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Multiple Criteria Analysis of Discrete Alternatives with a Simple Preference Specification: Pairwise-outperformance based Approaches

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Foreword

Practically all important decisions involve analysis of several (or even many), typically conflicting, criteria. Analysis of trade-offs between criteria is difficult because such trade-offs for most problems are practically impossible to be defined a-priori even by analysts experienced in Multi-Criteria Analysis (MCA). Therefore, the trade-offs emerge during an interactive MCA which actually supports a learning process about the trade-offs. Hence, effective MCA methods are important for actual support of decision-making processes, especially those related to policy-making.

IIASA has been developing novel methods for MCA since mid 1970s, and successfully applying them to many practical problems in various areas. However, there are new practical problems for which the existing MCA methods (developed not only at IIASA but also by many researchers all over the world) are not satisfactory. In particular, discrete decision problems with a large number of criteria and alternatives (the latter making pairwise comparisons by the users impracticable) demand new methods. For example, MCA analysis of future energy technologies involves over 60 criteria and over 20 discrete alternatives; a careful requirement analysis of this application has proven that none of the existing MCA methods is suitable for an effective analysis of the corresponding problem. Moreover, this analysis has been done by a large number of stakeholders with diverse backgrounds and preferences; most of them have no analytical skills, therefore the specification of preferences needed to be simple but still provide effective and intuitive analysis of the Pareto set.

The paper introduces new methods for MCA of discrete alternatives together with several associated concepts, including automated pairwise comparisons which lead to the corresponding pairwise outperformance aggregations. The methodological background for the developed methods is presented, and the methods are compared using a large sample of preferences coming from actual analyses made by a large and diversified set of stakeholders.

Abstract

Many methods have been developed for multiple criteria analysis and/or ranking of discrete alternatives. Most of them require complex specification of preferences. Therefore, they are not applicable for problems with numerous alternatives and/or criteria, where preference specification by the decision makers can hardly be done in a way acceptable for small problems, e.g., for pair-wise comparisons.

In this paper we describe several new methods implemented for a real-life application dealing with multi-criteria analysis of future energy technologies. This analysis involves large numbers of both alternatives and criteria. Moreover, the analysis was made by a large number of stakeholders without experience in analytical methods. Therefore, a simple method for interactive preference specification was a condition for the analysis. The paper presents a number of new methods based on the developed outperformance aggregations that take into account inter-alternative factors. Finally, a comparison of methods and experience of using them is discussed.

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Marek Makowski leads the IIASA Integrated Modeling Environment Project. His research interests focus on model-based support for solving complex problems, which incorporates three interlinked areas. First, integration of interdisciplinary knowledge and its representation by mathematical models. Second, creation of knowledge by comprehensive model analysis, including multicriteria methods. Third, tailoring the modeling process to meet the needs of decision-making processes. Thus Marek's research interests cover a cluster of areas relevant to the adaptation (whenever possible) or development (when needed) of methodology, algorithms, and software for model-based decision-making support. This includes more specific topics in Operations Research such as: multicriteria problem analysis, large scale optimization, optimization of badly conditioned problems, use of database management systems for complex models, decision analysis and support, user interfaces in decision support systems, effective treatment of uncertainty and risk.

Marek has published over 130 papers and book-chapters, co-edited four books, coordinated or led several scientific projects, and has been twice guest editor of the European Journal of Operational Research.

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Notation

Notation:

- attributes, indicators or outcomes, or criteria., c_i indexed by $i = 1, \ldots, n$ (also denoted $i \in I$).
- alternatives, options denoted by o_j and indexed by $j = 1, \ldots, m$ (also $j \in J$),
- attribute values denoted by $q_{ij} = c_i(o_j)$ specified for each pair $\{i, j\}$
- outcome vector $\boldsymbol{q}^j = (q_{1j}, q_{2j}, \ldots, q_{nj})$
- $a_i : R \to R$ - Individual Achievement (IA) functions measuring (for each criterion separately) the satisfaction level corresponding to a value of the criterion.
  Individual achievements $a_{ij} = a_i(q_{ij})$
  achievement vector $\boldsymbol{a}^j = (a_{1j}, a_{2j}, \ldots, a_{nj})$
- $S : R^n \to R$ - Scalarizing Function (aggregation) measuring satisfaction levels for each alternative.
  $S(\boldsymbol{a}^j)$ scalarized achievement for alternative o_j
- $w_i(\boldsymbol{ri})$ and $v_i(w_i)$ - relative importance criteria scaling/weighting;
  they can be used in definition of scalarizing function S or IA functions a_i.
- ordered achievements
  the ordering map $\Theta : R^n \to R^n$ such that $\Theta(\boldsymbol{y}) = (\theta_1(\boldsymbol{y}), \theta_2(\boldsymbol{y}), \ldots, \theta_n(\boldsymbol{y}))$, where $\theta_1(\boldsymbol{y}) \leq \theta_2(\boldsymbol{y}) \leq \cdots \leq \theta_n(\boldsymbol{y})$ and there exists a permutation τ of set I such that $\theta_i(\boldsymbol{y}) = y_{\tau(i)}$ for $i \in I$.
  Ordering operator defined by vector $\boldsymbol{a}^j$ can be applied to another vector, e.g., to vector of the corresponding weights, we will denote it as
  $\Theta^j(\boldsymbol{w}) = (\theta_1^j(\boldsymbol{w}), \theta_2^j(\boldsymbol{w}), \ldots, \theta_n^j(\boldsymbol{w})) = (w_{\tau(1)}, w_{\tau(2)}, \ldots, w_{\tau(n)})$, where τ is a permutation of set I ordering vector $\boldsymbol{a}^j$, i.e., such that $\theta_i(\boldsymbol{a}^j) = a_{\tau(i)j}$ for $i \in I$.

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1 Introduction

Multiple Criteria Analysis is a well established area of applied science, which has been developed in response of needs for problem analysis that could not be met by traditional Operational Research methods. A sample of diverse approaches and the corresponding tools can be found e.g., in [1, 3, 5, 7, 11, 14, 27, 29, 30, 31, 36, 37]. One can thus ask why still new methods need to be developed. Therefore we start this paper with summarizing the motivation for the reported research.

Multicriteria analysis was needed for supporting a large number of diversified stakeholders in individual analysis of preferences for diverse future energy technologies developed with the European Integrated Project NEEDS.[3] A concerted effort of European researchers resulted in defining over 20 technologies in each of the four analyzed European countries; each technology is characterized by about 40 attributes. The attributes were organized in a hierarchical structure composed of three subsets of criteria following the concept of sustainable development, i.e., environmental, economic, and social criteria. From a modeling point of view for each of the four countries a multicriteria analysis was done for a set of over 20 alternatives, each characterized by 61 criteria (composed of attributes, three top-level criteria, and intermediate criteria) organized in a hierarchical structure forming an unbalanced criteria tree.

Over 3,000 stakeholders invited to the analysis had diversified backgrounds and typically rather limited mathematical skills. Due to the number of stakeholders and their geographical dispersion as well as limited time, the analysis was done through the Web. Moreover, the users typically had little time to become familiar with the tool supporting the analysis, and to complete the analysis. Therefore the Web-based tool for multicriteria analysis had to be easy to use; in particular, specification of preferences had to be intuitive and the corresponding multicriteria analysis method needed to support an effective

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[3]Information about the NEEDS Project is available at `http://www.needs-project.org/2009/`, and e.g., [18, 28].

analysis of a large number Pareto efficient alternatives characterized by a large number of criteria organized in a hierarchical structure. A detailed requirement analysis of this problem is provided in [18].

There have been no multicriteria analysis methods meeting these requirements although various approaches to multicriteria analysis in energy have been used, see e.g., [9], [4]. A comprehensive justification of this statement is available in [8]. In order to provide adequate support for analysis of the class of problems outlined above over 30 new methods have been developed and tested; 12 of these methods are presented in this paper, other methods are described in [35] and [16].

The structure of the remaining part of the paper is as follows. Basic terminology is introduced in Section 2. Then the specification of preferences as well as the methods of preferences' aggregation are discussed in Section 3. The fundamental assumption for the designed methods was simplicity of the preference specification which resulted in using the relative importance of criteria for representation of preferences. Due to the strong demand for organizing the large number of criteria into three pillars of sustainability, the corresponding hierarchy of the criteria was implemented. The main scientific result is presented in Section 4. The formulated Pairwise Outperformance Measure takes into account the differences of the compared achievements as well as the modified absolute values of the achievements. Next, in Section 5 we define the Ordered Pairwise Outperformance Aggregation and show its applicability. Such an aggregation is based on comparison of pairs of achievements ordered (for each criterion) from the worst to the best. The transitivity property of the developed methods, and Net-Flow approaches are discussed in Section 6. The correspondence between the implemented methods and the described methodology is presented in Section 7. Section 8 summarizes extensive experiments with the developed methods. Finally, Section 9 contains conclusions.

2 Preliminaries

2.1 Problem definition

In this paper we focus on the problem of analysis of discrete set of alternatives (objects) o_j, $j \in J = \{1, 2, \ldots, m\}$. The set of all alternatives will be referred as $Q = \{o_j : j \in J\}$. Objects o_j are described by numerical attributes (or criteria, selected outcomes) c_i, $i \in I = \{1, 2, \ldots, n\}$. Attribute values are denoted by $q_{ij} = c_i(o_j)$ specified for each pair $\{i, j\}$.

object attribute	o_1	o_2	$\ldots$	o_m
c_1	$q_{1,1}$	$q_{1,2}$	$\ldots$	$q_{1,m}$
c_2	$q_{2,1}$	$q_{2,2}$	$\ldots$	$q_{2,m}$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
c_n	$q_{n,1}$	$q_{n,2}$	$\ldots$	$q_{n,m}$

In the process of problem analysis the user selects some of the attributes as criteria and decide on each criterion type (minimization or maximization). Optionally, the user can define hierarchical structure of criteria forming a tree, in which leafs are the criteria

defined by the selected attributes, and the higher-level criteria are defined to aggregate lower-level criteria, see [17] for details.

There are three basic types of multicriteria analysis:
- choice: select the most preferred object,
- ranking: order all objects from the most preferred to the last preferred,
- sorting: partition of the set of alternatives into several categories.

The essence of multiple criteria analysis is to help the user in finding a solution (either an object or a ranking or a sorting) that fits best her/his preferences. The basic function of multicriteria analysis is to support the user in an interactive modification of her/his preferences upon analysis of the corresponding solutions. This approach substantially differs from the classical (single-objective) optimization which requires a prior specification of one objective function (optimization criterion).

In order to facilitate the discussion we recall now the basic concepts of Pareto efficiency (Pareto-optimal solution) and preference models.

2.2 Basic concepts of Pareto efficiency

Pareto-optimal alternative: An alternative is called Pareto-optimal, if there is no other alternative which has a better value of at least one criterion while no criterion has a worst value. In other words (and assuming for the following definition that all criteria are maximized) alternative $o_l \in Q$ is Pareto-optimal if and only if:

$$\neg\exists o_j \in Q: \quad \{c_i(o_j) \geq c_i(o_l) \;\; \forall i \in I \;\; \text{and} \\ \exists k \in I : c_k(o_j) > c_k(o_l) \;\} \tag{1}$$

If such an alternative o_j exists than we say that it dominates o_l. A Pareto-optimal alternative is also called an *efficient* or non-dominated one.

