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DREWS INSTITUTIONALIZED DIVVY ECONOMY

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Background of the Problem

This is a simplified version of an economy considered by W.P. Drews[1] , in which he showed how a market-clearance mechanism could serve to define the objective optimized by the "invisible hand" of a competitive economy. More specifically, he worked with the patterns of expenditure by which each institutionalized "consumer group" would dispose of its surplus income, and showed that a set of equilibrium prices would exist such that resource earning power would equal expenditures of surplus disposable income, for each "consumer group".

In this paper, Drews' model is modified in such a way that the sizes of the institutionalized "consumer groups" and the prices charged by other institutions controlling "resources" are manipulated by these institutions in an effort for each to achieve its share of the total money flows as agreed upon by the "political process"--for example, by traditions and negotiations.

Thus this institutionalized view of the economy injects into the usual framework of technological relations an additional mechanism--the "invisible hand" or "political process", which can arbitrarily set the proportions of total money flows to different institutions. Our purpose is to show that once these are agreed upon, all other quantities, such as the levels of industrial

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[1] Private communication.

production, prices of consumer goods and resources, and the sizes of consumer groups can be determined. Among other things, it allows "Parkinson's Law" to operate with full force.

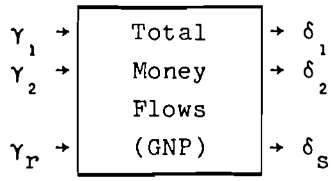
In this economy there is one type of institution that controls the resources like labor, land, oil, iron ore, capacity of various types of machine and plant capacity (e.g., refinery capacity). Nothing prevents each of them from charging any price they please for their "resource" except a kind of live-and-let-live understanding between institutions which stipulates the proportions of the total money flows from the purchases of their resources that each is entitled to receive. In a sense the proportions of the total money flows (revenues) that each institution receives is a measure of its influence and it is this that has been fixed by the "political process"--for example, by collective bargaining, by tradition, or by arbitration. It is then up to various institutions to manipulate the prices they charge for resources in order to try to achieve the stipulated money flows.

The revenues thus received by the resource institutions are transferred to a second type of institutions which we call consumer groups. As an example, workers associated with the labor-resource institution could be one such consumer group. Except for union dues, pension funds, and taxes, all other monies received for the sale of the primary resource labor would be given over to the workers as "wages". Administrative personnel (executives) associated with a resource would be a second example. Other examples of consumer groups would be government workers, research institutes, universities, and education generally which for the most part get their funding via taxation. Still other consumer groups (like private universities, banks, and little-old-ladies) own stock in various resource institutions as their share of profits after taxes. We assume that each consumer group in turn uses the money it receives to purchase consumer goods and each such group has a bill-of-consumer goods which is characteristic of the group. (Nothing, of course, prevents two consumer groups from having the same vector input of consumer goods per person. If so, then for the purpose of this analysis they could be combined into one group.) The transfer of funds from resource institutions to consumer groups is assumed to be given.

The size of a consumer group is free to vary. For example, if the U.S. Congress budgets a lot of tax money for Aerospace Research, then presto a lot of people become associated with this activity. If these funds should be reallocated elsewhere (say to Energy Research), then (like bees going to honey) people change jobs and associate themselves with the newly formed groups. Again, we assume that the proportions of the total money flows that each of the consumer groups has been allotted is fixed by the "political process", e.g., by changing national priorities, by stock ownership, by the tax structure, by bargaining. It is then up to the consumer groups to manipulate their sizes so that those people who remain associated (or become associated) with the consumer group, obtain a good standard of living (i.e., obtain their characteristic bill of goods, at a total cost that is within the budget allotted to the consumer group).

Thus the proportions of the total money flows for purchases of resources are stipulated by the "political process"--also stipulated are the proportions that each consumer group gets after transfer. Our problem is the following: Is there a set of prices for resources and sizes for the consumer groups that can achieve the stipulated proportions?

The proportions γ_i of total money flows to institutionalized resource groups are fixed by the political process.



Black Box

The proportions δ_j of the total money flows transferred to consumer groups are fixed by the political process.

We shall show for a Leontief type economy (which passively transforms resources into consumer goods) under the assumption of positivity of certain coefficients, that <u>prices</u> for resources and <u>sizes</u> for consumer groups do exist.

