external parameters. The knowledge of their bifurcation diagrams or discriminant sets (from Subsection 3.2) is necessary to control behavior of the system smoothly.

The propagation of various catastrophic disturbances (e.g. shock waves, emanations, epidemic or a flame) in certain media has many common features modeled by the theory of wavefronts and caustics (outlined in Subsection 3.3). This covers a vaste area of applications of singularity theory, including Y. Zeldovich's model of the catastrophic formation of the Universe.

An interesting new approach to the problem of the choice of a good decision is described in Subsection 3.4. Political and social sciences provide enough examples when the wrong decision leads to real catastrophe.

Different models of shock fronts are described in Subsection 3.5.

The implementation of the control parameters into a system does not lead automatically to the perishing of the catastrophes.

To avoid catastrophe in the behavior of a system one has to know the typical singularities of the control systems themselves. This is the subject of Section 4. Here we again meet the simplest models based on the singularities of families of functions (conflicts sets) and bifurcations of singular points of vector and direction fields (Subsection 4.2).

We have to emphasize that to suggest a detailed and adequate mathematical model for real life supporting systems and to prove their consistency is the subject of the specific sciences going beyond the aims of the present article.

2. Basic Notions and Examples

2.1. Catastrophe of the "Pleat"

Singularity theory provides the following general methodology for studying the qualitative properties of systems depending on parameters.

Assume that the systems under the consideration form a space S endowed with a metric or a topology. This space, generally speaking, may be very complicated (may have infinite dimension, may not be a manifold, etc).

Introduce a partition (called *equivalence relation*) of this space S into a certain collection of subsets. Two systems from one subset will be called *equivalent.*

Often such a natural partition corresponds to a certain group action on the ambient space S. In this case the equivalence classes are the orbits (or collection of orbits) of the systems under this action.

A certain property (which is common to all objects from certain equivalence classes) of a system is called *generic* if it holds for any system from an open and dense subset of the spaces S. In other words, by small perturbation of an arbitrary system one gets a system with this property. If the system is generic then all nearby systems are generic as well. One says then that this property is *structurally stable*, and a system with its generic (or *typical*).

The non-generic systems form a thin subset *D*.

Let us consider a family of systems depending on *m* auxiliary parameters. It may happen that it fails to be generic for certain values of these additional parameters.

Moreover this effect is inevitable by small modifications of a family within the space of all *m*–parameter families.

For example, let *D* be a line in a plane. Almost all points of the plane do not belong to *D*, but generically one can not avoid the intersections of *D* with one–parameter family of points (a curve) in the plane by means of arbitrary small modifications of the curve.

Thus for the generic families of systems for certain specific parameter value a particular non- generic system can occur.

The subset of the auxiliary parameters corresponding to degenerate (non-generic) systems is called *discriminant* or *bifurcation diagram* of the family.

If one can avoid (using small perturbations of arbitrary *m*-parameter family) the appearance of the objects from a certain much thinner subset $D_m \subset D,$ then D_m is said to have *codimension* at least *m*+1 in the ambient space of systems.

We can determine the hierarchy of successively degenerate systems $... \subset D_m \subset D_{m-1} \subset ... \subset D = D_1 \subset S$ formed by subsets of growing codimension.

Such a hierarchy provides a sequence of questions to answer while analyzing the space *S* with respect to a given property.

The first question is the description (classification) of the equivalence classes that are generic. In the next step the equivalence classes of the systems with the degenerations of codimension1. (i.e. the singularities that occur in generic one-parameter families) should be studied, then those of codimension 2, and so on.

This approach was outlined in the pioneering work of H. Whitney (1955), where smooth mappings of the plane into another plane were considered.

Whitney observed that generically only **two** kinds of singularities are encountered. All others disintegrate under small movements of the mapping.

The mappings of the smooth surface to the plane can be easily visualized. The visible contours of bodies are the projection of their bounding surfaces onto the retina of the eye. By examining the objects surrounding us we can study the singularities of visible contours. Other singularities become visible when the surface is transparent (for example, medical X-rays photos contain plenty of them).

The first kind of singularity (called a “fold”) is the singularity arising at equatorial points when a sphere is projected onto a plane, parallel to the equator.

In suitable coordinates x_1, x_2 on the surface and suitable coordinates y_1, y_2 on the target plane this mapping is locally given by $y_1 = x_1^2$ and $y_2 = x_2$.

Another singularity was named the *pleat* (or *cusp*). It arises in particular when a surface Γ in three –dimensional space $\mathbf{R}^3 = \{(a,b,x)\}$ defined by the equation

$$\Gamma = \{(a,b,x) \mid x^3 + ax - b = 0\} \qquad (1)$$

is projected onto the horizontal plane (a,b) along the vertical direction of x-axis (Figure 1).

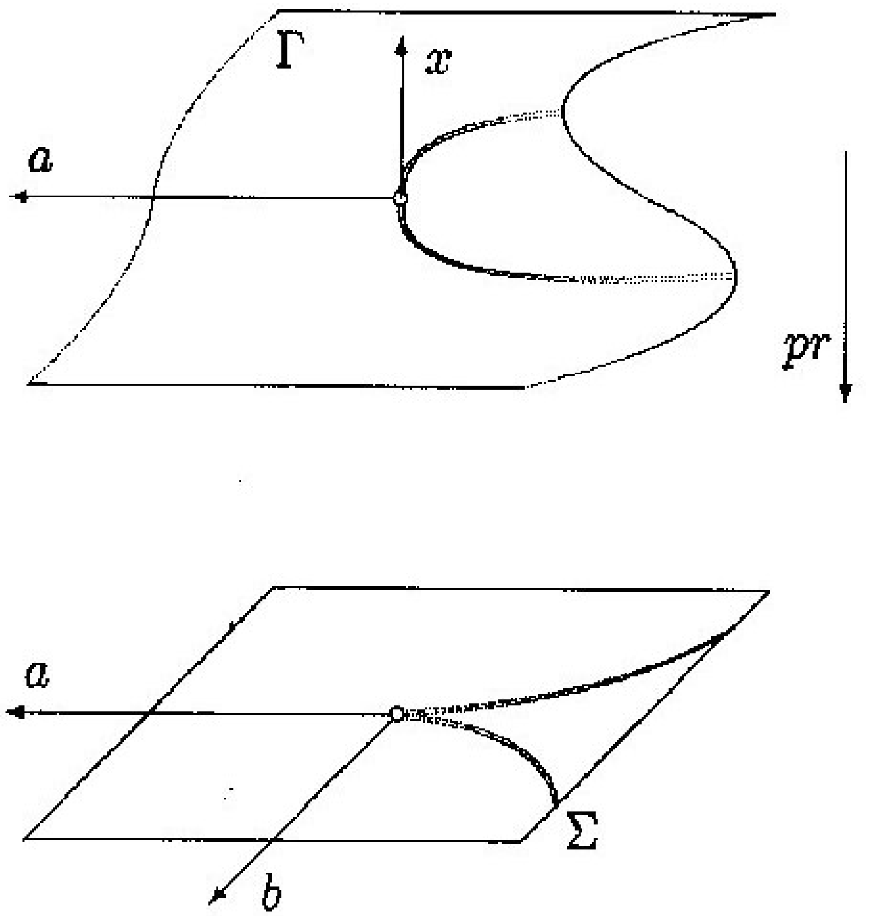

Figure 1: Pleat

Parameters (a,x) can be taken as coordinates on Γ and parameters (a,b) are the coordinates on the target plane.

The curve $\sum$ on the horizontal projection plane defined by the equation $4a^3 + 27b^2 = 0$ is the set of the images of the critical points of the projection of the surface Γ.

This curve is called *semicubical parabola* $\sum$ (with a cusp at the origin).

At these critical points (which form a smooth curve $3x^2 + a = 0, x^3 + ax - b = 0$ on the surface) the vertical direction is tangent to surface Γ.

The curve $\sum$ divides the horizontal plane into two parts: a smaller and a larger one. The points of the smaller part have three inverse images (three points of the surface project onto them), a point of the larger part has only one inverse image, and a generic point on the curve itself has two.

On approaching the curve from the smaller part (outside the origin), two of the inverse images (out of the three) merge together and disappear (here the singularity is a fold), and on approaching the origin (the cusp point) all three inverse images coalesce.

Whitney proved that the pleat is *stable*, i.e. every nearby mapping has a similar singularity at an appropriate nearby point. Here “similar”means that in suitable local coordinate systems in a neighborhood of the mentioned point and of its image the deformed mapping is described by the same formulas as those describing the original projection in the neighborhood of the origin (such two mappings are called *right-left equivalent).*

Whitney also proved that *every singularity of a smooth mapping of a surface onto a plane, after an appropriate arbitrary small perturbation, splits into folds and pleats.*

This initial result from the singularity theory has various applications.

Let us consider for example a system with two external parameters *a* and *b* and with one internal x. Assume that the relation between them is determined by the equation $f(x,a,b) = 0$.

The Whitney results imply that for generic relation f representing a smooth surface in (a,b,x) space only folds and pleats of its projections to external parameter plane may happen. When (a,b) parameter point passes through the regular part of the curve $\sum$ (say, from the smaller part of the complement to the $\sum$), the internal parameter x (assuming its value merges with another one) has to jump from the fold branch of the critical set to the remaining inverse image of (a,b) or disappear .

This simplest (structurally stable and inevitable) catastrophe model occurs in various elastic systems.

The following installation (called Zeeman’s “catastrophe machine”) demonstrates this phenomenon.

The system consists of a wheel, rotating about a fixed axis, and two elastic strings attached at a point on the rim of the wheel: one string has its other end fixed in the plane, the second string has its other end attached to the tip of a pencil so that, as the pencil moves, it traces a curve on a sheet of paper in the same plane.

As one changes the position of the pencil (defined by two coordinates-two external parameters), one can observe, generally speaking, that the wheel rotates, smoothly responding to the changes of the parameters; but in some cases the wheel changes its position with a jump. Such jumps occur for exceptional (bifurcation) positions of the pencil: on the sheet of paper they form a catastrophe curve with four cusps. If the pencil crosses this curve while moving, a catastrophe (jump of the wheel) may or may not occur, depending upon the pre-history of the motion.

The state space of this machine is 3-dimensional (an auxiliary internal parameter defines the rotation angle of the wheel). The potential energy of the system is a function of all three parameters. For fixed values of the external parameters the system minimizes (locally) its potential energy. The surface of equilibria is formed by the critical points of the potential energy.

As the values of the external parameters approach the catastrophe curve, the critical points of the potential energy, considered as a function on a circle, undergo a metamorphosis. Upon intersecting the catastrophe curve at a generic point, two critical points of the potential energy merge - a local maximum and a local minimum. The system, in a stable equilibrium state at a point where the potential energy has a local minimum, remains in this state up to the moment of bifurcation. At that moment the critical point becomes unstable, and the system jumps to another (stable) equilibrium state. Thus, whether or not there will be a jump at intersecting the catastrophe curve depends on which local minimum of the potential energy of the system lies before the intersection.

Similar bifurcations arise in the case of relaxation oscillations.

The ideas more or less equivalent to the catastrophe theory were used in thermodynamics from the time of J.C. Maxwell and J. W. Gibbs.

The metamorphosis of the isotherm of the Van der Waal's equation of state of a real gas is a typical example of an application of the pleat singularity. An analysis of the asymptotics in the neighborhood of the critical point quickly leads to the understanding that this geometry is independent of the exact form of the equation of the state: temperature, volume and pressure play the same roles as parameters a, b, x in the standard model.

The geometric investigations of singularities of generic multiparameter families might be found in the works on chemistry and mineralogy (the last in connection with investigations of process of crystallization of magma).

In N.N. Semenov's theory of thermal explosions (1929) the transitions that have jumps from one reaction mode to another ("ignition" and "extinction") arise under a smooth change of a parameter (for example the phase variable, describing the concentration of one of the chemical substances).

In elasticity theory the smooth surface of equilibria of a loaded elastic bar provides another example of pleat singularity.

Suppose that the bar is bent in the form of arc of a bridge and its ends are fixed.