Further on, a properly Pareto-optimal alternative will be simply called Pareto solution.[4]

Pareto-optimal point: Pareto-optimal point is composed of values of all criteria for a corresponding Pareto-optimal alternative.

Pareto set: Pareto-optimal set is composed of all Pareto-optimal alternatives.

It is clear that a dominated alternative is not a rational choice. Therefore, it is rational to analyze trade-offs between non-dominated alternatives only. Thus the purpose of multicriteria analysis is help the user to analyze the Pareto set in order to find either a Pareto efficient solution or a ranking of non-dominated alternatives.

2.3 Preference model

Preferences for alternatives can be analyzed in terms of the corresponding outcome vectors (shortly: outcomes), i.e., criteria values of the corresponding alternatives. Those approaches are equivalent, and we will use both of them interchangeably.

[4] In actual applications one usually deals with properly Pareto-optimal alternatives with a prior bound on trade-off coefficients (see [34] for more details). For the sake of brevity we don't exploit this concept here.

Let us consider now pairwise comparison, i.e., the problem of deciding which of two (say o_{j1} and o_{j2}) selected alternatives (or corresponding outcomes) is preferred. One can distinguish two situations:

- one of these alternatives dominates the other; in this case the dominating outcome is clearly preferred.
- if the alternatives do not dominate each other, then one cannot objectively decide which one is better; however the user either (subjectively) prefers one of them or cannot decide which one is preferred.

Generally, it is clear that if an outcome dominates another one, then it is better than the other one. In truly multicriteria problems however, there is no alternative that dominates all other alternatives. In other words, one cannot distinguish a best (in terms of strict mathematical relations) alternative because the nondominated outcomes are incomparable on the basis of the specified set of criteria. However, a user usually has preferences which help him/her to select an alternative that fits best these preferences.

A preference structure [24] (that can be used for definition of advanced preference models) is a collection of binary relations defined on the set of alternatives Q such that exactly one relation is satisfied.

The simplest preference model assumes that while comparing two different elements of the set Q we can distinguish only two situations:

- preference of one element to the other (relation $\succ$), or
- indifference of one element to the other (relation $\sim$).

Above simple preference model can be defined by *preference structure* composed of two disjoint binary relations on $Q \times Q$:

$$\langle \succ, \sim \rangle. \tag{2}$$

Note that $\succ$ is asymmetric while $\sim$ is reflexive and symmetric.

We summarize here the basic features of the preference model:

1. the preference model (2) is called complete if for any pair of alternatives (o_{j1}, o_{j2}) either $o_{j1} \succ o_{j2}$ or $o_{j2} \succ o_{j1}$, or $o_{j1} \sim o_{j2}$;
2. the preference model (2) is called transitive, if for any three alternatives o_{j1}, o_{j2}, o_{j3} the following implications hold:
   - if $o_{j1} \succ o_{j2}$ and $o_{j2} \succ o_{j3}$ then $o_{j1} \succ o_{j3}$, and
   - if $o_{j1} \sim o_{j2}$ and $o_{j2} \sim o_{j3}$ then $o_{j1} \sim o_{j3}$.

By extending properties of the binary relations one can define various more specific preference structures called *orders* e.g. total, weak, semi-order, interval order. For example outranking methods are based on preference structures called partial and quasi order. The details of various preference structures can be found e.g., in [24].

The preference models might have also numerical representations. The most common numerical representations of preference models is a value function $V : Q \to R$ defined for each alternative. In such cases while considering a pair of two alternatives (o_{j1}, o_{j2}):

- alternative o_{j1} is preferred to o_{j2} (i.e., $o_{j1} \succ o_{j2}$), if and only if $V(o_{j1}) > V(o_{j2})$;
- o_{j1} and o_{j2} are indifferent (i.e., $o_{j1} \sim o_{j2}$) if and only if $V(o_{j1}) = V(o_{j2})$.

The preference model defined by a value function is obviously complete and transitive. The value function can also be considered as a simple method of aggregation of the criteria.

A specific form of value function is so called Achievement Scalarizing Function (ASF) introduced by Wierzbicki, see e.g., [32, 33, 34, 36]. ASF can be written in a

form:

$$ASF = V(\boldsymbol{q}, \boldsymbol{q}^a, \boldsymbol{q}^r)$$

where $V(\boldsymbol{q}, \boldsymbol{q}^a, \boldsymbol{q}^r)$ is a strictly monotone function of criteria (increasing for maximized and decreasing for minimized), while $\boldsymbol{q}^a$ and $\boldsymbol{q}^r$ are user-defined parameters representing the values of the aspiration and reservation levels for the corresponding criterion, respectively.

3 Specification and aggregation of preferences

In order to support the user in analysis of Pareto-optimal alternatives one needs to provide an effective way for specification of his/her preferences, and to aggregate them in a way that results in finding a Pareto-alternative that possibly well fits the user preferences. We now discuss these two topics.

3.1 Specification of preferences

Preference information are generally considered in two categories:
- information between the criteria, e.g., relative importance of criteria;
- information within each criterion, e.g., satisfaction/utility levels for different values of a criterion.

Due to the requirements explained in Section 1, the developed methods had to use a very simple way of preference specification that is suitable also for users without analytical skills. For the considered problem the inter-criteria preferences need to be specified; therefore, we refrained from specification of preferences within each criterion. In order to rationally deal with criteria types (maximized or minimized) and very diverse orders of criteria value magnitudes, all criteria values are linearly mapped in the [0, 1] interval, where 0 and 1 correspond to the worst and best value, respectively. Moreover, the lack of specification by the user of intra-criterion preferences is to some extend compensated by the pairwise outperformance measures presented in Section 4.

It has been agreed that specification of relative importance of each criterion using the importance categories was the most suitable way for specification of preferences. Therefore, the preferences for each criterion are specified interactively by selecting one of eight levels which is interpreted as the corresponding value of $ri_i, i \in I$ as follows:
- $ri_i = 4$ denotes average importance;
- ri_i values 5 through 7: more, much more, vastly more, important than average, respectively;
- ri_i values 3 through 1: less, much less, vastly less, important than average, respectively;
- $ri_i = 0$ temporally ignore the criterion.

The non-zero values of $\boldsymbol{ri}$ are mapped into weights $w_i, i \in I$ in one (depending on the selected method) of the following ways. The ignored criteria (and their children in the criteria hierarchy, if one is specified) are excluded from the set of criteria used for defining the weights.

The first is the simplest linear (standard) mapping defined by:

$$\omega_i = ri_i/6, \qquad i = 1, \ldots, n \tag{3}$$

The second is the multiplicative mapping which is less popular than the linear one, but it has a number of advantages (see e.g., [14]); it is therefore used by all methods described in this paper. The multiplicative mapping is defined by:

$$\omega_i = (\sqrt{2})^{x_i} \tag{4}$$

where x_i is selected from the ri_i-th position from the following vector

$$\{-8, -4, -2, 0, 2, 4, 8\} \tag{5}$$

In other words, the values of weights can be selected from the vector

$$\{1/16, 1/4, 1/2, 1, 2, 4, 16\} \tag{6}$$

from the position defined by the relative importance button.

For both methods the vector ω is normalized to get

$$\bar{w}_i = \omega_i / \sum_{i=1}^{n} \omega_i, \qquad i = 1, \ldots, n \tag{7}$$

3.1.1 Weights for criteria hierarchy

If a criteria hierarchy is defined then the following procedure is applied:

1. Compute weights $\bar{\boldsymbol{w}}$ defined by (7).
2. Define sets $S_k, k = 1, \ldots, K$ composed of siblings (i.e., nodes having a common parent node) of criteria.
3. Normalize subsets of siblings:

$$\hat{w}_l = \bar{w}_l / \sum_{l=1}^{L_k} \bar{w}_l, \qquad l \in S_k, k = 1, \ldots, K \tag{8}$$

   where L_k is the number of elements in S_k.
4. For each leaf-criterion define

$$w_i = \prod_{k \in M_i} \hat{w}_k, \qquad i = 1, \ldots, n \tag{9}$$

   where set M_i is composed of indices of the following criteria:
   - i-th leaf criterion,
   - intermediate-levels criteria belonging to the branch of the active criteria tree leading to the i-th criterion.

Note that the weights $\boldsymbol{w}$ generated by the above procedure are already normalized in the sense that $\sum_{i=1}^{n} w_i = 1$.

3.2 Aggregation of preferences

In order to select a satisfactory efficient solution, most of multiple criteria methods aggregate the individual outcomes with some scalarizing functions or relations based on some aggregations. The scalarizing functions may have various constructions and properties depending on the specific approach to preference modeling applied in several methods. Nevertheless, most scalarizing functions can be viewed as two-stage transformation of the original outcomes:

- First, the individual outcomes are rescaled to some uniform measures of achievements with respect to several criteria and preference parameters. Thus, the individual achievement functions a_i are built to measure actual achievement of each outcome in a uniform scale [0, 1].
- Second, the outcomes transformed into a uniform scale of individual achievements are aggregated at the second stage to form a final scalarization. The aggregation may measure, for instance, the average or the worst individual achievement. Typically the aggregation is impartial or symmetric with respect to the individual achievements thus is treats all individual achievements as equally important as long as there is no criteria importance introduced.

In the methods presented in this paper the user specifies his/her preferences as relative criteria importance. Therefore, the corresponding weights are used in the aggregations of preferences. The weights representing criteria importance can be introduced into methods either within the aggregation level or within the individual achievement model. Here we outline both approaches. Each of them is used for the same three pairwise outperformance measures. In other words, we will present six methods organized into two sets characterized by the way in which the weights are used for aggregation of preferences. These two sets of pairwise outperformance methods are presented in Sections 4 and 5, respectively.

3.2.1 Weights within individual achievements

The traditional weighted sum aggregation

$$V = \sum_{i=1}^{n} w_i a_i \tag{10}$$

is one of the oldest approaches to multicriteria analysis. The weights are there typically interpreted in terms of a tradeoff preference model. That means an additional scaling of individual achievements is introduced to transform them into equally important units while the aggregation itself remains impartial (symmetric). Depending on the method (or later applied aggregation) the individual achievement are multiplied either by w_i or by $1/w_i$. This approach is still popular because it is believed to be simple, intuitive, and reliable. Actually however, the weights applied in the form of (10) support poorly analysis of Pareto sets, and are often contra-intuitive. The discussion of this approach is beyond the scope of this paper, but it can be found e.g., in [15, 19, 20].