Of course, in practice some of the prices as well as sizes may also be fixed by the political processes and some of the total money flows are free to vary. This suggests the study of a variety of related problems.

The Mathematical Model

We now make all this mathematically precise and then prove the theorem just outlined. The inter-industry relations of the consumer goods industry and the products they produce are defined by a square n x n Leontief Matrix L. These industries must purchase in addition r resources from the resource institutions. The r x n matrix R will denote the inputs coefficients of resources per unit level of production, e.g., row i of R (denoted $R_{i.}$) is the input vector of resource i into various consumer goods industries expressed in natural units, e.g., man hours, square feet, BTU's. The n x s matrix C will denote the consumption patterns (bill-of-goods) of the various consumer groups, e.g., column j of C (denoted by $C_{.j}$) is the desired bill-of-goods of consumer group j if its corresponding proportional size $\mu_j = 1$.

(1)

	x	μ
y :	L (n x n)	C (n x s)
λ :	R (r x n)	

GIVEN: L: Square Leontief Input-Output Matrix

R: Resource Input Matrix

C: Consumption-Pattern Output Matrix

TO BE DETERMINED: x: Levels of Consumer Industries

y: Prices of Consumer Goods

λ: Prices of Primary Resources

μ: Sizes of Consumer Groups

where prices and sizes are <u>relative</u> prices and <u>relative</u> sizes, so that

$$\sum_{i=1}^{r} \lambda_i = 1 \quad ,$$

$$\sum_{j=1}^{s} \mu_j = 1 \quad .$$

The average bill of goods is given by $C\mu = \Sigma C_{\cdot j}\mu_j$. Similarly, the cost of purchasing resources to produce 1 unit of consumer goods k is $\sum_i \lambda_i R_{ik}$. The vector of all these costs for the various production activities $k = (1,\ldots,n)$ at unit level is given by $\lambda R = \Sigma \lambda_i R_{i\cdot}$. Therefore, the production levels x and the consumer goods prices y are given in terms of λ and μ by (2).

(2) $$Lx = C\mu \quad ,$$

$$yL = \lambda R \quad .$$

Moreover $R_{i\cdot}x$ is the total amount of resource i purchased at price λ_i, so that (3) gives the total money flows to resource institution i which is stipulated to be proportional to γ_i.

(3) $\lambda_i R_{i\bullet} x = p \cdot \gamma_i$, $i = (1,\ldots,r),\ \sum_{i=1}^{r} \gamma_i = 1$

where γ_i is given and p is a scalar factor of proportionality that will need to be determined (p can be interpreted as the total money flows when we use relative prices λ_i and relative sizes μ_j both of which sum to one).

Recalling that $C_{\bullet j}$ is the bill of goods of the j-th consumer group, then $yC_{\bullet j}\mu_j$ is the cost of its bill of goods if its size is μ_j. The cost of the bill of goods of the j-th consumer group is also stipulated by the "political process" and letting δ_j be the stipulated proportion of total money flows we require that

(4) $yC_{\bullet j}\mu_j = p\delta_j$, $j = (1,\ldots,s),\ \sum_{j=1}^{s} \delta_j = 1$

where δ_j is given and p, the factor of proportionality, turns out to be the same as that of (3). The equality of the scalar factors in (3) and (4) is easily established by summing (3) and replacing λR by yL and by summing (4) and replacing $C\mu$ by Lx, see (2).

Substituting the values of x and y from (2) we have

(5) $\lambda_i R_{i\bullet} L^{-1} C\mu = p\gamma_i$, for $i = (1,\ldots,r)$.

(6) $\lambda R L^{-1} C_{\bullet j}\mu_j = p\delta_j$, for $j = (1,\ldots,s)$.

If we set

$$(7) \qquad [M_{ij}] = [R_i L^{-1} C_j] \qquad \text{where } M \text{ is } r \times s,$$

then (5) and (6) simply state that we seek a λ in the simplex $S = \{\lambda | \lambda_i \geq 0, \sum_{i=1}^{r} \lambda_i = 1\}$ and a μ in the simplex

$T = \{\mu | \mu_j \geq 0, \sum_{j=1}^{s} \mu_j = 1\}$ such that the <u>rescaled matrix</u> $[\lambda_i M_{ij} \mu_j]$ has row sums proportional to $\gamma = (\gamma_1, \ldots, \gamma_r)$ and column sums proportional to $\delta = (\delta_1, \ldots, \delta_s)$.