The dependence of the maximal load that the bar can carry on the magnitude of the shift of the application of the load (from the symmetry axis of the bridge) is determined by the cusp curve. This is the first result (due to W.T.Koiter 1945) in the theory of sensitivity of elastic models to imperfections. In terms of catastrophe theory it reduces to the investigation of the family of potentials $-x^4 + (F_0 - F)x^2 + \varepsilon x$

To carry out loading of various structures without dangerous snaps it is necessary to know the "catastrophe diagram" and the topology of how the sheets of the equilibrium surface are joined over it.

2.2. Introduction to Bifurcations in Dynamical Systems

2.2.1. Bifurcations of Equilibrium States

An *evolutionary process* is described mathematically by a vector field in a phase space. A point of the phase space defines the state of the system. The vector at this point indicates the velocity of change of the state.

The *equilibrium states* correspond to the zeros of velocity (the state of the system which does not change with time).

The curves in the phase space traced by the successive states of process are called *phase* curves. In the neighborhood of a non-equilibrium state the partition of the phase space into phase curves looks like a partition into parallel lines. In the neighborhood of an equilibrium point the picture is more complicated.

In generic systems only equilibrium points with non-degenerate linear part occur. In the phase plane typical phase portraits are (stable or unstable) Focus, (stable or unstable) Node and Saddle (Figure.2).

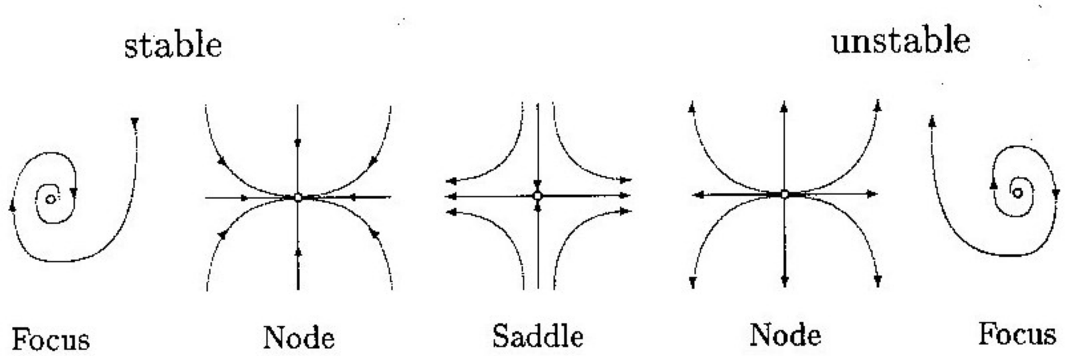

Figure 2: Typical equilibrium points

For a generic one-parameter family of vector fields, the equilibrium states for all values of the parameter form a smooth curve Γ in the product of the phase space and of the parameter space (in multiparameter case the dimension of the manifold of equilibrium states is equal to the number of parameters).

If the parameter space is one-dimensional, then the projection of Γ onto the parameter axis has singularities of the fold type only. (For more parameters the more complex singularities of mappings of manifold to a space of the same dimension appear: in generic two-parameter families the projection of the surface of equilibria onto the parameter plane can have pleats where three equilibrium states come together.)

Thus, as we change the parameter, we may single out the bifurcation values of the parameter (The word *bifurcation* means forking and is used in a broad sense for designating all sorts of qualitative reorganization or metamorphoses of various objects resulting from a change of the parameters). Those are the critical values of the projection of Γ to the parameter space.

Away from these values the equilibrium states depend smoothly on the parameters. When the parameter approaches a bifurcation value an equilibrium state “dies” by combining with another one (or going the opposite way, a pair of equilibrium states is born).

Of the two simultaneously appearing (or dying) equilibrium points in one-dimensional phase space one is stable and the other–unstable. (Here of course, *stability* means the “Lyapunov stability”. In other words, a trajectory with initial conditions close to the equilibrium point rests near it at any time).

At the instant of birth (death) both the equilibrium states move with infinite speed: when the parameter value differs from the bifurcations value by ε the distance between the two nearby equilibrium states is of order of $\sqrt{\varepsilon}$.

It turns out that in general all metamorphoses of equilibrium states can be obtained from one dimensional reorganizations by means of appropriate *suspensions* (separation of variables).

For example (Figure 3) the collision of a saddle and a stable node in the plane (at the instant of fusion a non-generic situation “saddle-node” arises) up to an appropriate diffeomorphisms of the plane corresponds to the collision of two equilibrium points of the vector field on one coordinate axis while the evolution along the other direction does not change at all.

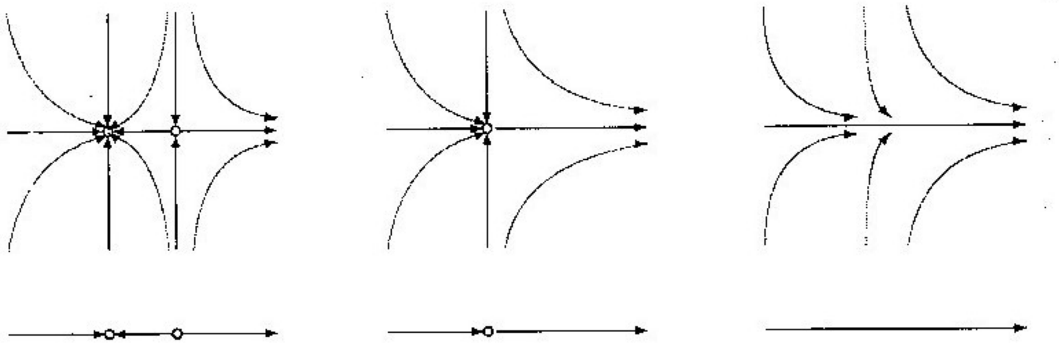

Figure 3: The saddle–node bifurcation

If a stable equilibrium state describes the established conditions in some real systems (in economics, ecology or chemistry) then when it merges with an unstable equilibrium state, the system must jump to a completely different state: as the parameter changes then the equilibrium condition in the corresponding neighborhood suddenly disappears. The equilibrium evolution suffers a catastrophe.

2.2.2. Loss of stability

Loss of stability of an equilibrium state is not necessarily associated with the birth or death of the equilibrium state: an equilibrium state can lose stability by itself.

Two versions are possible:

A. On change of the parameter the equilibrium state gives birth to a *limit cycle* of radius of order $\sqrt{\varepsilon}$, where the parameter differs from their bifurcation value by ε. The stability of the equilibrium is transferred to the cycle, and the equilibrium point becomes unstable.

In other words, after loss of stability of the equilibrium a *periodic oscillatory behavior* is established. The amplitude of the oscillation is proportional to $\sqrt{\varepsilon}$ (ε, recall, is the difference of the parameters from the critical value).

This form of the loss of the stability is called *mild* since the oscillating behavior for small ε differs a little from the equilibrium state (also it is often wrongly called *Hopf bifurcation).*

B. An *unstable limit cycle* collapses at the equilibrium state: the domain of attraction of the equilibrium state shrinks to nought with the cycle.

Before the established state loses the stability, the domain of attraction of the state becomes very small and ever–present random perturbations throw the system out of this domain even before the domain of attraction has completely disappeared. This form of loss of stability is called *hard* one.

It was observed by Poincaré and proved by Andronov's school for two dimensional case (and later for multidimensional systems) that no other forms of loss of stability apart from the described bifurcations (or their suspensions) are encountered in generic one-parameter families.

A behavior of motion which establishes itself is called *attractor* since it attracts neighbouring evolutions. An *attractor* is an attracting set in phase space. Attractors which are not equilibrium states or strictly periodic oscillations are called *strange* attractors and are connected with turbulence. Even in generic systems strange attractors can be very complicated (they can look like Kantor's set, have non integer dimension etc.)

Assume that after the loss of stability of an equilibrium state the mode of the behavior of the system is a strange attractor.

The transition of a system to such a behavior means that complicated non-periodic oscillations are observed in it. Their details are very sensitive to small changes of the initial conditions.

They look like a turbulent motion. It appears that the disordered motion of a fluid observed on loss of stability of laminar flow with an increase of the Reynolds number (*Re*) (i.e., with a decrease in viscosity) is described mathematically by just complex attractors in the phase space of the fluid. The dimension of the attractor happens to be finite. For two-dimensional fluid motion the dimension grows with a magnitude of order at most Re^4.

The transition from a stable equilibrium state to a strange attractor can be realized both by means of a jump (hard catastrophic loss of stability) or after a mild loss of stability. In the latter case the created stable cycle loses its stability.

The loss of stability of a cycle in a generic one-parameter family of systems can take place in number of ways: a *collision* with an unstable cycle, *doubling*, and the birth or death of a torus.

The details of these processes depend on the resonances (that is on the rationalitly of the ratio) between the frequencies of the motion along the meridian of the torus and along its parallels.

The behavior of the phase curves close to cycle can be described approximately with the aid of an evolutionary process for which the cycle is represented by an equilibrium state.

The most difficult case of 1:4 resonance in the theory of two-parameter bifurcations is still not investigated completely.

It is impossible to enumerate all the applications of this theory: parameter dependent systems in mechanics, physics, chemistry, biology and economics can lose stability.

For example, a stable steady–state mode of behavior (let us say, the working mode of a reactor or an ecological or economic system) usually perishes either by colliding with an unstable mode (where, as we have seen, at the moment of the collision the speed of the convergence is infinitely large), or as a consequence of the (again infinitely rapid) growth of self-sustaining oscillations.

This explains why it is so hard to fight a catastrophe once its symptoms have already become noticeable: the speed of the catastrophe approach grows unboundedly in proportion to the rate of a parameter change.

A catastrophic loss of stability may be the result of optimization and intensification.

Let us consider, for example, the simplest model of exploitation of certain resource (say, an agricultural product) taking into account the stabilization in the growth of the resource caused by natural limitations:

$$\frac{dX}{dt} = K_1 X\left(1 - K_2 X\right) - C. \qquad (2)$$

Here X is the current amount of the resource, C is a constant quota rate of exploitation while the term $K_1(1 - K_2 X)$ is the rate of the natural growth per unit of the resource. By an appropriate scaling of X and t one can normalize the coefficients K_1, K_2 and get the normalized equation

$$\frac{dx}{dt} = x - x^2 - c. \qquad (3)$$

The maximization of the quota rate $c = 1/4$ leads to instability of the steady state behavior of the model and to catastrophe – the annihilation of the resource by small random disturbances.

Stability will not be lost if we introduce *feedback*: for the rigid quantity C one substitutes a quantity proportional to the actually existing resource.

In this model with feedback

$$\frac{dx}{dt} = x - x^2 - kx \qquad (4)$$

the optimal value for the coefficient *k* is 1/2. With this choice a many–years' average result of $kx_0 = \frac{1}{4}$ will be established by itself.

This is the same exploitation result as the maximal rigid quota (a greater productivity is impossible in this system).

But while at the maximal rigid plan the system loses stability and destroys itself; small changes of the coefficient *k* (or other fortuities) lead only to a small decrease in productivity, but by no means to a catastrophe. The improved dynamical system is *structurally stable.*

Control without feedback always leads to catastrophes: it is important that persons and organizations making responsible decisions should personally and materially take the consequences of these decisions into account.

The following simple qualitative laws of the functioning nonlinear dynamical systems can be useful for those who are undertaking the crucial change in a complicated system.

Suppose that the actual state of the system corresponds to a local minimum of its potential, then:

1. Gradual motion in the direction of a better state (another local minimum with lower value of the potential) at once leads to deterioration. The speed of deterioration under uniform motion toward the better state is increasing.

2. As one moves from the worse state to the better one, the resistance of the system to change of its state grows.

3. The maximum of resistance is attained sooner than the worst state through which it is necessary to pass in order to reach the better state. After passing the maximum of the resistance the state continues to become worse.

4. As one approaches the worst state the resistance from a certain instant onwards begins to decrease, and as soon the worst state has passed, not only does the resistance completely vanish, but the system starts to be attracted towards the better state.