3.2.2 Weights at the aggregation level

Formula (10) may also be interpreted as the weighted average achievement with importance weights introduced on the aggregation level. This interpretation follows the rule that

the importance weights w_i define a repetition measure within the distribution (population) of achievement values while the impartial aggregation take into account this repetition measures. For example, let us consider two symmetric achievement vectors $\boldsymbol{a}^1 = (0, 1)$ and $\boldsymbol{a}^2 = (1, 0)$ and introducing importance weights $w_1 = 0.75$ and $w_2 = 0.25$ we replace $\boldsymbol{a}^1 = (0, 1)$ with the distribution taking value 0 with the repetition measure 0.75 and taking value 1 with the repetition measure 0.25 while $\boldsymbol{a}^2 = (1, 0)$ is replaced with the distribution taking value 0 with the repetition measure 0.25 and taking value 1 with the repetition measure 0.75. In this specific case, the distributions may easily be equivalently interpreted in terms of four dimensional space of equally important achievements (measure 1/4 each) where the original first achievement has been triplicated, thus $\bar{\boldsymbol{a}}^1 = (0, 0, 0, 1)$ and $\bar{\boldsymbol{a}}^2 = (1, 1, 1, 0)$.

Certainly, different interpretations of the weighted sum aggregation do not change its properties. It shows however, how the importance weights can be utilized in more complicated aggregations. We will use such an approach in Section 5 to exploit the importance weights for defining ordered achievements.

4 Pairwise outperformance aggregation

We present here the background, motivation and implementation of three methods based on the pairwise outperformance aggregation approach. Further on we assume the achievements are normalized to [0, 1], where 0 and 1 correspond to the worst and best values, respectively.

4.1 Standard component-wise aggregation

A natural improvement of the weighted sum aggregation is to transform individual achievements by a nonlinear (utility) function. The scalarizing functions is then defined by:

$$S(o_j) = \sum_{i=1}^{n} w_i u(a_{ij}), \qquad j \in J \tag{11}$$

The utility function $u(a_{ij})$ may be used to amplify the impact of increasing weak values (much) more than that of good values. A concave increasing utility function guarantees that an improvement of smaller value may result in a larger satisfaction increase than the same (in terms of the criterion value) improvement of a larger value. Further, a standard (user defined) importance weights w_i are applied on the aggregation level. Thus, the entire scalarization may be viewed as the weighted average of nonlinear utilities.

As already mentioned, such a scalarizing function can be used for defining outperformance aggregation. Let us consider two alternatives o_j and o_l and apply such a nonlinear aggregation to a simple preference model, e.g.:

$$\begin{aligned} o_j \succ o_l \quad &\Leftrightarrow \quad \sum_{i=1}^{n} w_i [u(a_{kj}) - u(a_{kl})] > 0 \\ o_j \sim o_l \quad &\Leftrightarrow \quad \sum_{i=1}^{n} w_i [u(a_{kj}) - u(a_{kl})] = 0 \end{aligned} \tag{12}$$

Such a preference model is based on scalarizing functions defined for each alternative separately, and therefore does not take into account inter-alternative factors. However, the latter is strongly desired for pairwise comparisons, and this observation motivated the authors to develop a new approach to pairwise outperformance aggregation. This approach assumes that due to the number of alternatives the user cannot make pairwise comparisons directly.

4.2 Outranking Procedures for Aggregation

Various approaches to aggregation of preference-relations are discussed in [3]. One of them is the widely used outranking procedure. Pirlot in [25] presented a common framework for defining some outranking procedures. These procedures use pairwise comparisons instead attempting to numerical evaluation of each alternative using a common scale. The ELECTRE methods are examples of outranking procedures and belong to the class of weighted majority relation with veto. In such procedures the statement *alternative* o_i *outranks* o_j is equivalent to the statement that it is at least as good as o_j. The procedure of checking if an alternative outranks another one is based on semiorder S_i and veto relation V_i. The semiorder S_i is determined in the following way:

$$q_{ij}\ S_i\ q_{il} \Leftrightarrow q_{ij} \geq q_{il} - \tau_i$$

and the veto relation V_i is defined as

$$q_{il}\ V_i\ q_{ij} \Leftrightarrow q_{il} > q_{ij} + \nu_i$$

o_j outranks o_l if the following condition is valid:

$$o_j\ \mathcal{S}\ o_l \iff \begin{cases} \sum_{i \in I: q_{ij} S_i q_{il}} w_i \geq \delta \\ \\ \text{and there is no} \quad i \text{ on which} \quad q_{il}\ V_i\ q_{ij} \end{cases}$$

where w_i denotes the normalized weights and δ stands for the majority threshold that belongs to the $[0.5,\ 1]$ interval. The above formula means that the sum of weights of the criteria for which the criteria values are better with respect to S_i are grater then a given threshold δ, and there is no veto (V_i) on any other criterion. The above procedure is called ELECTRE I. There are other more advanced definitions of the outranking relation $\mathcal{S}$, e.g., ELECTRE II and III, PROMETHEE I and II.

However, the outranking procedure of this type is not applicable to problems with many criteria; as pointed out in [6], the ELECTRE methods are suitable for decision models that include more than five criteria and preferably less than thirteen criteria. Moreover, for problems with many (practically more than 6) alternatives methods based on pairwise comparisons are not likely to be actually used.

4.3 Motivation and basic features

For pairwise comparison it is desired to evaluate i-th achievements from the perspective of both compared alternatives, and then to aggregate such evaluations for all criteria.

Let us consider two alternatives o_j and o_l. While evaluating the i-th achievement value of alternative o_l from the perspective of alternative o_j we consider the difference of the values relative to a_{ij}:

$$dc'_{jli} = \beta(a_{ij})(a_{ij} - a_{il}) \qquad i = 1, 2, \ldots, n \tag{13}$$

where $\beta(\cdot)$ is a convex, decreasing, and positive internal scaling function.

The role of $\beta(\cdot)$ is to amplify differently the impact of a given difference of a criterion values for both alternatives. The amplification for weak[5] criterion values is stronger than for strong (i.e., close to 1) values. To illustrate this feature let us consider equal weights/importance for all criteria, and alternatives o_1 and o_2 defined in Table 4.3.

alternatives achievements	o_1	o_2	. . .	o_m
a_1	0.0	0.1	. . .	1
a_2	1.0	0.9	. . .	1
a_3	0.5	0.5	. . .	0

Typically, o_2 is preferred to o_1 (although the sum of differences in criteria values is equal to 0) because the improvement of the worst value of a_1 is usually preferred over worsening the much better performing a_2 by the same value.

Coming back to comparing the alternatives o_j and o_l we shall also consider the comparison from the perspective of alternative o_l. Symmetrically to (13) we define

$$dc''_{jli} = dc'_{lji} = \beta(a_{il})(a_{il} - a_{ij}) \qquad i = 1, 2, \ldots, n \tag{14}$$

By aggregating both comparisons we define for each criterion the following components dc_{jli} of the outperformance aggregation:

$$dc_{jli} = dc'_{jli} - dc''_{jli} = (\beta(a_{ij}) + \beta(a_{il}))(a_{ij} - a_{il}) \qquad i = 1, 2, \ldots, n \tag{15}$$

Thus, the two factors of components dc_{jli} have the following roles:

- Factor $(a_{ij} - a_{il})$ is a difference between i-th criterion values of both compared alternatives.
- Factor $(\beta(a_{ij}) + \beta(a_{il}))$ averages the amplification of the difference of the compared achievements. The amplification depends on both achievement's values under comparison, and thus averages scaling the difference of the achievements in order to equally treat both alternatives.

One may also note the following properties of (15):

- For absolute large value of $(a_{ij} - a_{il})$ one element of $(\beta(a_{ij}) + \beta(a_{il}))$ is also large, and thus the value of dc_{jli} is large.
- For a small value of $(a_{ij} - a_{il})$ the value of dc_{jli} depends on whether the corresponding achievements are weak (small) or strong (large).

[5]For the applied criteria normalization weak means values close to 0.

4.4 Pairwise Outperformance Aggregation (POA)

Based on the discussion in Section 4.3 we aggregate the components dc_{jli} defined for each criterion by (15) into the following Pairwise Outperformance Measure $POA(o_j, o_l)$ to be used for comparing alternatives (o_j, o_l):

$$POA(o_j, o_l) = \overline{POA}(a^j, a^l) = \sum_{i=1}^{n} w_i dc_{jli} = \sum_{i=1}^{n} w_i(\beta(a_{ij}) + \beta(a_{il}))(a_{ij} - a_{il}) \quad (16)$$

In other words, we define pairwise outperformance aggregation as:

$$\begin{aligned} POA(o_j, o_l) &> 0 \quad \Rightarrow \quad o_j \succ o_l \\ POA(o_j, o_l) &= 0 \quad \Rightarrow \quad o_j \sim o_l \end{aligned} \quad (17)$$

The properties of $POA(\cdot)$ depend on the choice of $\beta(\cdot)$. Two forms of the function $\beta(x)$ (where $x \in [0, 1]$ stands for normalized values of criteria and the parameter $\lambda > 1$) have been analyzed and implemented, namely:

$$\beta(x) = \lambda^{-x} \quad (18)$$

$$\beta(x) = \frac{\lambda - 1}{1 + (\lambda - 1)x} \quad (19)$$

The choice of the form of $\beta(x)$ and its parameter λ not only implies the analytical properties of the $POA(\cdot)$ but also the behavior of the corresponding multicriteria method. From the point of view of multicriteria method implementation the following two elements are important:

- The ratio λ of values of $\beta(\cdot)$ for the worst and best values of normalized criteria:

$$\lambda = \frac{\beta(0)}{\beta(1)} \quad (20)$$

  which characterizes the amplification depending on the performance (weakness or strength) of the corresponding criterion. Note that for $\beta(x)$ defined by either (18) or (19) the ratio λ is actually equal to the parameter λ. Experiments show that values of λ about 10 are satisfactory. However, advanced users should have a possibility to control the value of λ.
- Consistency of the aggregation (16) in the sense of monotonicy with respect to the Pareto dominance relation, i.e.:

$$o_j \succ_P o_l \quad \Rightarrow \quad POA(o_j, o_l) > 0 \quad (21)$$

  If (21) does not hold then application of (18) does not guarantee that a non-dominated alternative will be selected. In order to avoid such situations a preprocessing of alternatives is needed for filtering-out the dominated alternatives before the pairwise outperformance aggregation (16) is applied. Such a preprocessing is very easy for discrete alternatives problems but cannot be applied for MCA of mathematical models (for which an auxiliary parametric optimization problem is generated for each specification of preferences).

Therefore, the analytical properties of $POA(\cdot)$ are discussed in Section 4.5, and its application to multicriteria analysis is presented in Section 7.

4.5 Properties of POA

In this Section we analyze the dependence between the form and parameter of functions $\beta(\cdot)$ defined above, the concavity and monotonicity properties of POA, as well as their relations to the corresponding valued preference relations. These properties of POA are illustrated by simple examples in Section 4.5.2. The transitivity properties are discussed in Section 6.

For the sake brevity we use in this Section a simplified notation for POA. For any alternative o_j we consider a relative outperformance function comparing any achievement vector $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)$ with the achievement vector $\boldsymbol{a}^j$ defined by the alternative o_j, and denote:

$$P_j(\boldsymbol{y}) = \overline{POA}(\boldsymbol{y}, a^j) = \sum_{k=1}^{n} w_k(\beta(y_k) + \beta(a_k^j))(y_k - a_k^j) \tag{22}$$

4.5.1 Concavity and monotonicity

The two propositions below deal with concavity of monotonicity properties of POA for $\beta(\cdot)$ defined by (18) and (19), respectively.