Note that $R \geq 0$, and $C \geq 0$, and $L^{-1} \geq 0$ because L is a Leontief Matrix, so that $M = RL^{-1}C \geq 0$. (The theorem that follows assumes $M > 0$.) If M has a column of all zeros one will not be able to rescale it to obtain a specified non-zero column sum. Our purpose here is not to specify the most general form of $M \geq 0$ for which the theorem holds, nor is it to develop the most efficient algorithm for finding the rescaling factors when they do exist, but rather to set down the main problem in a symmetric format with dual roles for prices and sizes and to show under reasonable conditions that these prices and sizes can be determined.

<u>Theorem</u>. Given $M > 0$, $r \times s$, and $(\gamma_i,\dots,\gamma_s) \geq 0$, $\Sigma\, \delta_s = 1$, $(\delta_i,\dots,\delta_s) \geq 0$, $\Sigma\, \delta_j = 1$, then there exist $\lambda \varepsilon S$, $\mu \varepsilon T$ and a scalar p such that

$$\sum_j \lambda_i M_{ij} \mu_j = p\gamma_i \quad , \qquad \text{for } i = (1,\dots,r), \tag{8}$$

$$\sum_i \lambda_i M_{ij} \mu_j = p\delta_j \quad , \qquad \text{for } j = (1,\dots,s).$$

where

$$S\ \underline{\text{Simplex}}: \quad \{\lambda \mid \sum_{i=1}^{r} \lambda_i = 1, \lambda_i \geq 0\} \quad , \tag{9}$$

$$T\ \underline{\text{Simplex}}: \quad \{\mu \mid \sum_{j=1}^{s} \mu_j = 1, \mu_j \geq 0\} \quad .$$

<u>Proof</u>. Starting with any $\lambda \varepsilon S$ determine a mapping of $\lambda \rightarrow \mu \varepsilon T$ by first setting

$$\mu'_j = \delta_j / \sum_{i=1}^{s} \lambda_i M_{ij} \tag{10}$$

and then normalizing μ' to obtain μ

$$\mu_j = \mu'_j / \sum_{j=1}^{s} \mu'_j \quad , \qquad \text{for } j = (1,\dots,s). \tag{11}$$

Next map back this $\mu \rightarrow \bar{\lambda} \varepsilon S$ by

$$\bar{\lambda}'_i = \gamma_i / \sum_{j=1}^{s} M_{ij} \mu_j \tag{12}$$

and then normalizing $\bar{\lambda}'$ to obtain $\bar{\lambda}$

$$\bar{\lambda}_i = \bar{\lambda}'_i / \sum_{i=1}^{r} \bar{\lambda}'_i \quad , \qquad \text{for } i = (1,\dots,r). \tag{13}$$

The two successive mappings are a mapping in S: $\lambda \to \bar{\lambda}$ which is clearly continuous in λ if $M_{ij} > 0$. By the Brouwer Fixed-Point Theorem, there exists a λ such that $\bar{\lambda} = \lambda$. See [4, 6].

What we have shown is that given any arbitrary allocation or "divvying" of the proportions of total money flows to resource institutions and reallocation to consumer groups, it is always possible for the economy to adjust its prices for resources and to adjust the sizes of its consumer groups so as to achieve the agreed-upon money flows.

An iterative method can be used to solve for λ_i and μ_j. Starting, for example, with arbitrary prices for resources $\lambda_i = \lambda_i^0 > 0$, (6) is solved for the sizes of consumer groups $\mu_j = \mu_j^0$. This can in turn be substituted into (5) to obtain the next iterate $\lambda_i = \lambda_i^1$, etc. The iterative procedure is the same as that used by A. Evans [2], S. Evans [3] and by A.G. Wilson [7]. Proof of convergence can also be found in a letter to the author dated 8 April 1974 by James Bigelow of RAND. The iterative procedure just outlined could be used with non-reduced data. For example, initial prices λ could be assigned to resources. The average cost of resources λR could be used to determine prices y of consumer goods using relation (2). Next y could be used to determine sizes of consumer groups μ using relation (4) with p = 1. Then relation (2) is used to determine levels of production x. Finally (3) with p = 1 is used to re-estimate λ, etc. iteratively.