5. The magnitude of the deterioration necessary for a transition to the better state is comparable to the final improvement and increase in proportion to the perfection of the system. A weakly developed system can go over to the better state almost without a prior deterioration, whereas a well-developed system, by virtue of its stability, is not capable of such a gradual continuous improvement.

6. If one manages to move the system out of the bad state at once, by jump and not continuously, near enough to the good state, then, subsequently, the system will evolve towards the good state by itself.

In Section 4 singularities in control systems are discussed. In particular, we show that, aiming to steer a given state of a system to another "desired" state, one has to choose the control in an appropriate way. For arbitrary (even generic) choice of control system the *attainability domain* of the initial state can be far from the state, which one wishes to achieve.

3. Singularity Theory.

3.1. Classification of Functions

In all the examples described above we actually deal with a family of functions *F* of one variable x depending on the parameters *a* and *b*

Local changes of variable x (*diffeomorphisms* of the x axis) and shifts by a constant in the target space constitute the group action, which splits the space of functions $F(x)$ (defined in a certain neighborhood of a distinguished point) into a series of equivalence classes:

$$A_0 \leftarrow A_1 \leftarrow A_2 \leftarrow A_3 \ldots \tag{5}$$

The class A_k contains all functions which are equivalent to the function $\pm x^{k+1}$ at the origin (if *k* is even then the signs $\pm$ correspond to one class). The arrows between classes mean that the class A_{k+1} belongs to the closure of the class A_k in the space of functions. In the space of the Taylor series of the functions at the origin, the class A_k forms a smooth subspace of codimension *k*.

Starting from $k = 1$ these functions have critical points at the origin with different orders of degeneracy.

This is the first simple example of the classification problem in singularity theory.

At a first glance the most natural classification principle is classifying by codimension. To classify the objects "*up to codimension* $\leq k$ "means to represent the entire space of the objects as a finite union of submanifolds of codimensions not greater than *k* (classes) and the remainder of codimension $\geq k+1$ so that within each class the object's properties that are of interest to us do not change. Then all objects in typical, no more than *k* –parameter families, belong to our classes: the remaining ones may be avoided by a small perturbation of the family (according to Sard or Thom transversality theorems).

Let us consider now smooth functions in several variables. We are interested in the local properties of function near its critical point, say, the origin. Introduce the space of germs of functions at the origin: two functions determine the same *germ* if they coincide in some small neighborhood of the origin.

Assume also that the function has critical point at the origin 0 and its critical value equals 0. The classification up to codimension 4 of the space of such germs is formed by the classes of Thom's seven "catastrophes"

$$A_i, \quad i = 1...5, D_4 : x^2y \pm y^3, \quad D_5 : x^2y \pm y^4 \tag{6}$$

Here the germs are classified according to *stable equivalence*. Any germ of a function $f(x), x \in \mathbf{R}^n$, by an appropriate change of variables $x \mapsto (y, z), \quad y \in \mathbf{R}^k, z \in \mathbf{R}^{n-k}$ can be reduced to the form $\tilde{f}(y) + Q(z)$, where Q is a non-degenerate quadratic form in z and all the first and second derivatives of $\tilde{f}$ vanish at the origin. Such a separation of variables follows from the Morse Lemma. In particular, if the second differential of a function at the origin in not degenerate then the function is equivalent to its quadratic part. (Analogous results for functionals in infinite number of variables are the key point of various branches of non-linear analysis).

Two germs f, g of functions (may be of different number of variables) are called *stably equivalent* if their corresponding functions germs $\tilde{f}, \tilde{g}$ are equivalent (in particular, if the corresponding dimensions of y - space should be equal).

Classification up to the codimension differs, generally speaking, from the classification by *codimension of orbits* of the diffeomorphisms group.

This is so because the orbits may form continuous families. In these cases, the appearance of objects whose orbits have codimension *k* may turn out to be unavoidable under classification up to some codimension less than *k*.

For example, in the space of three-jets (that is in the space of the Taylor polynomials of degree 3) of function in three variables at the origin the homogenous polynomials of degree 3 form a 10- dimensional subspace of codimension 6. The orbits of these polynomials under the group of (jets of) diffeomorphisms are at most 9 dimensional (because only linear parts of changes of variables, forming the space of 3×3 matrices, act nontrivially on these polynomials). This 10-dimensional subspace is split into a 1-dimensional family of orbits and a remainder of codimension 8.

The main measure of the complexity of a singular point of the function f is its *multiplicity* or *Milnor number* $\mu(f)$: for complex functions it can be defined as the maximal number of critical points into which our singularity can be decomposed by small perturbations, and for the singularities of real functions it provides an upper bound of the number of such points. The multiplicity is finite for all function germs except for a subset of infinite codimension. It is equal to 1 only for *Morse* critical points, whose quadratic parts of the Taylor expansions are non-degenerate.

The codimension of the orbit of f in the space of all germs of function $(\mathbf{R^n}, 0) \to (\mathbf{R}, \mathbf{0})$ with singularity at 0 is always equal to $\mu(f) - 1$.

It turns out however that the most natural classification is not by codimensions or multiplicities of degenerate critical point, but by their modalities.

The number of moduli or the *modality* of an object is the least number *m* for which some neighborhood of the objects can be covered by at most a finite number of at most *m*- parameter families of orbits. In the given case we are talking about a neighborhood of a jet of the function in the space of *k* –jets. (i.e. of the space of Taylor polynomials of degree *k*) with zero critical value at the origin. This finite number must remain bounded as $k \to \infty$.

Objects of modality zero are called *simple*. The neighborhood of a simple object is covered by a finite number of orbits. The germ of a function at a critical point is simple if it can be deformed only in a finite number of ways.

The list (obtained by V.I. Arnold in 1972) of simple critical points of holomorphic functions (up to stable equivalence) is formed by two series of classes of singularities and three exceptional classes.

$$\begin{aligned} &A_\mu, \quad \mu \geq 1: \quad x^{\mu+1}; \qquad\qquad D_\mu, \mu \geq 4: \quad x^2y + y^{\mu-1}; \\ &E_6: \quad x^3 + y^4; \quad E_7: x^3 + xy^3; \qquad E_8: \quad x^3 + y^5. \end{aligned} \tag{7}$$

(In all these formulas the lower index is the multiplicity of the corresponding singularity.)This list repeats the list of Weyl groups (the crystallographic groups of Coxeter, generated by reflections) without multiple links in their Dynkin diagrams (that is, with 90°and 120°angles between the generating mirrors (that is fixed point subspaces for reflections, which generate the group).

A connection between functions and reflection groups is the following. Let us include a function with a critical point of multiplicity μ in a generic $(\mu - 1)$-parameter family as the function that corresponds to the zero value of the parameter.

For a typical value of the parameter, sufficiently close to zero, the critical point of the function *f* is splitted into $\mu = \mu(f)$ nondegenerate (Morse) critical points with different critical values.

These critical values may be considered as a μ -valued function of μ -1 variables (the parameter of the family).The graph of this μ-valued function lies in μ- dimensional complex space and is called the *bifurcations diagram of zeros* or *discriminant* of the original singularity. For example, the bifurcation diagram of zeros of the singularity A_2 is a semicubical parabola, and that of A_3 is a swallowtail (see the left picture in Figure 4 below).

The investigation of the geometry of discriminants, their cross-sections and projections forms in a technical respect the most important part of the catastrophe theory.

It turns out that the discriminants (bifurcation diagrams of zeros) of the simplest singularities are diffeomorphic to the varieties of non-regular orbits of the corresponding reflection groups.

These diagrams provide examples of rather general important features of many classification problems in singularity theory (with a relatively good equivalence notion).

A family F of objects (e.g. Functions) depending on parameter λ is said to be induced from another family G depending on parameter ε, if there exists a smooth mapping $I:\lambda \mapsto \varepsilon$ such that the objects $G_{I(\lambda)}$ is equivalent to the objects F_λ (provided that this equivalence depends smoothly on parameter).

In many reasonable problems (including the case of classification of smooth functions or mappings) any singularity, whose orbit has finite codimension *k*, possesses a (uni) versal deformation (i.e. a family containing this singularity for zero value of parameters): any deformation of this singularity is induced from this (uni) versal one.

The method to construct versal deformation comes from finite dimensional case (the action of the finite dimensional Lie group on the finite dimensional manifold). A mapping $g: M \to N$ is said to be *transversal* to a submanifold $S \subset N$ at the point $g(x)$, if the tangent vectors to *S* at this point and the image under the derivative of *g* of tangent vectors to M at this point $x \in M$ span the tangent space to the total ambient manifold *N*.

A deformation of an object might be regarded as a mapping of the parameter space to the space of the objects.

It turns out that often the following statement holds even for infinite dimensional spaces:

The deformations, which are transversal to the orbit of the given object, are versal.

Roughly speaking, to obtain a versal deformation it is sufficient to perturb the singularity in *k* independent directions transversal to the orbit of the singularity provided that the codimension of orbit equals k .

The rigorous theory of versal deformations is based on B. Malgrange preparation theorem and results of J. Mather and J.C. Tougeron (we recommend also recent papers of J. Damon). The main theorem of this theory claims that any function germ of finite multiplicity μ admits versal deformations depending on μ parameters.

The versal deformations (if they do exist) are very useful: the bifurcation diagram of any particular family may be regarded as a transformation (or a section) of the unique bifurcation diagram of the versal deformations, whose numbers of parameters is equal to the codimension of the orbit.

More general setting when the relations between external and internal parameters are determined not by a hyper surface (of zeros of functions) but by a subset *C* of common zeros of several functions is related to the singularities of mappings $g: M^m \to N^n$ with $n \geq 1$. The natural equivalence here is the group of a *complet*e *intersection* .The transformations form this group send the equations, determining S to their linear combinations (with variable coefficients) and change the coordinates.

The corresponding bifurcation diagrams also arise in applied problems (singularities of diffraction on boundaries, projections of surface and so on).

However, if the corank (i.e. the codimension of the image of the derivative of the mappings g) equals to 1, the singularity is equivalent to a deformation of a single function singularity.

Generically the singularities of corank >1 of mappings g arise in the case $m \geq n$ only if $m \geq 4$.

3.2. Geometry and Topology of Discriminant Sets

In this subsection we study the geometrical properties of different discriminant sets in the space of functions, especially the properties of bifurcation sets of zero (or simply discriminants), i.e. the sets of functions with singular zeros level, and bifurcation sets of functions, i.e. spaces of non-Morse functions.

3.2.1. Discriminants

A family of functions $\mathbf{R}^n \to \mathbf{R}$ depending on m parameters can be considered as a function $F(x,\lambda): (\mathbf{R}^n \times \mathbf{R}^m) \to \mathbf{R}$. For any $\lambda \in \mathbf{R}^\mathbf{m}$, the restriction $F(\cdot,\lambda)$ of F onto the subspace $\mathbf{R}^n \times \{\lambda\}$ will be denoted by f_λ its *discriminant set* $\sum \subset \mathbf{R}^m$ is the set of all values $\lambda \in \mathbf{R}^m$ such that 0 is a critical value of f_λ.

For the family x^3+ax+b (of functions in one variable x depending on parameters a, b) this set already has been considered in Section 2.1: it is the semicubical parabola $4a^3+27b^2=0.$ The next complicated singularity, A_3 admits the versal deformation of the form

$$x^4+ax^2+bx+c. \tag{8}$$

Its discriminant set is called the *swallowtail*; it is shown in the left picture of Figure 4. Its consecutive sections by planes $\{a = \text{const}\}$ are shown in Figure 5: for $a>0$ the section is a smooth parabola –like curve, for $a<0$ it has two singular points (at any of which it is locally diffeomorphic to a semicubical parabola) and one self-intersection, and in the plane a=0 it coincides with the curve $(b/4)^4 = (c/3)^3$ having a singular point at the origin. (The sense of the thin line in Figure 5 will be explained later.)