Proposition 1 *For any achievement vector $\boldsymbol{a}^j$ the corresponding relative outperformance function P_j is concave and strictly increasing with respect to each achievement y_k whenever $\beta(x) = \lambda^{-x}$ with $1 \leq \lambda \leq e$.*

Proof. Calculating the partial derivatives of function P_j we get

$$\frac{\partial P_j(\boldsymbol{y})}{\partial y_k} = w_k[(1 - \mu y_k + a_{kj}\mu)\lambda^{-y_k} + \lambda^{-a_{kj}}], \quad k = 1, 2, \ldots, n$$

where $\mu = \ln \lambda$. If $1 \leq \lambda \leq e$, then $0 \leq \mu \leq 1$ and $1 - \mu y_k + a_{kj}\mu \geq 0$ for any $0 < y_k < 1$ and $0 \leq a_{kj} \leq 1$. Therefore,

$$\frac{\partial P_j(\boldsymbol{y})}{\partial y_k} > 0 \qquad \forall 0 < y_k < 1.$$

Further, calculating the second order partial derivatives we get obviously

$$\frac{\partial^2 P_j(\boldsymbol{y})}{\partial y_{k'} \partial y_{k''}} = 0 \qquad \forall k' \neq k''$$

and

$$\frac{\partial^2 P_j(\boldsymbol{y})}{\partial y_k^2} = w_k(\mu^2 y_k - 2\mu + a_{kj}\mu^2)\lambda^{-y_k}, \quad k = 1, 2, \ldots, n$$

If $1 \leq \lambda \leq e$, then $0 \leq \mu \leq 1$ and $(y_k - a_{kj})\mu \leq 2$ for any $0 < y_k < 1$ and $0 \leq a_{kj} \leq 1$. Therefore,

$$\frac{\partial^2 P_j(\boldsymbol{y})}{\partial y_k^2} \leq 0 \qquad \forall 0 < y_k < 1$$

thus guaranteeing the concavity properties. □

Summing up, the POA defined by (16) with $\beta(\cdot)$ defined by (18) is concave and strictly increasing for $\lambda \in [1, e]$. Such a rather small range of values of λ results is a rather small amplification of weak criteria values. Therefore, the corresponding method may either have undesired behavior for some problems (if applied with $\lambda < e$) or does not guarantee finding Pareto-efficient solution for $\lambda > e$.

Although the latter problem may be effectively addressed by filtering-out dominated alternatives in a preprocessing phase of the multicriteria analysis we have found an alternative form of $\beta(\cdot)$ which guarantees concavity and monotonicity of POA for any $\lambda > 1$.

To show this let us now consider $\beta(\cdot)$ defined by (19). By applying $\beta(\cdot)$ defined (19) to (13) one gets

$$dc'_{jlk} = \frac{\lambda - 1}{1 + (\lambda - 1)a_{kj}}(a_{kj} - a_{kl}) = \frac{1}{\tilde{a}_{kj}}(\tilde{a}_{kj} - \tilde{a}_{kl}) \tag{23}$$

where

$$\tilde{a}_{kj} = \frac{1}{\lambda} + (1 - \frac{1}{\lambda})a_{kj} \tag{24}$$

In other words, the criteria values are rescaled by (24) from [0, 1] to $[\frac{1}{\lambda},\ 1]$, which in turn allows for applying the standard inverse-proportional scaling.

Similarly,

$$dc''_{jlk} = dc'_{ljk} = \frac{1}{\tilde{a}_{kl}}(\tilde{a}_{kl} - \tilde{a}_{kj}) \qquad k = 1, 2, \ldots, n \tag{25}$$

and

$$dc_{jlk} = dc'_{jlk} - dc''_{jlk} = (\frac{1}{\tilde{a}_{kj}} + \frac{1}{\tilde{a}_{kl}})(\tilde{a}_{kj} - \tilde{a}_{kl}) = \frac{\tilde{a}_{kj}}{\tilde{a}_{kl}} - \frac{\tilde{a}_{kl}}{\tilde{a}_{kj}} \qquad k = 1, 2, \ldots, n \tag{26}$$

The corresponding relative outperformance function (22) comparing any achievement vector $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)$ with achievements of o_j takes then form

$$P_j(\boldsymbol{y}) = \sum_{k=1}^{n} w_k(\frac{\tilde{y}_k}{\tilde{a}_{kl}} - \frac{\tilde{a}_{kl}}{\tilde{y}_k}) = \sum_{k=1}^{n} w_k(\frac{1 + (\lambda - 1)y_k}{1 + (\lambda - 1)a_{kl}} - \frac{1 + (\lambda - 1)a_{kl}}{1 + (\lambda - 1)y_k}) \tag{27}$$

Proposition 2 *For any achievement vector $\boldsymbol{a}^j$ the corresponding relative outperformance function (27) is concave and strictly increasing with respect to each achievement y_k whenever β is defined by (19) with $\lambda > 1$.*

Proof. Calculating the partial derivatives of function P_j we get

$$\frac{\partial P_j(\boldsymbol{y})}{\partial y_k} = w_k[\frac{(\lambda - 1)(1 + (\lambda - 1)a_{kj})}{(1 + (\lambda - 1)y_k)^2} + \frac{\lambda - 1}{1 + (\lambda - 1)a_{kj}}], \quad k = 1, 2, \ldots, n$$

If $\lambda > 1$, then

$$\frac{\partial P_j(\boldsymbol{y})}{\partial y_k} > 0 \qquad \forall y : 0 \leq y_k \leq 1.$$

Further, calculating the second order partial derivatives we get obviously

$$\frac{\partial^2 P_j(\boldsymbol{y})}{\partial y_{k'} \partial y_{k''}} = 0 \qquad k' \neq k''$$

and

$$\frac{\partial^2 P_j(\boldsymbol{y})}{\partial y_k^2} = w_k \frac{-2(\lambda-1)^2(1+(\lambda-1)a_{kj})}{(1+(\lambda-1)y_k)^3}, \quad k = 1, 2, \ldots, n$$

If $\lambda > 1$, then

$$\frac{\partial^2 P_j(\boldsymbol{y})}{\partial y_k^2} \leq 0 \qquad \forall y : 0 \leq y_k \leq 1$$

thus guaranteeing the concavity properties. □

Note that by choosing a (very) large value of λ for β defined by (19) the achievements values rescaled by (24) can be made very close to the original achievements, and the POA aggregation will be driven by improving the worst values of achievements. This is, in a sense, consistent with the Rawlsian approach (improve the weakest) which is a methodological justification of using the max-min scalarizing functions in the reference point approaches.

4.5.2 Illustration of POA properties

Now we illustrate some properties of $P_j(\boldsymbol{y})$ using a sample problem with two criteria and nine alternatives. We focus our discussion on two pairs of alternatives $(o_6,\ o_8)$ and $(o_3,\ o_7)$. Equal relative importance of criteria is assumed, and the criteria values are normalized (value of 1 corresponds to the best value). These four alternatives differ substantially, but they were defined in such a way that the pairs $(o_6,\ o_8)$ and $(o_3,\ o_7)$ have the same differences of achievement values for criterion 1 and for criterion 2, respectively. The achievement values are shown in Table 1.

	o_6	o_8	o_3	o_7	o_6 - o_8	o_3 - o_7
a_1	0.80	1.00	0.20	0.40	-0.20	-0.20
a_2	0.05	0.00	0.75	0.70	0.05	0.05

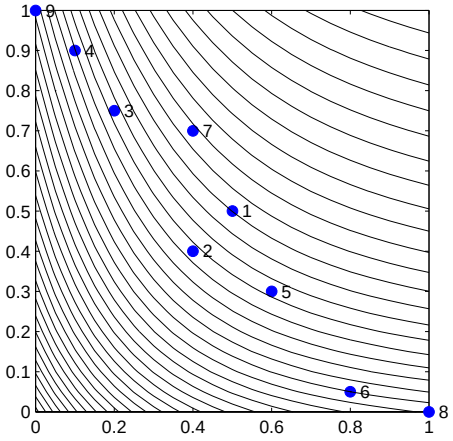
Table 1: Values of achievements a_1 and a_2 for alternatives o_6, o_8, o_3, o_7, and their differences for pairs $(o_6,\ o_8)$ and $(o_3,\ o_7)$.

We focus on two pairs of comparisons, namely $(o_6,\ o_8)$ and $(o_3,\ o_7)$. Let us observe that:

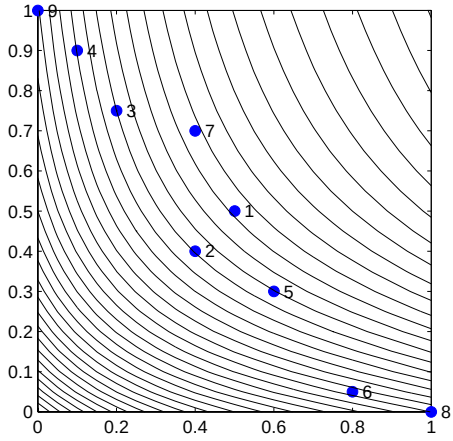
- both alternatives of the pair $(o_6,\ o_8)$ perform very well in respect of criterion 1 and very poorly on criterion 2;
- alternatives $(o_3,\ o_7)$ performs moderately on criterion 1 (20% to 40% of the best value, respectively) but rather well on criterion 2 (75% to 70% of the best value, respectively).

For both pairs, the trade-off (in terms of the difference of the criteria values) between the two corresponding alternatives is the same: 20% of improvement/worsening of criterion 1 for 5% of worsening/improvement of criterion 2. Therefore, any method that does not take into account inter-criteria relations[6] will result in either:

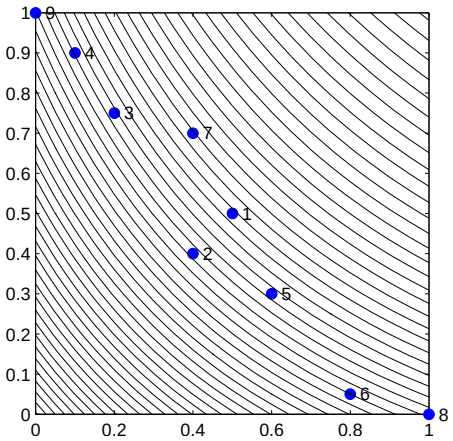
- $o_6 \succ o_8$ and $o_3 \succ o_7$, or
- $o_6 \prec o_8$ and $o_3 \prec o_7$.