As we noted earlier, in real situations often some of the prices for resources and the sizes of consumer groups are fixed by the political process instead of the proportions of money flows. These considerations give rise in practice to a variety of related models where the roles of the various variables and the fixed parameters which we considered here, get interchanged.

An Optimization Problem Associated with Drews' Divvy Economy

The basic relations are to find $x = (x_1, x_2, \ldots, x_n) \geq 0$, $y = (y_1, y_2, \ldots, y_n) \geq 0$, $(\lambda_1, \ldots, \lambda_r) \geq 0$, $(\mu_1, \ldots, \mu_s) \geq 0$ such that

$$(14) \qquad \sum_{j=1}^{n} y_i L_{ij} x_j + \sum_{\ell=1}^{s} y_i C_{i\ell} \mu_\ell = 0 \quad , \qquad i = (1,\ldots,n),$$

$$(15) \qquad \sum_{i=1}^{n} y_i L_{ij} x_j + \sum_{k=1}^{r} \lambda_k R_{kj} x_j = 0 \quad , \qquad j = (1,\ldots,n),$$

$$(16) \qquad \sum_{j=1}^{n} \lambda_k R_{kj} x_j = \gamma_k \quad , \qquad k = (1,\ldots,r),$$

$$(17) \qquad \sum_{i=1}^{n} y_i C_{i\ell} \mu_\ell = \delta_\ell \quad , \qquad \ell = (1,\ldots,s)$$

where $C_{i\ell} > 0$, $R_{kj} > 0$, $L_{ij} > 0$ for $i \neq j$ and $L_{ii} < 0$, $\gamma_k > 0$, $\delta_\ell > 0$ are given.

An Optimization Problem Associated with a Reduced Problem

The square matrix $[-L_{ij}]$ is assumed to have a sufficiently dominant diagonal that $[-L_{ij}]^{-1} > 0$. By factoring out y_i in (14) and x_j in (15), it is possible to solve $-Lx = C\mu$ for x and $-yL = \lambda R$ for y. Then, as we have noted earlier, by substituting for x and y we have reduced the original problem to the well-known problem of rescaling the rows and columns of a matrix M so as to have specified row and column sums, namely find $\lambda_i \geq 0$, $\mu_j \geq 0$ such that

$$(18) \qquad \sum_j \lambda_i M_{ij} \mu_j = \gamma_i \quad , \qquad i = (1,\ldots,r),$$

$$\sum_i \lambda_i M_{ij} \mu_j = \delta_j \quad , \qquad j = (1,\ldots,s)$$

where $M = RL^{-1}C > 0$. Professor A.G. Wilson (Leeds University, Great Britain) has pointed out (in an informal note dated 3 March 1974) that the latter problem is equivalent to solving the following convex program [5, 7]. Note that (18) differs from (8) in that we no longer normalize λ and μ.

$$(19) \text{ Min } G(X) = \sum_{i=1}^{s} \sum_{j=1}^{s} X_{ij} [-\text{Log } M_{ij} + \text{Log } (X_{ij}/\sigma)] ; \sigma = \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij}$$

subject to

$$(20) \qquad \sum_j X_{ij} = \gamma_i \quad , \qquad i = (1,\ldots,r) : \text{Log } \lambda_i ,$$

(21) $\sum_i X_{ij} = \delta_j$, $j = (1,...,s)$: Log μ_j,

(22) $X_{ij} \geq 0$

where the Optimal Lagrange Multipliers associated with the constraints (20) and (21) for the minimizing solution turn out to be Log λ_i and Log μ_j and satisfy $X_{ij} = \lambda_i M_{ij} \mu_j$.

Indeed, at a minimum for $X_{ij} \geq 0$, we have the Kuhn-Tucker Condition

$$\frac{\sigma G}{\sigma X_{ij}} = \text{Log } \lambda_i + \text{Log } \mu_j$$

where G(X), the well-known Gibbs Free Energy Function (of the Chemical Equilibrium Problem), has the property that

$$\frac{\sigma G}{\sigma X_{ij}} = -\text{Log } M_{ij} + \text{Log } [X_{ij}/\sigma] \quad .$$

Thus, by equating the relations above, we have

(23) $X_{ij} = \sigma \lambda_i M_{ij} \mu_j$.