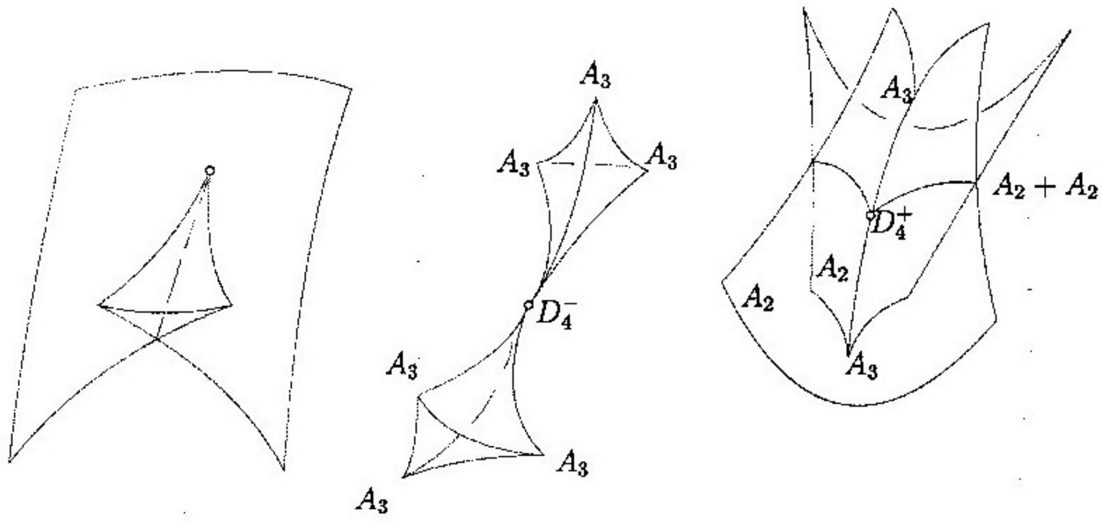

Figure 4: Generic caustics

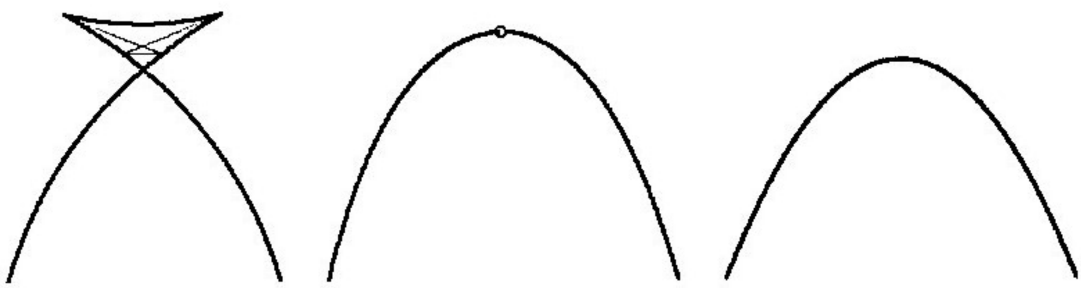

Figure 5: Sections of a swallowtail

The upper (in Figure 4 left) component of the complement of $\sum$ consists of all polynomials (8) having exactly two real roots, the lower one of the polynomial without real roots and the intermediate small pyramid consists of polynomials with 4 roots. Generic points of the discriminant surface are the functions with exactly one double root. The semicubical cuspidal edge consists of all functions with one root of multiplicity 3, and the selfintersection consists of polynomials with two double roots.

Similar strata arise on bifurcations varieties or arbitrary generic families of functions $\mathbf{R}^n \to \mathbf{R}$ Namely, such a variety is always a hypresurface in the space of parameters: it is only strata of codimension 1 that are cuspidal edges and transverse self- intersections and strata of codimension 2 are triple self–intersections, transverse intersections of cuspidal edge and a smooth piece of $\sum$, and strata A_3 at which $\sum$ looks like the direct product of the swallowtail and of $\mathbf{R}^{m\text{-}3}$.

Similar objects and facts hold for holomorphic complex functions and their complex families.

In the local singularity theory, one usually considers the families of functions f_λ which are the deformations (even *versal deformations* see Section 3.1) of some function singularities $f:(\mathbf{R}^n,\mathbf{R})\to(\mathbf{R},0)$ .In this case the local discriminant variety is defined as the set of such λ that $f\lambda$ has a critical point with zero value *close to the origin* $0\in\mathbf{R}^n$. Such a discriminant set has nice geometrical properties: it is irreducible and is swept out by smooth manifolds diffeomorphic to affine planes of low codimension. For example, the swallowtail is swept out by the one-parametric family of straight lines. Indeed, these discriminant sets look the same for stably equivalent functions; therefore we can assume that the Taylor expansion of *f* has zero quadratic part. Then for any point $x\in\mathbf{R}^n$ the condition " $f\lambda$ has a critical point with zero value at x "distinguishes an affine plane of the codimension $n+1$ in the space $\mathbf{R}^m$ of parameters λ of the deformation. All such planes over all $x\in\mathbf{R}^n$ form a smooth submanifold in the product $\mathbf{R}^n\times\mathbf{R}^m$; it is diffeomorpohic to $\mathbf{R}^n\times\mathbf{R}^{m\text{-}n\text{-}1}\equiv\mathbf{R}^{m-1}$ and, after the obvious projection to $\mathbf{R}^n$, provides a parametization of the discriminant $\sum$.

In particular we can go from any point of $\sum$ to any other inside $\sum$, only crossing finitely many times the transverse self-intersections and cuspidal edges. In the similar complex situation, the set of regular points of $\sum$ is path –connected.

E. Looijenga has proved that all the components of the complement of discriminant of the versal deformation of any simple real function singularity are contractible (i.e. homeomorphic to an open ball in $\mathbf{R}^\mu$); he also presented an algebraic description of all such components. For any function of type A_k (i.e. essentially the function x^{k+1} of the one variable) there are exactly $[(k+1)/2]+1$ such components: they are classified by the number of roots of corresponding polynomials.

3.2.2. Bifurcation Sets of Functions.

Given a smooth family of functions $F\equiv\{f_\lambda\},\ \lambda\in\mathbf{R}^m$, its *bifurcations set of functions* $\Delta(F)\ \in\mathbf{R}^m$ is the set of all $\lambda\in\mathbf{R}^m$ such that the function f_λ is not *strictly Morse*. By the definition of the last notion, this set consists of two components: the *caustic*, i.e. the set of the function having a non-Morse singular point (at which it is not equivalent to a non degenerate quadratic form), and the *Maxwell set*, i.e. the closure of the set of functions having several critical points with the same critical value. If our family is a deformation of a function singularity, then these sets are closely related with the discriminant set. Indeed, we always can choose the μ-parametric versal deformation of f, $\mu=\mu(f)$, in such a way that it contains along with any function f_λ also all function of the form $f_\lambda+const$. Adding constant functions preserves the bifurcations set. Therefore studying this set we can factorize trough such additions, or, which is the same, consider the *reduced deformation* consisting of functions f_λ, where λ runs over a hyperplane in $\mathbf{R}^\mu$ transversal to lines of the form $\{f_\lambda+t, t\in\mathbf{R}^1\}$. This factorization projects the discriminant set $\sum$ on the set of parameters of the reduced deformation. The caustic of this deformation is exactly the projection of the closure of the cuspidal edge of $\sum$ and the Maxwell set is the projection of the self intersection set. The shape of these bifurcation sets for such a reduced deformation is universal: for any generic deformation of the same (or stably equivalent) singularity depending on $m \geq \mu(f) - 1$ parameters, the corresponding bifurcation set of functions will be diffeomorphic to the product of this universal set and the space $\mathbf{R}^{m-\mu}$.

The caustic of any function of type A_k is diffeomorphic to the discriminant set of a function of type A_{k-1}: this diffeomorphism is provided by taking the derivative along x_1 and some scaling of coefficients.

So, the caustic of the singularity A_3 is diffeomorphic to the semicubical parabola, and that of A_4 is the swallowtail. The caustics of two more singularities of codimension 3, D_4^- and D_4^+ are shown in central and right–hand parts of Figure 4 and are called respectively *pyramid* and *purse*. In the case of functions of two variables, the large component of the complement of the pyramid consists of functions f_λ having exactly two saddlepoints and no other real critical points. In two other components f_λ has one additional saddle point and an extremum (minimum or maximum depending on the component). The lower (in Figure 4) component of the complement to the purse consists of functions f_λ having no real critical points, the upper one of points with one maximum, one minimum and two saddles. In two intermediate symmetric components the corresponding functions have one saddle point and either minimum or maximum, depending on the component.

The Maxwell set of the singularity A_3 is presented by the bissectral hemiline of the semicubical parabola (in the lower part of Figure 1 it is distinguished by conditions $b = 0, \quad a < 0$); the complementary semiaxis $\{b = 0, \quad a < 0\}$ corresponds to polynomials with two complex conjugate critical points with equal critical values. In general, the Maxwell set is a variety with boundary, and this boundary coincides with the stratum $\{A_3\}$ of the caustic.

It is easy to see that the Maxwell set of the deformation $x^5 + ax^3 + bx^2 + cx$ of the function x^5 (representing the class A_4) lies in the set of polynomials having at least three real roots of the derivative, i.e. in the closure of the small pyramid of the swallowtail. It is easy to see also that it is invariant under the action of the one-parametric group $\mathbf{R}_+^1$ of scalings, sending any point (a, b, c) to (t^2a, t^3b, t^4c), $t \in \mathbf{R}_+^1$. Thus this set is completely described by its intersection with the very left plane section of the swallowtail in Figure 5, this intersection is depicted there by the thin line.

For the singularity D_4^+ (see the right hand picture of Figure 4) the Maxwell set belongs to the "upper" component of the complement of the purse, is homeomorphic to the hemiplane and splits up this component into two symmetric parts. For singularity D_4^- the Maxwell set consists of three irreducible components homeomorphic to hemiplanes; these three components intersect one another along the axial line of the "pyramid". This line consists of functions with three coinciding critical values.

The complete set of strata of the bifurcation sets of functions for all simple singularities was obtained by O. Lyashko in 1976, similar results for real singularities are due to Yu.Chislenko.

3.3 Caustics, Wavefronts, and Symplectic Geometry.

We describe below one more important feature of the discriminants and bifurcation diagrams of functions. They can be visualized and recognized in many physical models.

Suppose that a disturbance (e.g. a shock wave, light, an epidemic or a flame) is being propagated in some medium. For simplicity let us start with the plane case. Suppose that at the initial instant the disturbance is on the initial curve and the speed of its propagation is constant at any point and in any direction.

To find out where the disturbance will be at time t (according to Huygens principle) we must lay out a segment of length t along every normal to the curve. The resulting curve is called *equidistant* or the *wave front* (Figure 6 left above).

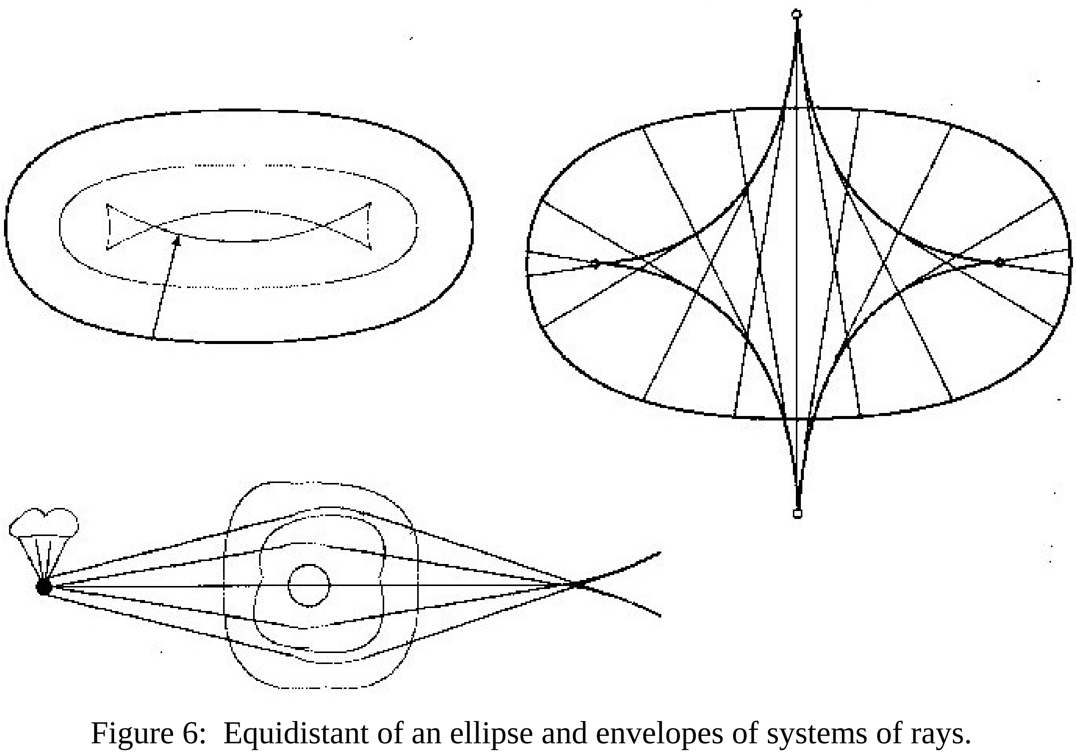

Figure 6: Equidistant of an ellipse and envelopes of systems of rays.