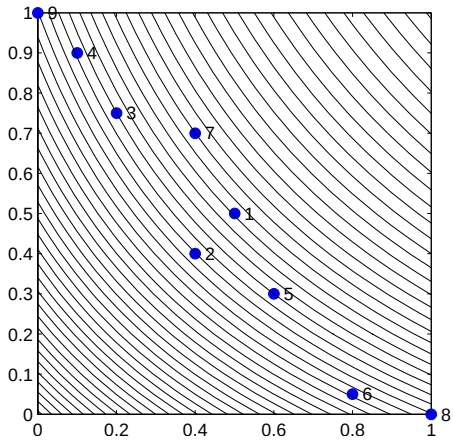
(a) Function $P_6(\boldsymbol{y})$, $\lambda = 10$



(b) Function $P_3(\boldsymbol{y})$, $\lambda = 10$



(c) Function $P_6(\boldsymbol{y})$, $\lambda = e$



(d) Function $P_3(\boldsymbol{y})$, $\lambda = e$

Figure 1: Isoline contours of functions $P_6(\boldsymbol{y})$ and $P_3(\boldsymbol{y})$ for λ equal to 10 and e.

All alternatives are shown in Figure 1 as points marked with the corresponding numbers 1 through 9. The coordinates of the points correspond to the criteria values (criterion 1 is shown on the horizontal axis). It is easy to see that all alternatives but o_2 are Pareto optimal.

[6] Such methods use separable component achievement scalarizing functions, i.e., functions built for each criterion separately.

Figures 1(a) through 1(d) provide isoline contours for different functions $P_j(\cdot)$; namely, from the perspective of o_6 ($P_6(\boldsymbol{y})$ Fig. 1(a) and 1(c)), and of o_3 ($P_3(\boldsymbol{y})$ Fig. 1(b) and 1(d)), respectively. These two pairs of Figures differ by the applied function $\beta(\cdot)$.

The values of functions $P_6(\cdot)$ and $P_3(\cdot)$ are in the ranges:

- $P_6(\boldsymbol{y})$: $[-1.021,\ 0.993]$, and $[-1.258,\ 1.412]$, for λ equal to 10 and e, respectively.
- $P_3(\boldsymbol{y})$: $[-1.21,\ 0.654]$ and $[-1.469,\ 1.161]$, for λ equal to 10 and e, respectively.

The contour lines are displayed for the values that differ by 0.05, and increase in the up and right direction. In other words, the outperformance relation can be easily seen by comparing any two alternatives and the corresponding isolines.

We now illustrate some of the properties of the POA by comparing the above defined two pairs of alternatives, and two forms of $\beta(\cdot)$. Let us first consider the isoline contours of function $P_6(\boldsymbol{y})$ for $\beta(x) = 10^{-x}$ shown in Fig. 1(a). From the isoline contours around alternatives 6 and 8 one can see that o_6 is preferred over o_8. However, the result of such a pairwise comparison is different for $P_6(\boldsymbol{y})$ that uses $\beta(x) = e^{-x}$; the latter is illustrated in Fig. 1(c). This example illustrates the scaling role of function $\beta(\cdot)$, and is easy to explain by considering the form of (22). The ratio of improving the value of criterion 1 to compromising the value of criterion 2 (between alternative 6 and 8) is equal to 4. To compensate this (in the sense of preferring the small improvement of the weakly performing criterion 2 over much the larger improvement of of very well performing criterion 1) the sum of $\beta(\cdot)$ for criterion 2 needs to be more than four times larger than for criterion 1. By easy calculations one can show that for $\lambda \geq 5$ this is the case, and $o_6 \succ o_8$, while $\lambda \leq 4.9$ results in $o_6 \prec o_8$.

Figures 1(b) and 1(d) show isoline contours of functions $P_3(\boldsymbol{y})$ for $\beta(\cdot) = 10^{-x}$, and $\beta(\cdot) = e^{-x}$, respectively. It is easy to see that in both cases $o_7 \succ o_3$. The explanation of this situation is obvious: for these two alternatives criterion 1 is weaker (than criterion 2), therefore there is a good justification[7] for such a preference.

4.5.3 POA and preference relations

The aggregated outperformance measure (16) allows us to build the corresponding valued preference relation. Note that values of the component measures dc_{jlk} and dc_{ljk} have different signs but equal absolute values. Similarly, $d_{jl} = -d_{lj}$. Hence, we can define the preference model (18) as

$$(o_j \succ o_l \quad \Leftrightarrow \quad d_{jl} > 0) \qquad \text{and} \qquad (o_j \sim o_l \quad \Leftrightarrow \quad d_{jl} = 0) \tag{28}$$

We will refer to this preference model as the outperformance relation. Respectively, we will say that alternative o_j weakly outperforms alternative o_l ($o_j \succeq o_l$) if

$$o_j \succeq o_l \quad \Leftrightarrow \quad d_{jl} \geq 0 \tag{29}$$

The weak outperformance relation might be considered an outranking relation with respect to the classical general definition of the outranking relation as a binary relation defined on $Q \times Q$ such that $o_j \succeq o_l$ if there are enough arguments to decide that o_j is at least as good as o_l, while there is no essential reason to refuse that statement [26, 30]. Although it is quite different from the commonly used outranking relations. Therefore, we use the different name.

[7] Let us recall that the relative importance of criteria is assumed to be equal.

The outperformance relation can be lexicographically enhanced by comparison of the original differences when the scaled once leads to the equal results, i.e.

$$\begin{aligned} o_j \succ_e o_l \quad &\Leftrightarrow \quad d_{jl} > 0 \quad \text{or} \quad \left(d_{jl} = 0 \quad \text{and} \quad \sum_{k=1}^{n} w_k(a_{jk} - a_{lk}) > 0 \right) \\ o_j \sim_e o_l \quad &\Leftrightarrow \quad d_{jl} = 0 \quad \text{and} \quad \sum_{k=1}^{n} w_k(a_{jk} - a_{lk}) = 0 \end{aligned} \tag{30}$$

Note that while the enhancement narrows the indifference relation it does not affect the weak outperformance relation as

$$o_j \succeq_e o_l \quad \Leftrightarrow \quad d_{jl} \geq 0 \quad \Leftrightarrow \quad o_j \succeq o_l$$

5 Ordered Pairwise Outperformance Aggregation (OPOA)

5.1 Background

Standard multiple criteria optimization problems with a general preference structure essentially assume the criteria to be incomparable, i.e. having no basis of comparison. Nevertheless, in our approach as in many typical multiple criteria optimization methods the individual achievement functions are built to measure actual achievement of each outcome with respect to the corresponding preference parameters. Thus, all the outcomes are transformed into a uniform scale of individual achievements within intervals $[0, 1]$. This allows one to compare achievement values for various criteria and, in particular, to compare each other small values and large values of achievements, respectively.

In the case of unweighted (equally important) attributes, the outperformance aggregation can easily be applied to the ordered achievement values thus guaranteeing comparison of the worst results, the second worst etc. This can be formalized as follows. First we introduce the ordering map $\Theta : R^n \to R^n$ such that $\Theta(\boldsymbol{y}) = (\theta_1(\boldsymbol{y}), \theta_2(\boldsymbol{y}), \ldots, \theta_n(\boldsymbol{y}))$, where $\theta_1(\boldsymbol{y}) \leq \theta_2(\boldsymbol{y}) \leq \cdots \leq \theta_n(\boldsymbol{y})$ and there exists a permutation τ of set I such that $\theta_i(\boldsymbol{y}) = y_{\tau(i)}$ for $i \in I$.

We define the single criterion outperformance components in a similar way as in Section 4.3:

$$odc_{jlk} = (\beta(\theta_k(\boldsymbol{a}^j)) + \beta(\theta_k(\boldsymbol{a}^l)))(\theta_k(\boldsymbol{a}^j) - \theta_k(\boldsymbol{a}^l)) \qquad k = 1, 2, \ldots, n \tag{31}$$

In particular the role of function $\beta(\cdot)$ is the same as discussed in Section 4.3, i.e., to amplify the influence of weak achievements values more than that of good ones.

The ordered pairwise outperformance relation is based on the aggregated quantities:

$$od_{jl} = \frac{1}{n} \sum_{k=1}^{n} odc_{jlk} = \frac{1}{n} \sum_{k=1}^{n} (\beta(\theta_k(\boldsymbol{a}^j)) + \beta(\theta_k(\boldsymbol{a}^l)))(\theta_k(\boldsymbol{a}^j) - \theta_k(\boldsymbol{a}^l)) \tag{32}$$

In the ordered outperformance aggregation (32) only distribution of achievements values is evaluated. When two alternatives o_j and o_l result in different achievement vectors

$\boldsymbol{a}^j$ and $\boldsymbol{a}^l$ that are built of identically distributed achievement values, they lead to the zero value of the ordered outperformance value. Indeed, two achievement vectors $\boldsymbol{a}^j$ and $\boldsymbol{a}^l$ which differ only with the order of individual achievement values, i.e.,

$$(a_{1j}, a_{2j}, \ldots, a_{nj}) = (a_{\sigma(1)l}, a_{\sigma(2)l}, \ldots, a_{\sigma(n)l})$$

for some permutation σ of set I, then $\Theta(\boldsymbol{a}^j) = \Theta(\boldsymbol{a}^l)$ and thereby $od_{jl} = od_{lj} = 0$. For instance, having $\boldsymbol{a}^j = (0.1, 0.2, 0.3)$ and $\boldsymbol{a}^l = (0.3, 0.1, 0.2)$ we get unordered outperformance measure $d_{jl} = 2\beta(-0.2) - \beta(-0.3) - \beta(-0.3)$ which is negative due to convexity of β while obviously for the ordered measure $od_{jl} = od_{lj} = 0$.

The ordered outperformance aggregation (32) is built for equally important achievements. Importance weights of achievements can be introduced into the aggregation following the rule that importance weights w_i define a repetition measure within the distribution (population) of achievement values, similarly to [21, 22, 23]. The outperformance components are then calculated within specific quantiles of this distribution, small enough to guarantee constant values of the ordered achievements for both alternatives. For instance, considering two symmetric achievement vectors $\boldsymbol{a}^1 = (0, 1)$ and $\boldsymbol{a}^2 = (1, 0)$ results in the corresponding ordered outperformance measure od_{12} equal 0. While introducing importance weights $w_1 = 0.75$ and $w_2 = 0.25$ we replace $\boldsymbol{a}^1 = (0, 1)$ with the distribution taking value 0 with the repetition measure 0.75 and taking value 1 with the repetition measure 0.25 while $\boldsymbol{a}^2 = (1, 0)$ is replaced with the distribution taking value 1 with the repetition measure 0.75 and taking value 0 with the repetition measure 0.25. In this specific case, the distributions may easily be equivalently interpreted in terms of four dimensional space of equally important achievements (measure 1/4 each) where the original first achievement has been triplicated, thus $\bar{\boldsymbol{a}}^1 = (0, 0, 0, 1)$ and $\bar{\boldsymbol{a}}^2 = (1, 1, 1, 0)$. The ordered outperformance aggregation calculated for subsequent quantiles of size $1/4$ results then in the value

$$\begin{aligned} od_{12} = \quad & 0.25(1+1)(0-0) + 0.25(1+0.1)(0-1) + 0.25(1+0.1)(0-1) \\ + \quad & 0.25(0.1+0.1)(1-1) = -0.55 \end{aligned}$$