It is convenient to normalize (18) so that

$$\sigma = \sum_i \sum_j X_{ij} = \Sigma \gamma_i = \Sigma \delta_j = 1 \quad ;$$

for otherwise the derived multipliers λ_i and μ_j would be proportional to λ_i and μ_j. Because the rank of (20) and (21), as a linear system in X_{ij}, is m + n-1, the multipliers are never uniquely determined; indeed λ_i can be replaced by $\theta\lambda_i$ and μ_j by μ_j/θ.

An Optimization Problem Associated with the Original Problem

We substitute for $y_i L_{ii} x_j = -W_{ii}$, $y_i L_{ij} x_j = W_{ij}$, $y_i C_{i\ell} \mu_\ell = V_{i\ell}$, $\lambda_k R_{kj} x_j = U_{kj}$ and seek a convex objective function in these variables whose variables at a minimum solution (subject to the constraints) turn out to have the required product format. Analogous to (19)...(22) we have the system (24)...(28) below which appears to have a non-convex objective but is in fact convex subject to (25)...(28).

$$(24) \qquad \text{Min } G(U,V,W) = \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \neq j}}^{n} W_{ij} [-\text{Log } L_{ij} + \text{Log } (W_{ij}/\sigma)]$$

$$- \sum_{i} W_{ii} [-\text{Log } (-L_{ii}) + \text{Log } (W_{ii}/\sigma)]$$

$$+ \sum_{i=1}^{n} \sum_{\ell=1}^{s} V_{i\ell} [-\text{Log } C_{i\ell} + \text{Log } (V_{i\ell}/\sigma)]$$

$$+ \sum_{k=1}^{r} \sum_{j=1}^{n} U_{kj} [-\text{Log } R_{kj} + \text{Log } (U_{kj}/\sigma)]$$

where, by definition,

$$\sigma = \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \neq j}}^{n} W_{ij} - \sum_{i=1}^{n} W_{ii} + \sum_{i=1}^{n} \sum_{\ell=1}^{s} V_{i\ell} + \sum_{k=1}^{r} \sum_{j=1}^{n} U_{kj} \quad .$$

Subject to

$$(25)\qquad \sum_{j\neq i} W_{ij} - W_{ii} + \sum_{\ell} V_{i\ell} = 0 \ , \qquad i = (1,\ldots,n) : \text{Log } y_i ,$$

$$(26)\qquad \sum_{i\neq j} W_{ij} - W_{jj} + \sum_{k} U_{kj} = 0 \ , \qquad j = (1,\ldots,n) : \text{Log } x_j ,$$

$$(27)\qquad \sum_{j} U_{kj} = \gamma_k \ , \qquad k = (1,\ldots,r) : \text{Log } \lambda_i ,$$

$$(28)\qquad \sum_{i} V_{i\ell} = \delta_\ell \ , \qquad \ell = (1,\ldots,s) : \text{Log } \mu_j ,$$

$$(U,V,W) \geq 0$$

where the Lagrange Multipliers associated with these relations are shown on the right.

To establish convexity, note that $\sigma = \Sigma\, \gamma_k = \Sigma\, \delta_\ell$ follows from (25)...(28). Without loss of generality, we can rescale so that $\sigma = 1$. We can now rewrite (24) using (25) to define W_{ii}.

$$(29)\qquad G(U,V,W) = \sum_{i=1}^{n} \left\{ \sum_{\substack{j=1 \\ j\neq i}}^{n} W_{ij}\left[-\text{Log } L_{ij} + \text{Log } (W_{ij}/W_{ii})\right] + \sum_{\ell=1}^{s} V_{i\ell}\left[-\text{Log } C_{i\ell} + \text{Log } (V_{i\ell}/W_{ii})\right] \right\}$$

$$+ \sum_{k=1}^{r} \sum_{j=1}^{n} U_{kj}\left[-\text{Log } R_{kj} + \text{Log } (U_{kj})\right]$$

$$+ \sum_{i=1}^{n} W_{ii} \text{ Log } (-L_{ii})$$

where the convexity of the bracketed terms of (29) follows from the well-known result that <u>Gibbs Free Energy Function</u>,

$$(30) \qquad G(x) = \sum_{j=1}^{n} x_j \operatorname{Log} (x_j/\sigma)$$

where $\sigma = \sum_{j=1}^{n} x_j$, $x_j \geq 0$, is convex.

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