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GAME THEORETICAL TREATMENT OF MATERIAL ACCOUNTABILITY PROBLEMS

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Abstract

In this paper, the problem of material accountability in industrial plants is analyzed. For this purpose a reference time is considered which contains a sequence of n inventory periods, i.e. during this reference time a physical inventory is performed n times and compared with the book inventory at that time. A decision problem arises if all necessary measurements can only be performed with limited accuracy as in this case one has to decide if a book-physical inventory difference is caused by missing material or simply by measurement errors.

In case it has to be assumed that there exists one party which may intend to divert material, the problem can be formulated as a two-person zero-sum inspection game, the payoff of which is the probability of detection.

In the first part of this paper the game theoretical model is established and the sets of strategies of both parties are given. In the second part the solutions of the game, i.e. saddlepoints, are analyzed: sufficient conditions in the form of systems of equations are given which also can be used for numerical calculations.

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1. Introduction

There are cases where the material to be processed by an industrial plant is so expensive that it is necessary to perform an accurate accountability in order to know whether in a given period of time material has disappeared or not. Here, accountability means the establishment of the so-called book inventory (i.e. starting inventory plus receipts minus shipments) over a certain period of time, and the comparison of the book inventory with the physical inventory (i.e. the material physically found in the plant) at the end of that period of time.

The problem of accountability becomes difficult in case the measurement of the material to be accounted for cannot be carried through without committing measurement errors. In this case, when some material seems to have disappeared, the problem arises of deciding if in fact some material has disappeared or if the material unaccounted for simply can be interpreted as being caused by measurement errors.

In addition, there arises the problem of the appropriate choice of the starting inventory for each inventory period: if it has been decided at the end of the foregoing inventory period that no material has disappeared, one could choose either the book or the physical inventory, or a combination of both as the starting inventory period for the next inventory period. In any case however, because of the starting inventory one gets stochastic dependencies between different inventory periods.

In some situations, it has to be assumed that there exists one party which intends to divert material. Furthermore, in case different modes of diversion exist, it has to be assumed that this party will select the optimal diversion "strategy"--whatever "optimal" means.

The problem as it has been sketched plays a very important role in the nuclear material safeguards developed in fulfillment of the Non-Proliferation Treaty for nuclear weapons [1]. In fact, the analysis presented in this paper has been performed on the basis of this specific problem. However, it is obvious that it can be applied to quite different problems.

In the first part of this paper, the principle of material accountability is sketched and the decision problem is treated for the case of a single inventory period. The problem of a sequence of inventory periods is formulated, the sets of strategies of the inspection authority as well as of the plant operator as a "would-be diverter" are described, and the game theoretical problem is formulated.

In the second part, the question of the existence of saddlepoints of the probability of detection--which plays the role of the payoff of a two-person zero-sum game for specific reasons--is analyzed. It is shown that sufficient conditions for the existence can be formulated and that these conditions are fulfilled by elements of specific sets of strategies. In addition, these conditions can be used for the numerical calculation of the saddlepoints.

The paper concludes with general remarks on the optimality criteria being used, and possible extensions of the considerations.

2. The Model: Sequence of Inventory Periods

2.1 One Inventory Period

Let I_0 be the starting inventory in a plant (or more generally, in a material balance area) at time t_0 and let D_1 be the sum of all material inputs and outputs in the interval of time (t_0,t_1). Then $B_1 := I_0 + D_1$ is called the book inventory at time t_1. Let I_1 be the physical inventory at time t_1. It is assumed that all material measurements have normally distributed measurement errors and that the variances of these errors are known.

If no material disappears in (t_0,t_1), the expectation value of the difference[1]

$$MUF_1 := B_1 - I_1 \tag{2-1}$$

is zero (Null hypothesis H_0). If the amount M_1 misses, the expectation value of MUF is M_1 (Alternative hypothesis H_1). In the formulae

$$E(MUF_1|H_0) = 0 \quad , \qquad E(MUF_1|H_1) = M_1 \quad . \tag{2-2}$$

In order to check at the end of one inventory period whether or not material has disappeared, a significance test is performed. Let α and β be the probabilities of the errors first and second kind,

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[1]The term "Material Unaccounted For" is not very precise, as a non-zero difference between B_1 and I_1 may also or mainly be caused by measurement errors, not only by missing material. However, it seems to be impossible to change this.