Certainly, one does not need to transform all the cases to equally important achievements in order to calculate appropriate aggregation value. The pairwise analysis may be split into (various size) quantile intervals of constant ordered achievements for both alternatives instead of quantile intervals of equal size. For our straightforward example it takes the following form

$$od_{12} = 0.25(1+1)(0-0) + 0.5(1+0.1)(0-1) + 0.25(0.1+0.1)(1-1) = -0.55$$

Actually, independently from the importance weighting patterns, there are no more than $2n$ such quantile intervals to be analyzed. It can also be mathematically formalized as follows. First, we introduce the right-continuous cumulative distribution function (cdf) of achievement values:

$$F_j(d) = \sum_{i=1}^{n} w_i \delta_{ij}(d) \tag{33}$$

where $\delta_{ij}(d) = 1$ if $a_{ij} \leq d$ and 0 otherwise. Next, we introduce the quantile function $F_j^{(-1)}$ as the left-continuous inverse of the cumulative distribution function F_j, ie.,

$F_j^{(-1)}(\xi) = \inf\{\eta : F_j(\eta) \geq \xi\}$ for $0 < \xi \leq 1$. Finally,

$$od_{jl} = \int_0^1 (\beta(F_j^{(-1)}(\xi)) + \beta(F_l^{(-1)}(\xi)))(F_j^{(-1)}(\xi) - F_l^{(-1)}(\xi))\, d\xi \tag{34}$$

Note that in the case of equal weights $w_i = 1/n$ one gets

$$F_j^{(-1)}(\xi) = \theta_k(\boldsymbol{a}^j) \quad \text{for } \frac{k-1}{n} < \xi \leq \frac{k}{n}, \quad k = 1, 2, \ldots, n$$

and similarly

$$F_l^{(-1)}(\xi) = \theta_k(\boldsymbol{a}^l) \quad \text{for } \frac{k-1}{n} < \xi \leq \frac{k}{n}, \quad k = 1, 2, \ldots, n$$

thus allowing us to reduce formula (34) to the unweighted formula (32). However, in general case the formula (34) cannot be simplified in this way. Nevertheless, since both $F_j^{(-1)}(\xi)$ and $F_l^{(-1)}(\xi)$ are stepwise functions with n breakpoints, the entire integrated function is also stepwise with no more than $2n$ breakpoints. Therefore, the ordered outperformance aggregation (34) can simply be computed as a sum of $2n$ terms. The computation procedure can be formulated as follows:

Data:

- ordered vector $OA_j = \Theta(\boldsymbol{a}^j)$ and respectively ordered weights vector $OW_j = \Theta^j(\boldsymbol{w})$;
- ordered vector $OA_l = \Theta(\boldsymbol{a}^l)$ and respectively ordered weights vector $OW_l = \Theta^l(\boldsymbol{w})$;

Initialize working variables:

- indices: $I_j = I_l = 1$;
- current value: $OD_{jl} = 0.0$;

Compute:

```
while ((I_j <= n)&&(I_l <= n)) do {
  if (OW_j[I_j] <= OW_l[I_l]) {
    OD_jl += OW_j[I_j](β(OA_j[I_j]) + β(OA_l[I_l]))(OA_j[I_j] − OA_l[I_l]);
    OW_l[I_l] −= OW_j[I_j];
    ++I_j;
  } else {
    OD_jl += OW_l[I_l](β(OA_j[I_j]) + β(OA_l[I_l]))(OA_j[I_j] − OA_l[I_l]);
    OW_j[I_j] −= OW_l[I_l];
    ++I_l;
  }
}
```

5.2 Properties

The ordered outperformance aggregation (34) retains the property that only distribution of achievements values is evaluated. When two alternatives o_j and o_l result in different

achievement vectors $\boldsymbol{a}^j$ and $\boldsymbol{a}^l$ but both built of identically distributed achievement values taking into account the importance weights w_i, then they lead to the zero value of the ordered outperformance value. Actually, two achievement vectors $\boldsymbol{a}^j$ and $\boldsymbol{a}^l$ result then in the same cumulative distribution functions F_j and F_l. Therefore, $F_j^{(-1)} = F_l^{(-1)}$ and thereby $od_{jl} = od_{lj} = 0$. Moreover, the ordered outperformance aggregation (34) has the properties of monotonicity and equitability [12, 13].

Proposition 3 *The ordered outperformance aggregation (34) is monotonic and equitable.*

Proof. The monotonicity follows simply from the fact that $F_j^{(-1)}(\xi) \geq F_l^{(-1)}(\xi)$ for all $0 < \xi \leq 1$ while functions β are positive. Hence, $\boldsymbol{a}^j \geq \boldsymbol{a}^l$ implies $od_{jl} \geq 0$.

In order to prove the equitability we need to reformulate the aggregation formula (34).

$$\begin{aligned} OPOA(o_j, o_l) = od_{jl} &= \int_0^1 \beta(F_j^{(-1)}(\xi))(F_j^{(-1)}(\xi) - F_l^{(-1)}(\xi))\, d\xi \\ &\quad + \int_0^1 \beta(F_l^{(-1)}(\xi))(F_j^{(-1)}(\xi) - F_l^{(-1)}(\xi))\, d\xi \end{aligned} \tag{35}$$

where

$$F_j^{(-1)}(\xi) = \theta_k(\boldsymbol{a}^j) \quad \text{for } \xi_{k-1}^j < \xi \leq \xi_k^j, \quad k = 1, 2, \ldots, n$$

with $\xi_0^j = 0$ and $\xi_k^j = \sum_{i=1}^k w_{\tau^j(i)}$ for $k = 1, 2, \ldots, n$, and

$$F_l^{(-1)}(\xi) = \theta_k(\boldsymbol{a}^l) \quad \text{for } \xi_{k-1}^l < \xi \leq \xi_{k-1}^l, \quad k = 1, 2, \ldots, n$$

with $\xi_0^l = 0$ and $\xi_k^l = \sum_{i=1}^k w_{\tau^l(i)}$. Hence,

$$\begin{aligned} od_{jl} &= \sum_{i=1}^n [\beta(\theta_k(\boldsymbol{a}^j)) \int_{\xi_{k-1}^j}^{\xi_k^j} (F_j^{(-1)}(\xi) - F_l^{(-1)}(\xi))\, d\xi] \\ &\quad + \sum_{i=1}^n [\beta(\theta_k(\boldsymbol{a}^l)) \int_{\xi_{k-1}^l}^{\xi_k^l} (F_j^{(-1)}(\xi) - F_l^{(-1)}(\xi))\, d\xi] \\ &= \sum_{i=1}^n [\beta_k^j \int_0^{\xi_k^j} (F_j^{(-1)}(\xi) - F_l^{(-1)}(\xi))\, d\xi] \\ &\quad + \sum_{i=1}^n [\beta_k^l \int_0^{\xi_k^l} (F_j^{(-1)}(\xi) - F_l^{(-1)}(\xi))\, d\xi] \end{aligned} \tag{36}$$

where

$$\beta_n^j = \beta(\theta_n(\boldsymbol{a}^j)), \quad \beta_k^j = \beta(\theta_k(\boldsymbol{a}^j)) - \beta(\theta_{k+1}(\boldsymbol{a}^j)) \quad k = 1, 2, \ldots, k-1$$

and

$$\beta_n^l = \beta(\theta_n(\boldsymbol{a}^l)), \quad \beta_k^l = \beta(\theta_k(\boldsymbol{a}^l)) - \beta(\theta_{k+1}(\boldsymbol{a}^l)) \quad k = 1, 2, \ldots, k-1$$

Hence, $\int_0^{\xi} F_j^{(-1)}(\xi)\, d\xi \geq \int_0^{\xi} F_l^{(-1)}(\xi)\, d\xi$ for all $0 < \xi \leq 1$ implies $od_{jl} \geq 0$. □

Similarly to the unordered approach, we introduce the ordered outperformance preference model as

$$o_j \succ o_l \quad \Leftrightarrow \quad od_{jl} > 0 \qquad \text{and} \qquad o_j \sim o_l \quad \Leftrightarrow \quad od_{jl} = 0 \tag{37}$$

The model can be lexicographically enhanced by comparison of the unscaled ordered differences when the scaled ones lead to the equal results. However, for unscaled ordered differences we get

$$\begin{aligned} \int_0^1 (F_j^{(-1)}(\xi) - F_l^{(-1)}(\xi))\, d\xi &= \int_0^1 F_j^{(-1)}(\xi)\, d\xi - \int_0^1 F_l^{(-1)}(\xi)\, d\xi \\ &= \sum_{k=1}^{n} w_k a_{jk} - \sum_{k=1}^{n} w_k a_{lk} \end{aligned}$$

Hence, comparison of the unscaled ordered differences is equivalent to the comparison of the average achievements and the enhanced preference model can be formalized as follows

$$\begin{aligned} o_j \succ_e o_l \quad &\Leftrightarrow \quad od_{jl} > 0 \quad \text{or} \quad \left(od_{jl} = 0 \quad \text{and} \quad \sum_{k=1}^{n} w_k (a_{jk} - a_{lk}) > 0 \right) \\ o_j \sim_e o_l \quad &\Leftrightarrow \quad od_{jl} = 0 \quad \text{and} \quad \sum_{k=1}^{n} w_k (a_{jk} - a_{lk}) = 0 \end{aligned} \tag{38}$$

The ordered outperformance relation defined by (38) used with the linear search algorithm has been implemented as three methods (OPOA, OPOA-E, OPOA-Inv) depending on the applied form of the $\beta(\cdot)$ function, see Section 7.

6 Transitivity property and Net-Flow approaches

6.1 Transitivity property of POA and OPOA

Unfortunately, neither standard nor enhanced preference models developed for POA and OPOA meet the transitivity requirement. This means that although all the alternatives are comparable there may not exist the best alternative (weakly) outperforming all the others either the maximal (nondominated) alternative not outperformed (strictly) by any other alternative. Such a situation is also known as the Condorcet paradox (see e.g., [3]), and it can be illustrated with a simple example of three alternatives with two attributes.

Let us consider alternatives o_1, o_2 and o_3 with the corresponding achievement vectors $\boldsymbol{a}^1 = (0.7255, 0.3110)$, $\boldsymbol{a}^2 = (1.0, 0.2285)$ and $\boldsymbol{a}^3 = (0.2230, 0.9992)$, respectively. Using the pairwise outperformance aggregation (16) with $\beta(x) = 10^{-x}$ we get:
$POA(o_1, o_2) = 0.01$, $POA(o_2, o_3) = 0.0101$, and $POA(o_3, o_1) = 0.01$.

Hence, $o_1 \succ o_2$, $o_2 \succ o_3$, and $o_3 \succ o_1$, which contradicts the transitivity. Indeed, alternatives o_1, o_2 and o_3 generate a cycle according to the pairwise outperformance aggregation (16), in which each alternative outperforms an alternative and is outperformed by another alternative.

The transitivity property is summarized by the following proposition.