Even if the initial wavefront has no singularities, after some time singularties will appear. For instance, upon propagation of the disturbance inside an ellipse the singularities arise when the time passes the value equal to the minimal radius of curvature of the ellipse. Immediately afterwards the wavefront contains four cusp points and two points of selfintersection.

These singularities are stable, not removable by all small perturbations of the initial wave front (and even of the speed of the propagation). All other singularities which can occur for non-generic initial curves decompose on a small perturbation of it into several singularities of standard type. Moreover, all the same facts hold for the wave propagation in the nonhomogenous media, where rays of propagation are not straight lines forcefully. For instance, if there is some domain where the wave propagation is comparatively slow, then the envelope of waves (where the shock is concentrated) will appear beyond it, and the edge of the envelope will be approximately directed into the core of his domain, see figure 6 left below. This picture should arise in the tomographical investigations (for example, of mines).

In three–space only cusp ridges and singularities of *swallowtail* type can appear, see the left picture of Figure 4.

Let us examine the intersections of the swallowtail with a system of parallel planes in generic position. These intersections are plane curves (Figure 5). As the plane is translated these curves change their form at the moment the plane passes through the vertex of the tail. The transformation here is exactly the same as the metamorphosis of a wave front on the plane (two cusp points and one self- intersection point appear of disappear).

We can describe the metamorphoses of wavefronts as follows. Consider the *space-time* which is the direct product of the initial space and of the time axis.

The wavefront being propagated in the plane sweeps out a surface in the space-time. It turns out that this surface can itself be regarded as a wave front in the space-time ("big-front"). In the generic case, the singularities of the big front will be swallowtails, cusp ridges and self–intersections, situated in the space–time in a generic way relative to the isochrones (which are made up of "simultaneous" points in the space-time). Now it is easy to understand which metamorphoses can be experienced by the momentary wave fronts on the plane in generic case: they are the changes in the form of the isochronic cross-sections of the big front.

The study of the metamorphoses of a wave front during its propagation in three-dimensional space leads in the same way to an investigation of the cross–sections of the big (three–dimensional) wave front in four-dimensional space-time by the three–dimensional isochrones.

Along with wave fronts, *ray systems* can be used to describe the propagation of disturbances. For example the propagation of a disturbance inside an ellipse can be described using the family of internal normals to the ellipse. This family has an envelope. The envelope of the family or rays is called *caustic* (i.e. "burning"), since lights concentrate at it. A caustic is clearly visible on the inner surface of a cup when the sun shines on it. A rainbow in the sky is also due to a caustic of a system of rays that have passed through a drop of water with complete internal reflection. (The use of the same word "caustic" for singularities of systems of rays and for component of bifurcation diagrams of function singularities, introduced above, is of course not occasional).

The caustic of an elliptic front has four cusps (see Figure 6). These singularities are stable: a nearby front has a caustic with the same singularities. All singularities of caustics resolve under a small perturbation into the standard ones: cusps and self–intersection points.

Recently V.I.Arnold (see “Uspekhi Phyicheskix Nauk”, 1999, n 12) observed that the collection of the normals to an ellipse determines an *anti-circle* on the dual plane that is a curve given by the equation.

$$x^{-2} + y^{-2} = 1 \qquad (9)$$

in appropriate affine coordinates. In particular, the anti–circle is projective dual to an astroid.

The system of normals to a surface in three-dimensional space also has a caustic. This caustic can be obtained by marking off on each normal the radius of principal curvature (a surface, in general, has two different radii of curvature at each point so that the normal has two distinct caustic points).

It is not easy to imagine the caustics (called also the “focal surface”) of even the simplest surfaces, a triaxial ellipsoid for instance.

Generic caustics in three-dimensional space have only standard singularities. Besides regular surfaces, cuspidal ridges and their generic (transversal) intersections these singularities are: the swallow-tail, the “pyramid” (or “elliptic umbilic”) and the “purse” (or “hyperbolic umbilic”), see Figure 4. Those are the strata of the corresponding bifurcation diagrams of the functions to types A_4, D_4^-, D_4^+, respectively.

The pyramid has three cuspidal ridges meeting tangentially at the vertex. The purse has one cuspidal ridge and consists of two symmetric boat bows intersecting in two lines.

These singularities are stable. All more complicated singularities of caustics in three dimensional space resolve into these standard elements on small perturbations.

Consider now for a given initial wavefront (for instance an ellipse in the plane), both its caustics and the fronts of the propagated disturbance. It is not difficult to see that the *singularities of the propagating wave front slide along the caustic and fill it out.*

The cuspidal ridge of the wave front moving in three-dimensional space sweeps out the surface of the caustic. Consider the case, when this surface is a swallowtail. This partition of the caustic into curves is not the same partitioning of the swallowtail surface into plane curves being its plane sections. The cusp ridge of the moving front does not have self intersections. The cusp ridge of the moving front passes twice through each point of the self-intersection line of the caustic. The time interval between these passing is very small (of order of $\varepsilon^{5/2}$ where ε is the distance from the vertex of the tail).

If the original front depends on a parameter, then its caustic also varies and during this movement it can undergo metamorphosis. The metamorphoses of a moving caustic can be studied by considering cross- sections of a big caustic in the space-time.

The cusp ridges of caustics moving in three-space sweep out the surface of a bicaustic. The generic singularities of bicaustics contain new stable classes of singularities.

In the examples considered above the disturbance from a point is propagated in all directions.

In other models the indicatrix of the disturbance rates can be shifted out of the origin. For example, the propagating front of the flame in a forest in windy weather moves at each point only in directions, which belong to certain sector at the tangent plane of all the velocities at this point.

Such families of fronts can have a non empty envelope. The generic singularities (apart from the *A* and *D* caustics, described above) of the union of caustics and the envelope in two and three spaces happen to correspond to bifurcation diagrams of functions of so-called boundary singularities B_3, C_3, B_4, C_4, F_4. The discriminant and the bifurcation diagram of the singularities C_3 are shown in Figure 7.

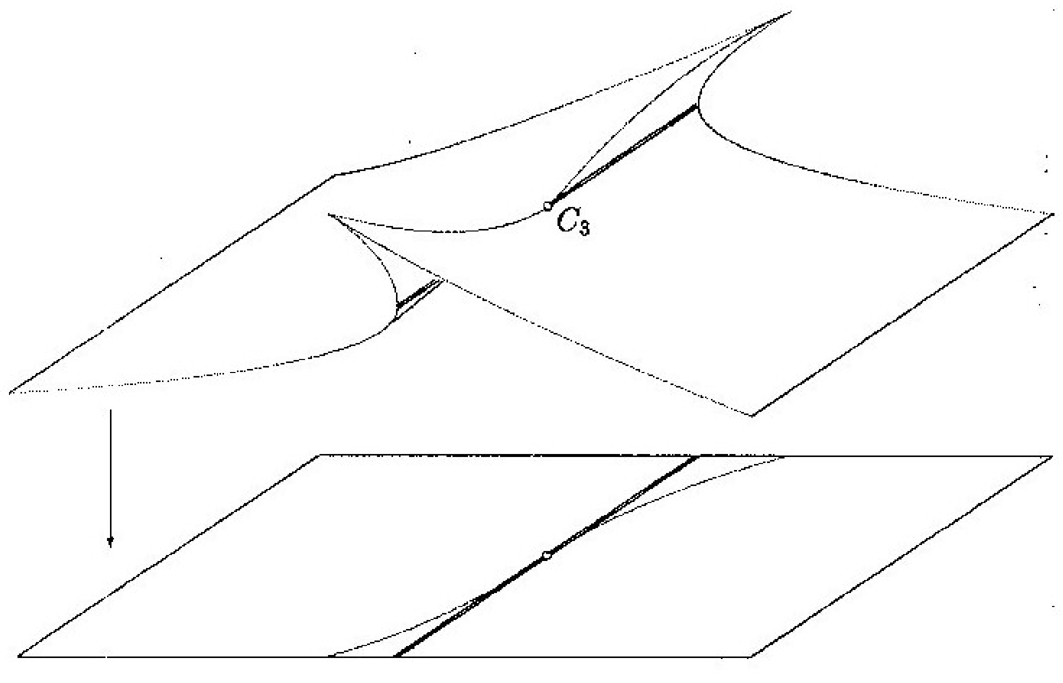

Figure 7: Bifurcation diagrams for the boundary singularity C_3

Let us consider a pair of functions $f(x),\ f(0)=0$ and $g(x),\ g(0)=0$ defined in the neighborhood of the origin of $\mathbf{R}^n$. Suppose the zero level hypersurface (called *boundary hypersuface*) of the function g is regular (in particular it is smooth).

The boundary singularities are the classes are the classes of such pairs with respect to the action of the group of diffeomorphisms of x-space, preserving the origin.

The simple boundary singularities naturally correspond to Coxeter groups of types $A, D, E,$ and also B, C, F_4 generated by reflections.

As is well known, rays describe the propagation of waves (say, light) only as a first approximation ; for a more precise description of a wave one has to introduce a new essential parameter, the wavelength (the ray description is satisfactory only when the wavelength is small in comparison with the characteristic geometric dimension of the system).

The intensity of light is greater near a caustic and still greater near its singularities. The coefficient of intensity amplification is proportional to $\lambda^{-\alpha}$, where λ is the wavelength and the index α is a rational number depending on a nature of the singularity.

For the simplest singularities the values of α are as follows:

Smooth caustic	cuspidal ridge	swallowtail	pyramid	purse
1/6	1/4	3/10	1/3	1/3

Thus the brightest–shinning are the point singularities of pyramid and purse type. In the case of a moving caustic, at isolated moments of time even brighter singularities A_5 and D_5 can appear.

If the light is so intense that it can destroy the medium, then the destruction will start at the points of greatest brightness. So the index α determines how the intensity for destruction to occur depends on the frequency of the light.

The predictions by the theory of singularities of the geometry of caustics, wave fronts and their metamorphoses have been completely confirmed in experiments and it seems strange now that this theory was not constructed long ago. However, the fact is that the mathematical apparatus needed is not trivial and is connected with such diverse areas of mathematics as the classification of simple Lie algebras and of Coxeter's crystallographic groups, the theory of braids, the theory of integrals depending on parameter's etc.

It turns out that the singularities of caustics and wavefronts are the loci of critical values of very special non-generic mappings of the manifolds of equal dimensions or of the mapping form *n*-dimensional manifold to $(n+1)$-dimensional one.

The mappings of the same type arise in many other applied problems.

For example, the gradient mapping (associating to each point the gradient of a certain function at that point), and the Gauss mappings (associating to each point of a hypersurface in Euclidean space the unitary vector, which is normal to the hypersurface at this point) yield the same generic singularities as caustics.

According to Ya. B. Zeldovich(1970), these singularities are responsible for the origin of large scale inhomogeneities in the distribution of matter in the Universe from initial small and smooth fluctuations of density and the initial velocity field of the potential flow of non-interacting particles of the “pro-matter”.