$$\alpha_1 := \text{prob}\,\{MUF_1 > x_1 | H_0\} \quad , \qquad \beta_1 := \text{prob}\,\{MUF_1 \leq x \,| H_1\} \,, \tag{2-3}$$

where x is the significance threshold. One obtains

$$\beta_1 = \Phi(U(1 - \alpha_1) - \frac{M_1}{\sigma_1}) \quad . \tag{2-4a}$$

Here, Φ is the Gaussian distribution function, U its inverse and

$$\sigma_1^2 := \text{var } I_0 + \text{var } D_1 + \text{var } I_1 \tag{2-4b}$$

the sum of all measurement variances. In the following, $1-\beta$ is called probability of detection, and α is called false alarm probability.

2.2 Starting Inventory for the i-th Inventory Period

At the beginning of the inventory period for the time interval (t_{i-1}, t_i) the problem arises how to choose the starting inventory S_{i-1} for this period. In order to make the best use of the information obtained so far Stewart [2] has proposed to use an unbiased minimum variance estimate formed from the book and ending physical inventories of the foregoing inventory period:

Theorem 2.1. The unbiased minimum variance estimate S_{i-1} for the starting inventory of the i-th inventory period is given by

$$S_{i-1} = a_{i-1} \cdot B_{i-1} + (1 - a_{i-1}) \cdot I_{i-1} \, , \qquad (2\text{-}5a)$$

where

$$a_{i-1} = \frac{\text{var } I_{i-1}}{\text{var } B_{i-1} + \text{var } I_{i-1}} \, . \qquad (2\text{-}5b)$$

The variance of this estimate is given by

$$\frac{1}{\text{var } S_{i-1}} = \frac{1}{\text{var } B_{i-1}} + \frac{1}{\text{var } I_{i-1}} \, . \qquad (2\text{-}6)$$

Proof. An unbiased estimate for the starting inventory is given by

$$\tilde{S}_{i-1} = \tilde{a}_{i-1} \cdot B_{i-1} + (1 - \tilde{a}_{i-1}) \cdot I_{i-1} \; ; \qquad 0 \leq \tilde{a}_{i-1} \leq 1 \, .$$

The variance of this estimate is

$$\text{var } \tilde{S}_{i-1} = \tilde{a}_{i-1}^2 \text{ var } B_{i-1} + (1 - \tilde{a}_{i-1})^2 \text{ var } I_{i-1} \, .$$

The value of $\tilde{a}_{i-1}$ which minimizes var $\tilde{S}_{i-1}$ is determined by the equation

$$a_{i-1} \cdot \text{var } B_{i-1} - (1 - a_{i-1}) \cdot \text{var } I_{i-1} = 0 \, . \qquad (2\text{-}7)$$

■

As can be seen easily the variance of this estimate is smaller than both the variances of the ending book and physical inventories of the foregoing inventory period.

It should be noted that this estimate is not necessarily the best estimate from the point of view of detecting missing material, see e.g. Ref. [3].

The estimate as given above however, has a further property which is important for the following:

Theorem 2.2. If for every inventory period the starting inventory is chosen as described in Theorem 2.1, then the book-physical inventory differences of different inventory periods are uncorrelated.

Proof. The book-physical inventory difference for the j-th inventory period is given by

$$MUF_j = a_{j-1} \cdot B_{j-1} + (1 - a_{j-1}) \cdot I_{j-1} + B_j - I_j .$$

Let be $j > i$. Then S_{j-1} can be written as

$$S_{j-1} = c_{j-1} + b_{j-1} \cdot S_i ,$$

where the c_{j-1} term involves I's and B's with subscripts larger than i, and b_{j-1} involves only constants. Then

$$\operatorname{cov}(MUF_j, MUF_i) = \operatorname{cov}(S_{j-1} + D_j - I_j, S_{i-1} + D_i - I_i)$$

$$= b_{j-1} \operatorname{cov}(S_i, S_{i-1} + D_i - I_i)$$

$$= b_{j-1} \operatorname{cov}(a_i I_i + (1 - a_i)(S_{i-1} + D_{i-1}), S_{i-1} + D_i - I_i)$$

$$= - b_{j-1} \cdot [a_i \operatorname{var} I_i - (1 - a_i) \cdot \operatorname{var}(S_{i-1} + D_{i-1})] \quad .$$

Use of eq. (2-7) completes the proof. ■

As it has been assumed in the beginning that all measurement errors are normally distributed, it follows from Theorem 2.2 that the book-physical inventory differences of different inventory periods are independent.