Proposition 4 *Any alternative o_b selected by the linear search algorithm either weakly outperforms all the alternatives o_j ($j \in J$) or belongs to a cycle $o_b \succeq o_{j_1} \succeq o_{j_2} \succeq \ldots \succeq o_{j_k} \succeq o_b$ (with possible alternative repetitions $o_{j_{i'}} = o_{j_{i''}}$) such that for any alternative o_j ($j \in J$) there exists (weakly) outperforming it alternative $o_{j'}$ belonging to the cycle.*

Proof. If o_1 remains the selected alternative after the linear search algorithm then obviously $o_j \not\succ o_1$ for any $j = 2, 3, \ldots, m$ and thereby $o_1 \succeq o_j$ for any $j \in J$. Otherwise, the algorithm builds the sequence of subsequent outperforming alternatives $o_1 = o_{j_1} \prec o_{j_2} \prec \ldots \prec o_{j_p} = o_b$ and it identifies the following relations:
$o_{j_1} \succeq o_{j_1+1}, o_{j_1} \succeq o_{j_1+2}, \ldots\ o_{j_1} \succeq o_{j_2-1}$,
$o_{j_2} \succeq o_{j_2+1}, o_{j_2} \succeq o_{j_2+2}, \ldots\ o_{j_2} \succeq o_{j_3-1}$,
$\ldots$,
$o_{j_p} \succeq o_{j_p+1}, o_{j_p} \succeq o_{j_p+2}, \ldots\ o_{j_p} \succeq o_n$.
If o_b does not outperform weakly all the alternatives then there exist an alternative outperforming o_b. If alternative o_{j_i} outperforms o_b then we get cycle $o_b \prec o_{j_i} \prec o_{j_{i+1}} \prec \ldots \prec o_b$. If alternative o_{j_i+t} outperforms o_b then we get a longer cycle $o_b \prec o_{j_i} \prec o_{j_{i+1}} \prec \ldots \prec o_b$. If some alternative o_{j_k} ($k < i$) outperforms all alternatives of the cycle then we need to extend the cycle with additional cycle $o_{j_i} \prec o_{j_k} \prec o_{j_{k+1}} \prec \ldots \prec o_{j_i}$. If some alternative o_{j_k+t} ($k < i$) outperforms all alternatives of the cycle then we need to extend the cycle with additional cycle $o_{j_i} \prec o_{j_k+t} \preceq o_{j_k} \prec o_{j_{k+1}} \prec \ldots \prec o_{j_i}$. After possible more extension we get finally a cycle with possible repetitions such that for any alternative o_j ($j \in J$) there exists (weakly) outperforming it alternative $o_{j'}$ belonging to the cycle. □

Generally, for large problems it is difficult to either proof or disproof the transitivity property for pairwise outperformance methods (both POA and OPOA approaches). Actually, for some methods rather extensive tests (see Section 8) were needed to detect the Condorcet paradox. Therefore, use of these methods is risky in the sense that either dominated alternatives may be returned as Pareto-efficient or the algorithm may loop infinitely. On the other hand, the pairwise comparison methods are attractive because of their convincing background.

Fortunately, it is possible to exploit the advantages of the pairwise outperformance method by applying the approach outlined below.

6.2 Net-Flow methods

The pairwise outperformance relations are built as the corresponding valued preference relations. Therefore, in order to guarantee the existence of a best alternative one may use the standard way to obtain a ranking method associated with valued preference relations, the so-called Net Flow Method. Actually, the Net Flow method is the only ranking method that is neutral, strongly monotonic and independent of circuits [2].

For each alternative o_j we define the aggregate outperformance measure ds:

$$ds_j = 0.5 \sum_{l \in J} (d_{jl} - d_{lj}) \tag{39}$$

where, depending on the method, either

$$d_{jl} = POA(o_j, o_l) \tag{40}$$

or

$$d_{jl} = OPOA(o_j, o_l) \tag{41}$$

Note that the symmetry property of $POA(\cdot)$ defined by (16) also holds for $OPOA(\cdot)$, i.e.:

$$OPOA(o_j, o_l) = -OPOA(o_l, o_j) \tag{42}$$

Therefore, ds_j defined by (39) can be redefined as:

$$ds_j = \sum_{l \in J} d_{jl} \tag{43}$$

Measure (43) assigns a real number to each alternative, therefore this can be treated as a scalarizing function and used for generating a complete ranking. Indeed, the preference model based on comparison of the measure values

$$(o_j \succ_n o_l \quad \Leftrightarrow \quad ds_j > ds_l) \qquad \text{and} \qquad (o_j \sim_n o_l \quad \Leftrightarrow \quad ds_j = ds_l) \tag{44}$$

is complete and transitive thus allowing us to identify the best alternative. In particular, for our three alternative cycle we get $ds_1 = 0.02 - 0.02 = 0$, $ds_2 = -0.02 + 0.0202 + 0.0002$ and $ds_3 = 0.02 - 0.0202 = -0.0002$ and the final ranking $o_2 \succ_n o_1 \succ_n o_3$ with o_2 as the best alternative. Usage of the linear search algorithm with relation $\succ_n$ allows us to always identify the best alternative o_b such that $o_b \succeq_n o_j$ for all $j \in J$.

Proposition 5 *If the outperformance functions $P_j(\cdot)$ are strictly monotonic, then the best solution o_b selected according to the net flow ranking $\succeq_n$ is Pareto-optimal.*

Proof. Suppose there exists an alternative $o_{\hat{j}}$ dominating o_b, that is $a_{k\hat{j}} \geq a_{kb}$ for all $k \in I$ and $a_{\hat{k}\hat{j}} > a_{\hat{k}b}$. Due to strict monotonicity of functions P_j, we get then $d_{j\hat{j}} < d_{jb}$ for all $j \in J$. Hence,

$$ds_b = \sum_{j \in J, j \neq b} 2d_{bj} = -\sum_{j \in J, j \neq b} 2d_{jb} < -\sum_{j \in J, j \neq b} 2d_{j\hat{j}} = \sum_{j \in J, j \neq b} 2d_{\hat{j}j} = ds_{\hat{j}}$$

which contradicts the optimality of o_b according to ranking $\succeq_n$. □

The corresponding methods are summarized in Section 7. The results of experiments presented in Section 8 show that the pairs of methods (either a POA or an OPOA method, and its Net-Flow version) have similar properties (in the sense of returning the same Pareto-optimal solution for a given preferences).

7 Methods implemented for multicriteria analysis

For the users of the MCA software we summarize here the correspondence between the method's acronym and the described methodology. Note that the first three methods are described in Section 4, the next three in Section 5, and the last six in Section 6.

POA uses the pairwise outperformance aggregation defined by (16) with $\beta(\cdot)$ defined by (18) for $\lambda = 10$. Thus alternative o_j dominates o_l, if

$$\sum_{i=1}^{n} w_i(10^{-a_{ij}} + 10^{-a_{il}})(a_{ij} - a_{il}) > 0$$

Actually, the safeguard defined by (30) is implemented in all POA methods to deal with the unlikely cases in which the above expression is equal to 0.

POA-E differs from POA only by the value of $\lambda = e$. Thus alternative o_j dominates o_l, if

$$\sum_{i=1}^{n} w_i(e^{-a_{ij}} + e^{-a_{il}})(a_{ij} - a_{il}) > 0$$

POA-Inv uses the pairwise outperformance aggregation defined by (16) with $\beta(\cdot)$ defined by (19) for $\lambda = 10$. Thus alternative o_j dominates o_l, if

$$\sum_{i=1}^{n} w_i(\frac{1 + (\lambda - 1)a_{ij}}{1 + (\lambda - 1)a_{il}} - \frac{1 + (\lambda - 1)a_{il}}{1 + (\lambda - 1)a_{ij}}) > 0$$

OPOA differs from the POA method by the applied outperformance method; instead of the pairwise outperformance aggregation defined by (16) it uses the ordered pairwise outperformance aggregation (35).

OPOA-E differs from the POA-E method by the applied outperformance method; instead of the pairwise outperformance aggregation defined by (16) it uses the ordered pairwise outperformance aggregation (35).

OPOA-Inv differs from the POA-Inv method by the applied outperformance method; instead of the pairwise outperformance aggregation defined by (16) it uses the ordered pairwise outperformance aggregation (35).

POA-NF, POA-E-NF, POA-Inv-NF, OPOA-NF, OPOA-E-NF, OPOA-Inv-NF are the Net-Flow modifications of the corresponding (i.e., having acronyms without the -NF suffix) methods defined above.

8 Case study: experiences and results

The new methods described above have been developed and modified successively upon analysis of the features and performance of the earlier developed and/or modified methods. We summarize here the experience with the described methods in order to illustrate some of the methodological issues discussed above, and to provide a justification of the methods selected for the public version of the MCA.

The methods have been implemented as the Web-based application called MCA, using the client-server architecture, see [17] for details. The application is available free of charge for research and educational purposes at `http://www.iiasa.ac.at/~marek`.

8.1 Classification of the methods

A key feature of each method is the mapping of the preferences (specified as relative importance of each criterion) into the corresponding outperformance measure. We discuss here six different outperformance measures reflected by the corresponding method acronym: POA, POA-E, POA-Inv, OPOA, OPOA-E, and OPOA-Inv.

There are two types of similarities between these methods:

- The first three methods (POA, POA-E, POA-Inv) use linear aggregations while the other three (OPOA, OPOA-E, OPOA-Inv) use quantile aggregations. We will refer to these subsets of methods as LA (Linear Aggregation) and QA (Quantile Aggregation), respectively.
- Pairs of methods (POA, OPOA), (POA-E, OPOA-E), and (POA-Inv, OPOA-Inv) share the same representation of key elements of the corresponding outperformance measure.

Another key feature is the form of the corresponding scalarizing function. Here we distinguish two groups of methods:

- Local (pairwise) - the scalarizing function uses only comparisons of pairs of alternatives
- Global (NF) - the scalarizing function based on the Net-Flow approach (see Section 6).

For the sake of discussing the features of the twelve methods we group them in two ways:

- Six pairs of methods, each composed of a method using the local scalarizing function and the corresponding NF method.
- Two subsets of methods using the Local (pairwise) and Global (NF) scalarizing functions, respectively.

8.2 Problems used for exploring the features of the methods

The features of the developed methods have been studied using the following five real-world problems of multicriteria analysis:

Future energy technologies developed by the EU funded project NEEDS[8] for each of the four countries denoted in the follow-up discussion by the corresponding code of the internet:

- ch - Switzerland,
- de - Germany,
- fr - France,
- it - Italy.

	ch	de	fr	it	robot
number of criteria	61	61	61	61	5
number of alternatives	19	25	26	21	184
number of preferences	235	96	179	60	32

Table 2: Summary of problems.

The details of these problems are summarized in Table 2. The problems are documented (and available for further testing and use) through the dedicated appli-

[8]Details are available at: `http://www.needs-project.org/`.

cation called MCA-NEEDS which is linked to `http://www.iiasa.ac.at/~marek`.

These problems have about[9] 20 alternatives, and 61 criteria organized in hierarchical structures. Over 3000 stakeholders from several countries were invited to make individual analysis using the Web-based application. Finally, 348 stakeholders initialized the analysis, and 162 actually completed it. Out of all preferences specified by these 162 stakeholders 570 were unique; these preferences were extracted from the database, and used (with the corresponding problem) for exploring properties of all described methods.