Moreover, many issues in singularity theory (for instance the classification of the singularities of caustics and wave fronts and also the investigation of the various singularities in optimization and variational calculus problems as well as the singularities of the special mappings mentioned above) become understandable only within the framework of the geometry of symplectic and contact manifolds.

Symplectic geometry is the geometry of the phase space (the space of positions and momenta of classical mechanics). It represents the result of the long development of mechanics, the variational calculus, etc.

The definition of a *symplectic structure* in a vector space is analogous to the definition of Euclidean structure: it is a skew-symmetric (while a Euclidean structure is symmetric) function of a pair of vectors, linear in each argument, which is non-degenerate (no non-zero vector is skew–orthogonal to every vector). Odd-dimensional spaces do not admit symplectic structures.

All symplectic vector spaces of the same dimension are isomorphic. It is easy to construct a symplectic structure in an even-dimensional space by representing the space as a sum of two-dimensional planes: the skew–scalar product is splitted into a sum of the areas of the projections onto these planes.

In sympletic space the *skew-orthognal complement* of a vector subspace consists of all vectors, whose skew–scalar products with all vector of the subspace are zero. The dimension of the skew- orthogonal complement (similarly to the Euclidean case) is equal to the codimension of the original subspace.

A vector subspace, which is its own skew-orthogonal complement, is called a *Lagrangian subspace*. Its dimension equals one-half of the dimension of the original symplectic space.

A *symplectic structure* on a manifold is given by the choice of a symplectic structure on each tangent space. However there is additional condition connecting the symplectic structures in different spaces: the “area” of the whole boundary of any three-dimensional figure should be equal to zero.

Unlike Riemannian manifolds, all symplectic manifolds of a given dimension are locally isomorphic (a neighborhood of any point can be mapped onto another one with preservation of symplectic "area"). Thus, each symplecitc manifold is locally isomorphic to a standard symplectic vector space. In such a space one may introduce coordinates $(p_{1,...,}p_{n,}q_{1,...,}q_n)$ such that the skew–scalar product equals the sum of the oriented areas of the projections onto the planes $(p_1\, q_1),...,(p_{n,}q_n)$.

A submanifold of a symplectic space is called a *Lagrangian* submanifold if its tangent space at each point is Lagrangian.

A fibration of symplectic space into submanifolds is called *Lagrangian fibration* if all the fibers are Lagrangian submanifolds.

Any Lagrangian fibration is locally isomorphic to the standard fibration of the phase space over the configuration space $(p,q) \to q$ (the fiber are the space of momenta). The base q-space of this fibration is called the configuration space.

Suppose now that in the space of a Lagrangian fibration we are given yet another Lagrangian manifold. Then we get a smooth mapping of this Lagrangian submanifolds to the base space of the Lagrangian fibration.

This mapping between manifolds of the same dimension n is called a *Lagrange mapping* and its singularities are called *Lagrangian singularities.*

These singularities form a special class of singularities of smooth mappings between manifolds of the same dimensions. Starting from $n = 3$ they are not generic singularities of general mappings.

It turns out that the theory of Lagrange mappings (with respect to the *Lagrangian equivalence*, which is a group of diffeomorphisms respecting the structure of fibrations and symplectic structure) corresponds to the theory of families of functions depending on the parameters (with respect to changes of variables depending on parameters and addings smooth functions on parameters).

It can be shown that the gradient, normal and Gausssian singularities are Lagrangian.

The caustics of ray systems are the critical values loci of Lagrangian projections.

The solution set of any variational problem (or, in general, the solution set of Hamiltonian equation, with a fixed value of Hamiltonian) forms a symplectic manifold, which is very useful for investigating the properties of these solutions.

In symplectic geometry (in contrast to Euclidean and Riemannain ones) the intrinsic geometry (the restriction of the symplectic structure to the set of tangent vectors to the submanifold) determines the local external geometry. In other words, submanifolds with the same intrinsic geometry can be locally transformed into one another by a diffeomorphism of the ambient space, which preserves the sympletic structure.

The situation becomes quite different if we embed not a submanifold but a variety, which itself has singular points. At these singular points certain information on the embedding into the ambient symplectic structure is conserved. Recent investigations (V.I. Arnold, 1999) of non-equivalent embeddings of curves (with singularities) into symplectic space show in particular that there are several (but finitely many) different embeddings of cusps.

R. Melrose in his papers on diffraction noted the importance of investigation of singularities of the disposition of submanifolds in a symplectic space.

M.Zhitomirski and R. Montgomery (1999) related a Lagrangian singular curve to so called Goursat non-holonomic distributions, which determines, for example, the motion of a chained trolleys (say, of the airport luggage trains).

Singular Lagrangian varieties are generic singularities in families of geodesics on a Riemannian manifold with a boundary (this setting is often called *the problem of bypassing an obstacle* or *variational problem with one–sided constrains*). The low dimensional classification of their projections provides once again a relation with the complete list of Coxeter groups.

A *contact structure* in an odd-dimensional space is a field of hyperplanes which is generic near each point.

For example the manifold of all line elements in the plane is a contact (three-dimensional) manifold.

The role of the Lagrangian manifolds passes over in the contact case to *Legendre manifolds* (that is integral submanifolds of the hyperplane field, which have the greatest possible dimension: it equals to m in a contact manifold of dimension $2m+1$).

The singularities of wavefronts, of Legendre transformation (useful in particular in thermodynamics), of surfaces dual to the smooth ones are Legendre singularities. The entire symplectic theory has contact counterpart which is extremely useful for the investigations of singularities in variational problems.

For example, in mathematical economics the contact geometry helps to solve the market disaggregation problem formulated by I. Ekeland.

3.4 Bifurcations and the Problem of choice

Our life is full of decision problems. Usually we solve them by analogy: there is an appropriate solution for a tentative problem, then if its initial data change slightly then we expect to find a solution of the new problem as a small perturbation of the previous one. However, if our problems (moving in the space of all problems of their class) come too close to a discriminant variety, then this strategy can fail because the problem of choice becomes nontrivial.

A model example of such situations is as follows. For the space of problems we take that of all *complex* polynomials

$$f_a(x) \equiv x^d + a_1 x^{d-1} + \ldots + a_{n-1}x + a_n; \tag{10}$$

it is to find (up to a sufficiently small constant $\varepsilon > 0$) one root of any such polynomial given by its coefficients. The above analogy method (in applied mathematics, it is called the “path–following method”) acts as follows. There are polynomials (say, $x^d - 1$) whose roots are well–known. Let us join such a model polynomial with one to be solved in the space $\mathbf{C}^d$ of all polynomials (10) by a path (say, by a line segment). Then we move along this path in small steps. By the inductive conjecture, before any step we know a root of the corresponding polynomial .Then we consider this root as initial data for the Newton iterative method for the polynomial obtained after this step. (This method produces the sequence of points z_i, $i = 0,1...$, where z_0 is some initial point and z_{i+1} is the solution of the equation $(z_i - z_{i+1}) f'(z_i) = f(z_i)$, i.e., it replaces the function f by the linear function, having the same value and first derivative at the point z_i and defines z_{i+1} as the root of this linear function). However, the Newton method with starting point z_0 surely converges only if the absolute value of $f(z_0)$ is sufficiently small and that of $f'(z_0)$ is *comparatively large*. When we approach the discriminant set (i.e., the set of polynomials with multiple roots), this condition can fail, and the Newton method can send us far away from the neighborhood of any root. In fact, this inconvenience has a topological origin and cannot be removed by replacing the Newton method by any other. Indeed, when we approach the discriminant variety, some other root becomes very close to the one we follow. Therefore, at the next step we obtain a problem of choosing one of two roots of the corresponding polynomial: solving it takes a lot of time.

S. Smale (1987) has related this problem with algebraic topology, namely with the notion of the *genus* of A. S. Shvartz and cohomology of braid groups. (The starting model observation here consists in the fact that equation $x^2 = a$ has no sufficiently good solutions depending continuously on all complex a). Smale proved that the worst number of choices in this problem grows to infinity when *d* does. Later V. Vassiliev has improved his lower estimate and found also an upper one, in particular he proved that this number grow asymptotically as *d*.

In fact, almost all difficulties in solving problems by analogy (by “path–following method” developed in particular in the works of M. Shub and S. Smale) follow from the presence of discriminants: it forces us to go by very small steps, or to go simultaneously along many paths, or to watch on any step whether we are too close to the discriminant.

For more complicated (and thus more realistic) problems these difficulties are even stronger than for the problem of solving (10). E.g., if we consider the problem of finding approximate roots of systems of n polynomial equations of degree *d* in n complex variables, then the degree of the discriminant set grows exponentially with *n*. For large n this set looks like a deep forest, and we need to pay a lot of power and time to go through it from the model system to the system to be solved. This forest is especially dense close to the subspace of real polynomials or systems (which, in fact are to be solved in a majority of applied problems).

The “path-following” method often is used to solve problems of even more complicated nature, say nonlinear differential equations. Of course, in these cases the harmful influence of the discriminant set becomes still worse.

However, the principal (of small codimensions) singular strata of such discriminant sets usually are standard, i.e. belong to a finite list of model degenerations. Only such strata have good chance to appear close to a generic path in the space of problems, thus the study and tabulation of the behavior of our problems in such model situations should help solve our problems in general.

3.5. Monodromy of Complex Singularities and Shock Fronts

3.5.1. Integral Representations and Complex Fiber Bundles

The most important special functions of mathematical physics are given by integral representatations: they include the fundamental solutions of all principal classes of partial differential equations. Newton–Coulomb potentials, Fourier integrals, Feynman integrals, generalizations of Gauss hypergeometric functions, and many others.

The general construction of such functions is as follows. We have a differentiable fiber bundle $p: E \to B$, a differential form ω on its total space E and a family of singular cycles in fibers $p^{-1}(b)$ of this bundle, depending continuously on the corresponding base point b. Then the integrals of the form ω along these cycles define a function $I(b)$ on the base B.

The principal qualitative and analytical properties of this function $I(b)$ can be formulated in terms of its asymptotic behavior as b tends to irregular limit points in some closure $\overline{B}$ of the domain B, over which the bundle p fails to be locally trivial.

Even if our applied problem and the integral function $I(b)$ are completely real, their qualitative properties can be well understood only if we enter into the complex domain (as it generally happens in the mathematics). In fact, usually it is possible to extend our fiber bundle to its “complexification” $p_{\mathbf{C}}: E_{\mathbf{C}} \to B_{\mathbf{C}}$; where $E_{\mathbf{C}}$ and $B_{\mathbf{C}}$ are complex manifolds whose real parts are E and B. Then the irregular points of $\overline{B}_{\mathbf{C}}$ will form a subvariety $\sum$ of complex codimension 1, i.e. of real codimension 2, and instead of approaching it we can go around it in the space B_C of regular complex values of b. The family of integration cycles also can be included into a similar family depending continuously on the points of $B_{\mathbf{C}}$. However this family is usually multivlaued: if we go around a piece of $\sum$ at its point β we can deform the initial cycle in the fiber to some different cycle, so that the value of the integral also can change. But this new value is nothing other than the analytical continuation of the initial function $I(b)$. In particular, if the value of the integral actually changes, we can be sure that this function is irregular at this point β even in the real domain.

The type of the "ramification" of integral cycles (and hence also the analytical properties of the function $I(b)$ at its suspicious points) is controlled by the *monodromy theory* or *Picard –Lefschetz theory.*

The importance of complex discriminants $\sum = \sum(F)$ studied in the previous subsections consists in the fact that they provide a wide class of model ramification sets. Indeed, over the space $\mathbf{C}^m$ of parameters λ of a family of holomorphic functions f_λ the *Milnor bundle* is defined: its fiber over a point λ is the hypersurface $f_\lambda^{-1}(0)$. The discriminant set $\sum$ is exactly the set of parameters λ over which this bundle fails to be locally trivial, and hence going round its pieces defines generally a nontrivial ramification of cycles in the fibers. The ramification of integration cycles, arising in many natural problems of mathematical physics and tomography, can be related to this model situation, and hence the ramification of corresponding integrals is controlled by the *local Picard –Lefschetz theory of function singularities*. However, there are more complicated problems, in which more degenerated situations arise; they are investigated in *generalized Picard –Lefschetz* theories, due in particular to F. Pham.