According to the choice of the starting inventory, the expectation value of the book-physical inventory difference at the end of an inventory period is not simply given by the amounts of material disappearing in these inventory periods:

Theorem 2.3. Under the assumption that in the j-th inventory period (j = 1...i) the amount M_j disappears (Alternative hypothesis H_1) the expectation value of the book physical inventory difference of the i-th inventory period is determined by the recursive relation

$$E(MUF_i|H_1) = a_{i-1} \cdot E(MUF_{i-1}|H_1) + M_i \; ; \quad E(MUF_1|H_1) = M_1 \; . \tag{2-8}$$

<u>Proof</u>. If one defines

$$EI_j =: E_j \; , \qquad j = 0,1,\ldots,n \; ,$$

one has according to the assumption

$$E_{j-1} + ED_j - E_j = M_j \; .$$

Therefore,

$$\begin{aligned} E\,(MUF_i|H_1) &= ES_{i-1} + ED_i - E_i \\ &= a_{i-1}\,(ES_{i-2} + ED_{i-1}) \\ &\quad + (1 - a_{i-1}) \cdot E_{i-1} + ED_i - E_i \\ &= a_{i-1}\,(E(MUF_{i-1}|H_1) - ED_{i-1} + E_{i-1} \\ &\quad + M_{i-1} - E_{i-2} + E_{i-1}) \\ &\quad + (1 - a_{i-1})\,E_{i-1} + ED_i - E_i \\ &= a_{i-1} \cdot E\,(MUF_{i-1}|H_1) + M_i \; . \end{aligned}$$

■

In the following the abbreviation

$$E(MUF_i | H_1) =: y_i \qquad (2\text{-}9)$$

will be used. Then one has instead of (2-8)

$$y_i = a_{i-1} \cdot y_{i-1} + M_i \; ; \qquad y_1 = M_1 , \qquad (2\text{-}8')$$

or in matrix language,

$$(y_1, \ldots, y_n)^t = A \cdot (M_1, \ldots, M_n)^t \; , \qquad (2\text{-}8'')$$

where $A: = \{a_{ij}\}$ and

$$a_{ij} : = \begin{cases} 0 & \text{for } j > i \\ 1 & \text{for } j = i \\ \prod_{k=j}^{i-1} a_k & \text{for } j < i \, . \end{cases}$$

Let us consider a reference time interval (O,T) containing n inventory periods. Then the probability to detect missing material in case the amount M_i is missing in the i-th inventory period is given by

$$1 - \beta = 1 - \text{prob} \, \{MUF_1 \leq x_1 \wedge \ldots \wedge MUF_n \leq x_n | H_1\} \quad . \qquad (2\text{-}10)$$

According to the foregoing considerations one obtains

$$1 - \beta = 1 - \prod_{i=1}^{n} \Phi(U(1-\alpha_i) - \frac{y_i}{\sigma_i}) \; , \qquad (2\text{-}11a)$$

where y_i is given by equation (2-8'), where α_i is the false alarm probability for the i-th inventory period, and where σ_i^2 is given by

$$\sigma_i^2 = \text{var } S_{i-1} + \text{var } D_i + \text{var } I_i \quad . \tag{2-11b}$$

The resulting false alarm probability α is determined by the relation

$$1 - \alpha = \prod_{i=1}^{n} (1 - \alpha_i) \quad . \tag{2-12}$$

2.3 Strategies, Two-person Zero-sum Inspection Game

For the purpose of optimization of the material accountability procedure as described above two boundary conditions have to be agreed. It is assumed in the following that

(i) the resulting false alarm probability α is fixed, and

(ii) the "sensitive" amount of missing material is either zero or $M = \sum_{i=1}^{n} M_i$ and fixed.

In order to determine the optimal probability of detection with respect to all possible "strategies"

$$S_1 : = \{s_1 = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n, \prod_{i=1}^{n} (1 - \alpha_i) = 1 - \alpha \quad ,$$

$$0 \leq \alpha_i \leq \alpha \; \forall i = 1, \ldots, n\} \; , \tag{2-13}$$

one has to take into account all possible strategies of the adversary party

$$S_2 : = \{s_2 = (M_1,\ldots,M_n) \varepsilon \mathbb{R}^n, \sum_{i=1}^{n} M_i = M\} \quad . \qquad (2\text{-}14)$$

<u>Note</u>: In the definition of S_2 the possibility is included that in some inventory periods there is more material than accounted for. In the last section of the next chapter the special case $M_i \geq 0$, $i = 1 \ldots n$ is considered.