Robot, the acronym used for the path-design problem for remote control of a partly autonomous space robot, see [10]. This is a pretty complex engineering problem for which a large number of instances has been generated (each instance corresponds to a specific area of the asteroid for which the robot[10] was sent). The instance of this problem selected for comparing the developed MCA methods has a different characteristic than the future energy technologies problems, namely it has 183 alternatives and only five criteria. MCA of this instance had been made by analysts who specified 32 unique preferences during the analysis.

The approach outlined above resulted in using a pretty large and diversified set of actual preferences of stakeholders having different backgrounds and priorities. The numbers of the unique preferences specified for each of the problems is presented in Table 2. We note that such a sample of actual preferences is both very valuable and rather rarely available.

8.3 Transitivity properties for the designed methods

A common feature of the methods described here is an automated pairwise outperformance approach. Pairwise comparison had to be automatized because of the number of alternatives which makes human pairwise comparison not practical. A natural and strong requirement for such procedure is assuring the transitivity of preferences (see Section 6), a lack of which is known as the Condorcet paradox (see e.g., [3]).

In Table 3 we summarize for each method and each of the problems the percentage of preferences for which the Condorcet paradox occurred. For each problem we provide two numbers:

- All - denotes all occurrences during the process of determining the ranking of alternatives. The ranking was based on an iterative procedure, in which the chosen Pareto alternative was removed from the set of alternatives, and the next best (for the same preferences) Pareto solution was found in such a smaller set.
- Par. - denotes the occurrences during searching for the Pareto alternative.

Actually, the results collected during the ranking procedure can be considered as a generation of a large (over 10 times larger than the original problem) number of subproblems derived from each of original problems. However, we did not use these results for the comparison of methods, because the preferences were specified by the users for the full sets of alternatives only.

[9] The numbers slightly differ amongst countries.

[10] The robot has the form of cube of 10cm size; it is a ”jumping” robot, thus difficult to control.

	ch		de		fr		it		robot	
method	All	Par.	All	Par.	All	Par.	All	Par.	All	Par.
	%	%	%	%	%	%	%	%	%	%
POA	27.7	18.3	50.0	31.3	46.4	22.9	35.0	23.3	100.0	75.0
POA-E	2.1	1.3	6.3	2.1	4.5	2.8	3.3	0.0	56.3	28.1
POA-Inv	71.9	50.2	93.8	76.0	85.5	63.1	76.7	63.3	100.0	84.4
OPOA	1.7	0.4	3.1	1.0	4.5	1.7	0.0	0.0	90.6	43.8
OPOA-E	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	25.0	9.4
OPOA-Inv	7.2	3.8	13.5	8.3	23.5	9.5	3.3	1.7	100.0	62.5
POA-NF	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
POA-E-NF	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
POA-Inv-NF	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
OPOA-NF	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
OPOA-E-NF	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
OPOA-Inv-NF	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Table 3: Summary of experiments related to the Condorcet paradox.

It is not surprising that the last six methods listed in Table 3 (i.e., the methods belonging to the NF subset) conform to the transitivity requirement, i.e., the Condorcet paradox was not detected. However, it is worth to note that detecting the Condorcet paradox for some of other the methods is not easy. In particular for the OPOA-E the transitivity problem was not detected for any of the 570 preferences (which is equivalent to analysis of about 6000 combinations of problems and preferences) of the four energy problems. This observation implies also a general recommendation for analyzing features of the methods based on also a rather large number of tests. In this particular case, one may tend to believe that the OPOA-E method is likely to have the transitivity property because such a hypothesis was not falsified by over 6000 of tests.

In our opinion the transitivity property is a necessary condition for any method to be recommended for a wide use. Such a property obviously cannot be proven for the six methods belonging to the subset called *Local*. Therefore these methods has been shown for comparison with other approaches although the net flow methods are preferred from the point of view of applications.

8.4 Pairwise comparisons of methods

For the reasons explained in Section 8.3 we recommend for actual use only the six methods that form the subset called *NF*. However, we found it interesting to explore the similarities of each of these methods with the corresponding method from the subset *Local* in order to provide indications how often the results (for each of the considered 602 preferences) differ depending which method from each of such pairs is used. The results of such comparisons are summarized in Table 4.

It is interesting to note that the corresponding pairs of the methods are rather likely (on average in more than 90% of cases) provide the same Pareto alternative. Thus replacing a *Local* outperformance measure by its corresponding *NF* scalarizing function does not

1st method	2nd method	ch	de	fr	it	robot	Average
		%	%	%	%	%	%
POA	POA-NF	93.2	93.8	93.3	81.7	87.5	91.9
POA-E	POA-E-NF	87.7	83.3	81.6	95.0	68.8	84.9
POA-Inv	POA-Inv-NF	91.5	89.6	86.6	93.3	84.4	89.5
OPOA	OPOA-NF	95.3	93.8	93.3	93.3	87.5	93.9
OPOA-E	OPOA-E-NF	99.6	100.0	98.9	93.3	96.9	98.7
OPOA-Inv	OPOA-Inv-NF	94.9	96.9	91.1	93.3	93.8	93.9

Table 4: Pairwise comparison of methods (without and with the Net-Flow).

change the characteristics of the method (understood as the correspondence between the specified preferences and the resulting Pareto solution).

8.5 Net-Flow based methods

Considering the properties of the methods confirmed by the experience summarized in Section 8.3 we recommend to use the six *NF* methods, and to refrain from using methods belonging to the subset of methods denoted by *Local*. The latter recommendation is also supported by the results presented in Section 8.4.

1st method	2nd method	ch	de	fr	it	robot	Avarage
		%	%	%	%	%	%
POA-NF	POA-E-NF	100.0	100.0	98.3	100.0	100.0	99.5
POA-NF	POA-Inv-NF	88.5	75.0	91.6	80.0	93.8	86.7
POA-NF	OPOA-NF	94.9	89.6	84.9	91.7	100.0	91.0
POA-NF	OPOA-E-NF	86.0	82.3	78.8	95.0	68.8	83.2
POA-NF	OPOA-Inv-NF	94.0	90.6	84.4	95.0	100.0	91.0
POA-E-NF	POA-Inv-NF	88.5	75.0	91.6	80.0	93.8	86.7
POA-E-NF	OPOA-NF	94.9	89.6	84.9	91.7	100.0	91.0
POA-E-NF	OPOA-E-NF	86.0	82.3	78.8	95.0	68.8	83.2
POA-E-NF	OPOA-Inv-NF	94.0	90.6	84.4	95.0	100.0	91.0
POA-Inv-NF	OPOA-NF	83.4	64.6	81.6	71.7	93.8	79.2
POA-Inv-NF	OPOA-E-NF	75.3	61.5	75.4	75.0	65.6	72.6
POA-Inv-NF	OPOA-Inv-NF	83.4	67.7	81.0	75.0	93.8	79.9
OPOA-NF	OPOA-E-NF	88.9	89.6	85.5	93.3	68.8	87.4
OPOA-NF	OPOA-Inv-NF	98.3	96.9	97.8	96.7	100.0	97.8
OPOA-E-NF	OPOA-Inv-NF	88.5	88.5	84.9	90.0	68.8	86.5

Table 5: Comparison of the Net-Flow methods.

There is no clear recommendation for a choice of any of these six methods. Users with analytical skills may have personal preferences based on the methodological background of a particular method. However, from the point of view of mapping the preferences (specified as relative importance of criteria) into the selected Pareto solutions most of

the methods are similar (i.e., most likely providing the same Pareto solution for a given preferences). To justify this statement we summarize in Table 5 the results of 15 pairwise comparisons of the six net-flow based methods, each done for 602 unique preferences specified for the five problems used for analysis of the methods' properties. One can observe that especially two pairs of methods, namely (POA-NF, POA-E-NF) and (OPOA-NF and OPOA-Inv-NF) provide the same Pareto solution for a given preferences in 99.5% and 97.8% of preferences, respectively. On the other hand, the pair (POA-Inv-NF, OPOA-E-NF) has the smallest (72.6%) similarity.

Considering the trade-off between providing diversity of methods and problems related to a rational choice from a large number of methods, we made two of the *NF* methods (namely POA-NF and OPOA-Inv-NF) available in the standard configuration[11] of the MCAA (see [17] for details). However, all methods can be made available for testing (within a customized version of the MCAA) upon request.

8.6 Summary of the experience

All methods described in the report have also been tested on several other problems, including the two problems described in [35], and several small (in terms of numbers of both criteria and alternatives) problems. All performed experiments show that the methods support analysis of all Pareto alternatives in an intuitive and easy (in terms of criteria specification) way. This is especially important for problems with large numbers of criteria and alternatives. Thus the methods conform to the basic necessary conditions of multicriteria analysis, namely the requirements for an effective analysis of the whole Pareto set.

An easy way for specification of preferences has clear advantages for users without analytical skills and/or needs. However, analysts may prefer more advanced ways for specification of preferences, which provide also possibilities of more advanced explorations of certain parts of the Pareto set. Some of such methods have also been developed and implemented in MCA, see [16, 17] for details.

9 Conclusions

The newly developed methods described in this paper support effective multiple criteria analysis of problems with many alternatives and many criteria. Specification of preferences is done in very simple way that is especially suitable for the users having limited analytical skills. Yet the methods and the way they have been implemented support an effective analysis of the whole Pareto-set.

The methods exploit the pairwise comparison approaches. A key feature of these methods is that the users are not asked for making comparisons for each pair of alternatives, which would not be practicable for more than several alternatives.

The pairwise comparison methods generally do not possess the transitivity property, which is needed for guaranteeing uniqueness of the solution. Therefore, the Net-Flow method has been applied to various pairwise comparison techniques to guarantee a unique selection of a best alternative corresponding to the specified preferences. Extensive tests

[11] Several other methods described in [17] are also included in the standard configuration of the MCAA.

using a large and diverse sample of actual user preferences have shown that the behavior (in the sense of mapping the preferences into the corresponding Pareto solution) of the pairs of methods (defined as a given method implemented with and without the Net-Flow approach) is rather similar.

Within our implementation all dominated alternatives are eliminated in the preprocessing phase of the solver, thus always guaranteeing efficiency of the best alternative selected according to the Net-Flow method. If not eliminated, a method may select a dominated alternative if the corresponding pairwise outperformance relation is not (strictly) monotonic (Proposition 5). One may notice that in the case of unordered pairwise outperformance relations their monotonicity is related with properties of the $\beta(\cdot)$ functions. For some $\beta(\cdot)$ functions their parameter have to conform to quite strong restrictions in order to result in the strict monotonicity of the outperformance relation, while for other forms of $\beta(\cdot)$ there are no such restrictions. On the other hand, the ordered pairwise outperformance relations preserve monotonicity for any positive function $\beta(\cdot)$. Thus the presented properties of the developed methods offer quite a lot of flexibility in modeling of amplification of differences within specific intervals of achievements.

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