3.5.2. An Example : Shock Fronts

An important application of these methods is in the theory of shock waves of hyperbolic partial differential equations (in particular of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}, \tag{11}$$

describing the propagation of waves with velocity *c*). A *fundamental solution*, i.e. an elementary wave arising from a point instantaneous perturbation, has singularities on the cone defined by the condition $c^2 t^2 = \sum_{i=1}^{n} x_i^2, t \geq 0$. The asymptotic behavior of the wave, for the argument (t, x) tending to this cone, depends on the parity of the number of variables. In our four-dimensional space-time (and more generally for any odd $n > 1$) the signal is noted only instantly, when passes a fixed observer. On the contrary, in the odd-dimensional case the signal continues to sound the whole time after the instant of the first meeting. The first circumstance allows us to communicate via the sound, while as the consequence of the second, the "acoustical layer" in the ocean, being an excellent conductor of individual signals, is badly suited for fast transmission of any complicated information. Both variants of the phenomenon of sound waves have analogues for arbitrary hyperbolic equations: in the language of the general theory we say that in the case of an odd *n* the interior component of the complement to the cone of the singularities (= wave front) is a *lacuna* (and the front is *sharp* from the side of this component) and in the case of even *n* there is diffusion of waves from the side of this component ; the exterior component is a lacuna for any dimensions (and arbitrary processes described by hyperbolic equations).

This phenomenon was investigated by J. Hadamard, and then by G. Herglotz and I.G. Petrovskii, who in particular have found an essential integral representation for the fundamental solutions of hyperbolic equations. In particular, Petrovskii related the sharpness with the triviality of the corresponding integration cycles. Further important investigations are due to M. Atiyah, R. Bott and L. Gårding, who have studied the *local sharpness* at particular points of wave fronts(of more complicated equations, where the front can be non-smooth, and looks like the real discriminant set studied in the Subsection 3.3). In particular, Gårding has found all domains of sharpness at the singularities of types A_2 and A_3 (where the wave front can be reduced to the semicubical parabola and the swallowtail respectively). Similar results exist also for all simple singularities.

4. Singularities in Optimization Problems.

4.1. Conflict Sets and Maxwell Strata

Problems in optimization, control theory and decision theory provide singularities, bifurcation diagrams, and conflict (or Maxwell) sets due to certain counterpart of the problem of choice.

Let us consider for example a swimmer in a still lake who would like to reach the shore as soon as possible. Denote by y his current position in the lake, denote by x a point of the shore and by $f(x,y)$ the distance between these points.

The problem is to find x providing the minimum of the function $f(x,y)$ for given y. For a generic point y such point x is defined uniquely, but for some states the minimum is achieved in several points of the shore. For example, if the lake is a line segment then such state is the center of this segment. Inside the lake the distance to the shore is a smooth function everywhere except this central point, at which it is equal to $a-|y|$ (where $2a$ is the length of the lake, and the origin on the y-axis is taken at this central point), and thus has a singularity. This point is a *conflict set* where the swimmer has a not unique optimal solution (but two in this example).

For two-dimensional lakes the conflict set can have a more complicated structure. It is clear that singularities of conflict sets are related to the singularities of wave fronts (see section 3). The level curves of the function

$$F(y) = \min_x f(x,y) \tag{12}$$

coincide with the leading fronts of the disturbance extending from the shore with the same velocity in any direction.

In a generic situation (for a lake of “general position”) the function F can have only standard singularities. Near any state y by appropriate choice of coordinates in the y-space and adding a smooth function of y to F the last function can be reduced to zero or to one of the following normal forms near the origin:

in the case of one parameter:

$$-|y| \tag{13}$$

in the case of two parameters:

either $-|y_1|$ or $\min(y_1, y_2, y_1+y_2)$ or else $\min_x\left(x^4 + y_1x^2 + y_2x\right)$. (14)

Here the conflict set is the set of points where the function F is not differentiable due to the existence of at least two optimal solutions. Note, that for the case of many variables x the list of generic singularities of F is the same. For the last singularity of this list the graph of the function F is the part of swallowtail surface obtained by the cutting of its pyramid.

For the case of 3, 4, 5, and 6 parameters such a list contains 5, 8, 12 and 17 singularities. Starting from the seven parameters the number of nonequivalent singularities is infinite, and the corresponding normal forms include the functions of the parameters. But even in these cases the function F is equivalent to a smooth one up to continuous deformation of the parameter space, provided that the system is in general position (V.I. Matov, 1981).

In the nature the water level in the lake depends usually on the season, and so the shore does, too. Thus, if each concrete time instant one has to solve the problem (12), then, in general, one needs to study an *extremal problem with constraints* $F(y,h) = \min\{f(x,y) | L(x) = h\}$ where h is the level of the lake surface over the sea and L is the respective level function on the earth surface, and now x is two-dimensional.

In a more general situation the constraint map L depends on both x and y, and there are several “level” parameters h. So, the problem takes the form

$$F(h) = \min\{f(X) | L(X) = h\}, \tag{15}$$

where $X = (x,y)$. Here the solution F is called the *relative minimum.* The list of generic singularities of the realative minimum is wider. Certainly, it includes all singularities of the case (12) but also many others. For some of them the function F is even discontinuous.

The generic relative minimum singularity is equivalent (up to more wide equivalence than considered above, namely, up to diffeomorphisms of the graph space of function F

preserving the natural fibration over the h–space) to zero or toone of the following singularities near the origin:

for one parameter, one of the following three singularities:

$$-|h|, \quad -\sqrt{h}, \quad \min\left\{-\sqrt{h},1\right\}, \tag{16}$$

and for two parameters one of the following twelve:

$$-\sqrt{h_1} \tag{17}$$

$$\min\left\{w \mid w^3 + wh_1 + h_2 = 0\right\} \tag{18}$$

$$\min_w\left\{w^4 + w^2h_1 + wh_2\right\} \tag{19}$$

$$-\sqrt{h_1}h_2 \tag{20}$$

$$\min\left\{-\sqrt{h_1}h_2,1\right\} \tag{21}$$

$$-|h_1| \tag{22}$$

$$\min\left\{h_1,h_2,h_1+h_2\right\} \tag{23}$$

$$\min\left\{-\sqrt{h_1},1\right\} \tag{24}$$

$$\min\left\{1-|h_1|,-\sqrt{h_2}\right\} \tag{25}$$

$$\min\left\{1-\sqrt{h_1},\sqrt{h_2}\right\} \tag{26}$$

$$\min\left\{2,1-\sqrt{h_1},-\sqrt{h_2}\right\} \tag{27}$$

$$\min\left\{-\sqrt{h_1},h_2\right\} \tag{28}$$

When the number of parameters in the problem (15) is equal to three there are 34 generic singularities. Starting from 4 parameters here we have the same picture as starting from seven parameters above: the number of nonequivalent singularities is infinite, and their normal forms include an arbitrary function of certain parameter.

The relative minimum singularities can be separated into two parts. One part includes the *point* singularities defined by the behavior of the *objective* (or *minimized)* function *f* and the *constraint* L near a point of X -space. Another part is formed by all generic superpositions of the point singularities. In the latter the first four lines represent point singularities, and all others are their superpoisitions.

In the case of a lake with variation of its level during the time we have four variables (point x of the earth and the point of y of the lake surface) and three parameters (the level h and again the point y). For such numbers of variables and parameters there are only six generic point singularities of the relative mimimum. Up to appropriate choice of smooth coordinate system and up to adding smooth functions of the parameter $\Lambda = (h, y)$, any such a singularity is either one of the first three singularities of the previous list or one of the following three singularities defined near the origin by functions.

$$\min\left\{w \mid w^4 + \lambda_1 w^2 + \lambda_2 w + \lambda_3 = 0\right\}, \tag{29}$$

$$\min_{v,w}\left\{\left(v-\lambda_1\right)^2 + \left(w - \lambda_2\right)^2 \mid v^2 - w^2 = \lambda_3\right\}, \text{ or} \tag{30}$$

$$\min_{v,w}\left\{\left(v-\lambda_1\right)^2 + \left(w - \lambda_2\right)^2 \mid v^2 + 2w^2 = \lambda_3\right\} \tag{31}$$

The last two singularities have a simple geometrical illustration. The function of (30) is equal to the square of the minimum distance from the point (λ_1, λ_2) of the plane to the hyperbola shore $v^2 - w^2 = \lambda_3$. .Here the distance to the shore is calculated both from the surface of the lake and from the earth surface, too. The conflict set of the hyperbola consists of the axis of its symmetry separating its branches and of two infinite rays emanating from the points $(\pm 2\sqrt{|\lambda_3|})$ on another axis of symmetry of this hyperbola. When level λ_3 varies, the conflict set sweeps two orthogonal planes in the space of parameters with some “cuts”. The relative minimum loses its smoothness exactly on this set. The singularity of this type is observed on an isthmus at the moment of its birth in the withering lake if the minimized function is the square of the distance to the shore.

Analogously, the function of the singularity (31) gives the square of the minimum distance to the ellipse $v^2 + 2w^2 = \lambda_3$. Here, for negative value of λ_3, the relative minimum is not defined at all, and for the positive one this minimum loses the smoothness on the segment connecting the points $(\pm\sqrt{\lambda_3}/2, 0)$. When the “profundity of the water” λ_3 varies from 0 to $+\infty$, this segment sweeps the convex hull of the parabola $\lambda_2 = 0, \lambda_3 = 42\lambda_1^2$. Here the set of non–smoothness of the relative minimum consists of this hull and the plane $\lambda_3 = 0$.

4.2. Singularities of Controllability

Implementation of control parameter into an evolving system helps sometimes to avoid its catastrophic behavior (see the example of feedback system in Subsection 2.2 above).

A *control system* is defined by vector fields with a control. At a point of the phase space all values of the control define the indicatrix of admissible velocities for the evolution of this state.

Taking every time admissible velocity we define an *admissible control* of the system. It implies an admissible evolution of the system.

A state of the system is *attainable from another one* (*in time T*) if there exists admissible evolution of the system steering out of the second state to the first one (in time *T*, respectively).

All states attainable from a given one form the *attainable set* of this state, and the set of states from any of which the given state is attainable is called the *control* set of this state.

In our model of exploitation with feedback the attainable set of any initial positive volume x_0 of population is the union of the interval (0, 1) and the interval $\left(0, x_0\right)$, and its control set is the interval $(x_0, +\infty)$ if $x_0 \geq 1$ and it is the interval $(0, +\infty)$ if $x_0 < 1$.

A maximal domain of phase space coinciding with the intersection of the control set and the attainable set of any its point is called a *transitivity set*. The control system describing the evolution of population has one transitivity set. This set is the interval (0, 1).

Inside the transitivity set any two points can be carried out one to another in finite time by appropriate choice of admissible evolution of the system. So, until the system is inside such a set, its possible future state can be arbitrary from this set, and in that the sense we need not be afraid of the possibility of destroying it. So, in some sense a transitivity set plays the same role as an attractor of a vector field. But sometimes the transitivity sets have no stationary states of the system as well as the cyclic motion of a dynamical system discussed above. Leaving such a set the system has no possibility to come back under any of its admissible evolution. After that the system can pass to some other stationary regime (=transitivity set) or it can attain such states when the system will be destroyed.

Control sets, attainable sets and transitivity sets are very important characteristics of any control system. On the line these sets have very simple structure, namely, all of them are intervals for a generic system with field of velocity indicatrix which is independent of time. In our model of exploitation with feedback all these sets have small variations under a small perturbation of the vector field of evolution $x - x^2$, and for any sufficiently close vector field they can be carried out to the respective sets of the initial field by a small deformation of the phase space.

On the plane (or a surface) the structure of control sets, attainable sets and transitivity sets is more complicated. But in generic cases they have some important properties of stability.

For example, for a generic system on the usual two-dimensional sphere the attainable set of a typical initial state (or typical initial start curve) is an attractor. That means that at any admissible evolution the distance between the current state of the system and this set tends to zero for any initial state sufficiently close to the attainable set. The same is true for the control set if we change the time direction.