According to the foregoing considerations the optimal guaranteed probability of detection is given by

$$1 - \min_{\substack{\alpha_1,\ldots,\alpha_n \\ \prod_{i=1}^{n}(1-\alpha_i) = 1-\alpha}} \; \max_{\substack{M_1,\ldots,M_n \\ \prod_{i=1}^{n} M_i = M}} \prod_{i=1}^{n} \Phi(U(1-\alpha_i) - \frac{y_i}{\sigma_i}) \quad . \qquad (2\text{-}15)$$

It should be mentioned that this optimization problem had to be considered in the course of the establishment of the nuclear material safeguards system of the International Atomic Energy Agency in fulfillment of the requirements of the Non-Proliferation Treaty. There, it had to be assumed that in case a diversion was planned it was planned in the most effective way. The value of the false alarm probability for a reference period of time had to be not larger than a given value for political reasons. The value of the sensitive amount of material was agreed to be not smaller than a given value in that specific connection. A two-person zero-sum game was constructed with the sets of strategies as given above and with the following payoff to the operator

0	in case of no diversion and no false alarm
0	in case of a false alarm (which was expected to be identified as a false alarm in a second action level)
-c	in case of detected diversion
d	in case of not detected diversion.

Therefore the expected gain a of the operator is

$$a = \begin{cases} 0 & \text{in case of no diversion} \\ -c \cdot (1-\beta) + d \cdot \beta & \text{in case of diversion .} \end{cases}$$

As the game (S_1,S_2,a) is strategically equivalent to the game (S_1,S_2,β) in the sense that the optimal strategies of both games are the same only the latter one was considered as a) it was not possible to agree on values for the payoff parameters and b) only the operational strategies, not the value of the game, were interesting.

Therefore, the general problem of the optimal choice of the significance thresholds leads to the problem of the solution of the two-person zero-sum game (S_1,S_2,β).

3. Saddlepoints

3.1 Formulation of the Problem

A saddlepoint is defined by the following

Definition 3.1. Let H be a real valued function defined on the nonempty set C x D. Then (c_0,d_0) is called a saddlepoint of H on C x D if

$$H(c_0,d) \leq H(c_0,d_0) \leq H(c,d_0) \qquad (3\text{-}1)$$

holds for all $c \in C$ and $d \in D$.

For saddlepoints the following theorem is valid (cf. [4], Th. 6.29)

Theorem 3.2. Let H be a real valued function defined on the nonempty set C x D.

a) If H has a saddlepoint $(c_0,d_0) \in C \times D$, then there exist $\min_{c \in C} \max_{d \in D} H(c,d)$ and $\max_{d \in D} \min_{c \in C} H(c,d)$ and it is

$$\min_{c \in C} \max_{d \in D} H(c,d) = H(c_0,d_0) = \max_{d \in D} \min_{c \in C} H(c,d) \quad . \qquad (3\text{-}2)$$

b) If (c_1,d_1), $(c_2,d_2) \in C \times D$ are saddlepoints of H then also (c_1,d_2), (c_2,d_1) are saddlepoints of H on C x D.

Therefore, the problem formulated at the end of the foregoing chapter can be formulated in the following way:

Determine the saddlepoints of the function $\beta(s_1,s_2)$, defined by eqs. (2-11), the region of definition of which is given by the Cartesian product $S_1 \times S_2$, eqs. (2-13) and (2-14).

Instead of $\beta(s_1,s_2)$ defined on $S_1 \times S_2$, the function

$$F(x,y) := \sum_{i=1}^{n} \ln \Phi \left(U(e^{x_i}) - \frac{y_i}{\sigma_i}\right) \qquad (3\text{-}3)$$

with the region of definition $X \times Y$,

$$X = \{x = (x_1,\ldots,x_n) \in \mathbb{R}^n, \sum_{i=1}^{n} x_i = \ln(1-\alpha) \quad ,$$

$$0 \geq x_i \geq \ln(1-\alpha)\ \forall i = 1,\ldots,n\} \qquad (3\text{-}4a)$$

$$Y = \{y = (y_1,\ldots,y_n) \in \mathbb{R}^n,\ y_n + \sum_{i=1}^{n-1} (1-a_i)\, y_i = M\} \qquad (3\text{-}4b)$$

will be considered in the following.

As the condition $\sum_{i=1}^{n} M_i = M$ is equivalent to the condition $y_n + \sum_{i=1}^{n-1} (1-a_i)\, y_i = M$, the following lemma is evident:

<u>Lemma 3.3</u>. If $(s_1^*,s_2^*) \in S_1 \times S_2$ is a saddlepoint of β on $S_1 \times S_2$, then $(x^*,y^*) = ((\ln(1-\alpha_1^*),\ldots,\ln(1-\alpha_n^*))^t,$ $A \cdot (M_1^*,\ldots,M_n^*)^t) \in X \times Y$, where A is defined by eq. (2-8"), is a saddlepoint of F on $X \times Y$.

If on the other hand $(x^*,y^*) \in X \times Y$ is a saddlepoint of F on $X \times Y$, then $(s_1^*,s_2^*) = ((1-e^{x_1^*},\ldots,1-e^{x_n^*}),\ A^{-1}(y_1^*,\ldots,y_n^*)^t)$ $\in S_1 \times S_2$ is a saddlepoint of β on $S_1 \times S_2$.

With the help of this lemma, the problem formulated above may be formulated in the following way:

> Determine the saddlepoints of the function F(x,y) defined by eq. (3-3) the region of definition of which is given by the Cartesian product X x Y, eqs. (3-4).

The advantage of this formulation is that X x Y is a convex set whereas this is not the case for the set $S_1 \times S_2$.

In the following we will prove the existence of a saddlepoint of F on X x Y as well as its uniqueness. For this purpose, in the next section convexity and concavity properties of F are derived. With the help of these properties in section 3.3 we will give sufficient conditions for the saddlepoint of F on X x Y and then show that these conditions can always be fulfilled by some point of X x Y. This way we also obtain a simple method for the calculation of the saddlepoint. We will use the following theorem for Lagrange multipliers (cf. [5]):

Theorem 3.5. Let $C \subset \mathbb{R}^n$ be an open and convex set. Let

$$G: \; C \longrightarrow \mathbb{R}$$

be a real valued, convex differentiable function, and let

$$g: \; C \longrightarrow \mathbb{R}$$

be a real valued, convex differentiable function.

If there is a $c_0 \varepsilon C$ and a $\lambda \geq 0$ with

$$\text{grad } G(c_0) + \lambda \text{ grad } g(c_0) = 0$$

$$g(c_0) = 0$$

then c_0 is a minimum of G on the set $\{c \varepsilon C, g(c) = 0\}$.

It should be noted that the properties of F derived in section 3.2 permit the application of the following theorem (cf. [4], Th. 6.3.7):

<u>Minimax-Theorem of Sion-Kakutani 3.4</u>

Let $C \varepsilon \mathbb{R}^k$, $D \varepsilon \mathbb{R}^\ell$ be convex compact sets. Let

$$H: C \times D \longrightarrow \mathbb{R}$$

be a real valued continuous convex-concave function on C x D, i.e. let $H(\cdot,d)$ be a convex function on C for every $d \varepsilon D$ and $H(c,\cdot)$ be a concave function on D for every $c \varepsilon C$. Then H has a saddlepoint on C x D.

This way the existence of a saddlepoint of F on X x Y can be proven in a very simple way; however, no idea is given how the saddlepoint could be determined.

In the last section, we will consider the saddlepoint problem on the special set X x Y' where Y' is given by

$$Y' := \{y = (y_1,\dots,y_n) \varepsilon \mathbb{R}^n, y_n + \sum_{i=1}^{n-1} (1 - a_i) y_i = M \ ,$$

$$A^{-1}(y_1,\dots,y_n)^t \geq (0,\dots,0)^t\} \quad , \qquad (3\text{-}5)$$

i.e. the case $M_i \geq 0$ for $i = 1,\dots,n$ is considered.

3.2 Convexity and Concavity Properties of F; Existence of Saddlepoints

In order to show that F is convex-concave on $X_t \times \mathbb{R}^n_+$ where

$$X_t := \{x = (x_1,\dots,x_n) \in \mathbb{R}^n: \ln(1-\alpha) \leq x_i \leq 0 \, \forall i = 1,\dots,n\} \quad (3\text{-}6)$$

$$\mathbb{R}^n_+ := \{r \in \mathbb{R}^n: r_i \geq 0 \, \forall i = 1,\dots,n\} \quad (3\text{-}7)$$

(in fact this is not true for $X_t \times \mathbb{R}^n$), the following lemma will be used.