For a generic two-dimensional control system the boundary of the attainable (control) set is some smooth curve with isolated singularities from a finite list. If the indicatrix of admissible velocities is a smooth closed curve (like a deformed circle) then there are only four generic singularities of this boundary.

Three of them are very simple; they can be expressed by the formulas

1) $y=|x|$; 2) $y=x|x|$; 3) $y=x^2|x|$ (32)

near the origin $(x=0, y=0)$ after an appropriate choice of smooth coordinates.

The fourth singularity is more complicated. It is related to an implicit differential equation of the first order. Such an equation is defined by its *surface* in the three-dimensional space of directions on the plane. In a generic situation near any point of this surface such an equation takes the form $F(x,y,p)=0$, where $p=dy/dx$ and F is some sufficiently “good” function. The saddles, nodes and foci of usual vector fields take form of folded ones for the implicit differential equations. In a generic case near such a folded singular point such an equation is reduced to the equation.

$$(p+kx)^2=y \tag{33}$$

near the origin $(x=y=p=0)$ after an appropriate smooth choice of coordinates x and y. The fourth singularity is defined near the origin by two typical integral curves of the last equation entering the origin from the opposite directions.

All the four described singularities have stable realizations. The first three of them can be observed on the boundary of attainability of the control system defined on the (x,y)–plane by two fields of admissible velocities $v_1=(-1,0)$ and $v_2=(1,x^2-y)$ and by the start set equal to the unit circle centered at the point (2, 2).

The fourth singularity is observed in the model “swimmer in the water” in which the stream of water has the velocity field $(-x,-\beta y), \beta>2,$ and the swimmer himself/herself can move in standing water in any direction with a velocity not exceeding 1. The admissible velocities of the swimmer are defined by the inequality $(\dot{x}+x)^2+(\dot{y}+\beta y)^2\le 1.$ Here the attainable set from the origin includes the set $x^2 + \beta^2 y^2 < 1$ and coincides with the transitivity set of the system. The boundary of this set has singularities of fourth type at points $(0, \pm 1/\beta)$.

Note, that in the last model the transitivity set is a global attractor: the distance between it and the current state of the system tends to zero in time for any initial state of the system. But generally the transitivity set can have no such property. This depends on the system and its concrete transitivity set.

Besides, on any closed orientable surface (like a sphere or a torus) all control sets and attainable sets of points (orbits of states under admissible evolutions of the system back and forth), transitivity sets of typical control system are stable under small perturbation of this system, and so the system is *structurally stable* (as well as a generic vector field on such a surface).

This means that for any control system sufficiently close to that typical one, the orbits of points and transitivity sets of one of these systems can be carried out to the respective sets of the another one by some small continuous deformation of the surface.

Sometimes the last statement is trivial. For example, the admissible velocities of the simple motion are the union of unit velocities in all directions. Therefore the transitivity set of the simple motion on a connected phase space coincides with this space, and so the attainable (control) set of any point does too. So, the respective small deformation of the phase space is just identity.

But in some cases a structurally stable control system has more complicated set of orbits. Consider an object on a sphere admitting motions with two structurally stable vector fields. One field has a stable node at the south pole and unstable node at the north pole. The remaining phase curves of this field go from the north pole to the south and coincide with the meridians outside a sufficiently small neighborhood of the poles.

Another field of admissible velocities has two cycles that are parallels and that are located near the equator somewhere to the north and to the south of it. The northern cycle is unstable. The phase curve unwinding from it either winds around the southern cycle or enters a nondegenerate stable focus near the north pole. The southern cycle is stable. The phase curve winding around it either unwinds from the northern cycle or outgoes from an unstable focus near the south pole (See Figure 8).

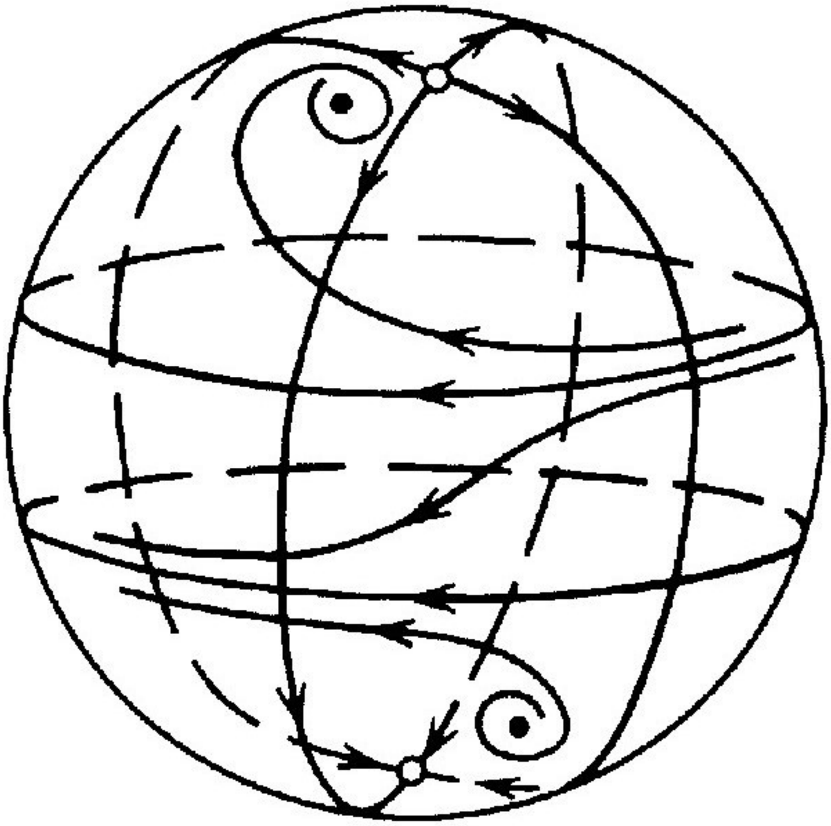

Figure 8: Structurally stable system on sphere

This control system has two transitivity sets, the northern and southern. They lie above and below the northern and southern cycles, respectively. Between them there is an equatorial annulus where the phase curves of one of the fields of admissible velocities are meridians and the phase curves of another field unwind from the northern cycles of this field and wind around the southern cycle.

For a point in northern (southern) transitivity set, the control (attainable) set coincides with this set whereas the attainable (control) set coincides with the entire sphere. For a point of the boundary of this transitivity set, one of the orbits is the closure of the set and another is the closure of this set complement.

For a point of equatorial annulus let us consider phase curves of our fields passing rough the point and taking two sections of the positive and negative semi-trajectories for each of them. These sections lie between the considered point and two nearest (along the trajectories) intersection points of these curves. The positive (negative) orbit of the considered point lies below (above) the union of two of these four sections belonging to the positive (negative) semi-trajectories.

It is clear that the considered system is structurally stable. Really, the structure of the point orbits for the considered control system remains the same after a small perturbation of the system, and therefore there exists a small deformation of the sphere carrying out the point orbits of perturbed system in the point orbits of the initial system.

In the multidimensional case the attainable sets, control sets and transitivity sets have been investigated weakly. But in a generic case the boundary of the attainable (control) set is a hypersurface in the phase space.

5. Conclusions

The models of real process which are important for the life of our civilization involve many parameters.

Even the creation of an adequate model is a great problem. Sometimes even the determination of the spaces of internal and external parameters, the existence and smoothness of the corresponding relation or properties of dynamical systems are completely unclear.

The qualitative analysis based on the simplest models might be useful. As soon as a mathematical description of the system is found the Bifurcation and Singularity Theories furnish quantitative models, but the qualitative deductions seem to be more important and at the same time more trustworthy: they do not depend on the details of the functioning system, whose mechanism and numerical parameters may be insufficiently known.

Napolean criticized Laplace for his “attempt to introduce the spirit of infinitesimals in government”. The mathematical theory of singularities is this part of the contemporary infinitesimal analysis, without which a conscious management of complicated and poorly known nonlinear systems is practically impossible.

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Glossary

Attainable set: consists of all the states of a control system which can appear during an *admissible evolution.*

Attractor: is a set of the states of a dynamical system which attracts all the nearby states: a trajectory issuing from any initial state from a certain vicinity of the attractor passes arbitrary close to each point of the attractor at certain moments of time. **Strange** one is an attractor which is not equilibrium state or strictly periodic evolution of a system.

Bifurcation diagram: is a geometrical characteristic of a collection of objects depending on parameters. It consists of parameter values, which correspond to degenerate (in an appropriate sense) objects. Often it is a hypersurface with singularities dividing the parameter space into domains corresponding to non degenerate objects with different properties.

Caustic: of a system of optical rays is the collection of points where the intensity of the light energy is much higher than usual. A caustic is also a critical value set of a Lagrangian projection. A set of parameters for which the parameter depending function has a non-Morse (= degenerate) singular point is also a caustic.

Codimension: of a subspace is the difference between the dimension of the ambient space and the dimension of the subspace. Often the singularities are classified by their codimensions. According to transversality theorem the objects from a subset of large codimension can be eliminated by small perturbations of a given family of objects depending on lower number of parameters.

Control system: is a dynamical system whose evolution (velocity) depends not only on its current state but also on the choice of a certain parameter *Control parameter*. The time dependent choice of this parameter gives an *admissible control* of the system and its *admissible evolution*

Cusp: is a small piece near the origin of a plane curve given by the equation $x^2 - y^3 = 0$ (semicubical parabola). Cusp-like singularity arises in almost all classification problems.

Diffeomorphism: is a differentiable invertible map such that its inverse is also differentiable.

Discriminant: see *bifurcation diagram.*

Equivalence relation: is a partition of the space of objects into certain collection of subsets (= equivalence classes). Often these subsets are the orbits of a certain group action.

Equivalence left-right: is a partition of space of maps into the orbits of the action of the *diffeomorphisms* of the sourse and the target spaces.

Generic: see *typical.*

Lacuna: is a set in a phase space in which a solution of wave-type equation has no prolongation to. The complement of a wave front singularity splits into lacuna domains and the domains where the solution exist.

Lagrangian submainfold: is a submanifold of maximal possible dimension of a *symplectic space* such that the symplectic form vanishes when restricted to any tangent space of this submanifold. Lagrangian submanifolds are encountered in various settings of mathematical physics, analysis and geometry.

Legendre submainfold: is a submainfold of a maximal possible dimension of a contact space such that the restriction of the contact form to any tangent space to this submainfold vanishes.

Loss of stability: of an equilibrium state is called a **hard** one if the trajectories of the system go far away of the equilibrium as soon as the parameter passes the bifurcation value. The loss of stability is called **mild** if the nearby trajectories still remain in a certain neighborhood (depending on a parameter) of an equilibrium state while the parameter has already passed the bifurcation value.

Milnor Number: of an isolated critical point of a complex function is a number of Morse critical points into which the singularity splits under small generic perturbations.

Milnor bundle: of an isolated critical point of a complex function consists of the restrictions of the generic level sets of the function to a certain ball centered at this critical point. Milnor bundle inherits important topological information of the initial singularity. Milnor bundle can be defined in various more general settings.

Modality: of a singularity is the smallest number *m* such that a certain neighborhood of the singularity splits into a finite number of at most *m*-parameter families of equivalency classes (orbits).

Pleat: is one of the two stable singularities of the mappings from plane to plane. Its critical value locus is a cusp.

Maxwell Set: is a closure of set of parameters for which the parameter dependent function has several critical points with the same critical value.

Structural stability: is a property of objects to be equivalent to all nearby ones.

Swallowtail: is a singular surface in three-space being a stable singularity of wavefronts. It is isomorphic to the *discriminant* of the family of functions $x^4 + ax^2 + bx + c$ in one (real) variable x with (real) parameters a, b, c.

Symplectic structure: is a closed nondegenerate external two-form on an even dimensional manifold.

Typical: object is an object from certain open and dense subset in the space of all objects (equipped with an appropriate topology).

Wave front: is the image of a projection to the base of a Legendre submanifold of a projectivized cotangent bundle. Another definition: a wave front is the collection of points in phase space reached at a certain instant by a propagating disturbance issuing from certain initial subset.

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