Lemma 3.6. Let $Q(x), x \in \mathbb{R}$, be defined by

$$Q(x) := \frac{\Phi'(x)}{\Phi(x)} \quad . \quad (3\text{-}8)$$

Then it is for $x \in \mathbb{R}$

$$-1 < Q'(x) < 0 \quad . \quad (3\text{-}9)$$

Proof. Let $R(x), x \in \mathbb{R}$, be defined by

$$R(x) := e^{\frac{x^2}{2}} \cdot \int_x^\infty e^{-\frac{t^2}{2}} dt \quad . \quad (3\text{-}10)$$

R is called "Mills Ratio." Then one can show (cf. [6], or [7], p.177) that for $x \in \mathbb{R}$

$$0 < \left(\frac{1}{R(x)}\right)' < 1 \quad .$$

Since $Q(x) = \frac{1}{R(-x)}$, the proof is completed. ∎

For the purposes of section 3.3 and in order to apply the Minimax Theorem 3.4, the following two theorems have to be established.

Theorem 3.7. For every $y \varepsilon \mathbb{R}_+^n$ the function $F(\cdot,y)$ is convex on X_t where X_t is given by (3.6).

Proof. We have to show

$$F(\lambda x' + (1 - \lambda)x'',y) \leq \lambda F(x',y) + (1 - \lambda)F(x'',y) \qquad (3\text{-}11)$$

for all $x' \neq x''$ with $x',x'' \varepsilon X_t, y \varepsilon \mathbb{R}_+^n$ and $\lambda \varepsilon (0,1)$. With the definition

$$g(t) := F(x'' + t(x' - x''),y)$$

this is equivalent to showing

$$g(\lambda) \leq \lambda g(1) + (1 - \lambda)g(0) \quad , \qquad \text{for } \lambda \varepsilon (0,1) \quad .$$

Therefore, it is sufficient to show that $g(t)$ is convex in $[0,1]$. This will be done by showing that

$$\frac{\partial^2}{\partial t^2} \ell n\ \Phi(u(e^{x_i'' + t(x_i' - x_i'')}) - \frac{y_i}{\sigma_i}) \geq 0 \quad , \qquad \text{for } t \varepsilon (0,1) \qquad (3\text{-}12)$$

as from this inequality one obtains immediately

$$\frac{d^2}{dt^2} g(t) \geq 0 \quad , \qquad \text{for } t \varepsilon (0,1) \quad .$$

With the definition

$$h(z) := \ln \Phi(U(e^z) - \frac{y_i}{\sigma_i}) \quad , \qquad \text{for } z < 0 \quad ,$$

one obtains

$$\frac{d}{dz} h(z) = e^z \cdot e^{-\frac{1}{2}(\frac{y_i}{\sigma_i})^2} \cdot \frac{e^{\frac{y_i}{\sigma_i} \cdot U(e^z)}}{\Phi(U(e^z) - \frac{y_i}{\sigma_i})}$$

$$\frac{d^2}{dz^2} h(z) = \frac{d}{dz} h(z) \cdot [1 + \sqrt{2\pi} \cdot e^z \cdot e^{\frac{1}{2}U^2(e^z)} (\frac{y_i}{\sigma_i} - Q(U(e^z) \frac{-y_i}{\sigma_i})] \ .$$

Now by Lemma 3.6 the function $\frac{y_i}{\sigma_i} - Q(U(e^z) - \frac{y_i}{\sigma_i})$ is monotonously increasing in y_i. As we have by assumption $y_i \geq 0$ we obtain

$$\frac{y_i}{\sigma_i} - Q(U(e^z) - \frac{y_i}{\sigma_i}) \geq - Q(U(e^z)) \quad .$$

This yields (with strict inequality for $y_i > 0$)

$$\frac{d^2}{dz^2} h(z) \geq \frac{d}{dz} h(z)[1 - e^z \sqrt{2\pi}\, e^{\frac{1}{2}U^z(e^z)} Q(U(e^z))] = 0 \quad . \tag{3-13}$$

Since we have

$$\frac{d^2}{dt^2} \ln\Phi(U(e^{x_i'' + t(x_i' - x_i'')}) - \frac{y_i}{\sigma_i})$$

$$= (x_i' - x_i'')^2 \cdot \frac{d^2}{dz^2} h(z) \Big|_{z = x_i'' + t(x_i' - x_i'')} \geq 0$$

(with strict inequality for $x_i' \neq x_i'', y_i > 0$), the proof of (3-12) and therefore, the proof of the theorem is completed. ∎

Theorem 3.8. For every $x \in X_t$ the function $F(x,\cdot)$ is concave on $\mathbb{R}^n$.

Proof. We have to show

$$F(x,\lambda y' + (1-\lambda)y'') \geq \lambda F(x,y') + (1-\lambda)F(x,y'')$$

for all $y' \neq y''$ with $y',y'' \in \mathbb{R}^n$, $x \in X$, $\lambda \in (0,1)$.

As in the proof of the foregoing theorem it is sufficient to prove the concavity of the function

$$f(t) := F(x,y'' + t(y' - y'')) \quad \text{on } [0,1] \; .$$

Now, we have

$$\frac{d^2}{dt^2}\left(\ln \Phi(U(e^{x_i}) - \frac{1}{\sigma_i}(y_i'' + t(y_i' - y_i''))\right)$$

$$= \left(\frac{y_i' - y_i''}{\sigma_i}\right)^2 \cdot Q'(U(e^{x_i}) - \frac{1}{\sigma_i}(y_i'' + t(y_i' - y_i'')) \leq 0 \quad (3\text{-}14)$$

for all $t \in [0,1]$ since $Q' \leq 0$ according to Lemma 3.6. Therefore, we have

$$\frac{d^2}{dt^2} f(t) \leq 0 \quad , \qquad \text{for all } t \in [0,1]$$

and $f(t)$ is concave in $[0,1]$ which completes the proof. ∎

With the help of these two theorems we can establish the following

Theorem 3.9. Let $X_1 \subset X_t, Y_1 \subset \mathbb{R}^n_+$ be closed convex sets, Y_1 bounded. Then F has a saddlepoint on $X_1 \times Y_1$.

Proof. X_1 and Y_1 are compact convex sets. F is convex-concave on $X_1 \times Y_1$ because of Theorems 3.7 and 3.8. Therefore, the application of Theorem 3.4 completes the proof. ■

3.3 Sufficient Conditions for the Saddlepoint, Uniqueness

In the following theorem a sufficient condition for a saddlepoint of F on X x Y is established.

Theorem 3.10. Let $(x^*,y^*) \varepsilon \overset{0}{X}_t \times \overset{0}{\mathbb{R}}{}^n_+$, where $\overset{0}{X}_t$ and $\overset{0}{\mathbb{R}}{}^n_+$ denote the open core of X_t and $\mathbb{R}^n_+$ respectively, be a solution of the following two systems of equations:

$$\frac{e^{-(x_i + \frac{U^2(e^{x_i})}{2})}}{\sigma_i(1 - a_i)} - \frac{e^{-(x_{i-1} + \frac{U^2(e^{x_{i-1}})}{2})}}{\sigma_{i-1} \cdot (1 - a_{i-1})} = 0 \quad , \qquad i = 2,\ldots,n-1,$$

$$\frac{e^{-(x_n + \frac{U^2(e^{x_n})}{2})}}{\sigma_n} - \frac{e^{-(x_{n-1} + \frac{U^2(e^{x_{n-1}})}{2})}}{\sigma_{n-1} \cdot (1 - a_{n-1})} = 0 \ , \tag{3-15}$$

$$\sum_{j=1}^{n} x_j = \ln(1 - \alpha) \ ;$$

$$e^{x_i + \frac{U^2(e^{x_i})}{2}} \cdot Q(U(e^{x_i}) - \frac{y_i}{\sigma_i}) - e^{x_{i-1} + \frac{U^2(e^{x_{i-1}})}{2}}$$

$$\cdot\ Q(U(e^{x_{i-1}}) - \frac{y_{i-1}}{\sigma_{i-1}}) = 0 \quad , \qquad i = 1,\ldots,n, \tag{3-16}$$

$$y_n + \sum_{j=1}^{n-1} (1 - a_j)\, y_j = M \ .$$

Then (x^*,y^*) is a saddlepoint of F on X x Y.

Proof. Let us regard the following systems of equations:

$$\frac{1}{\sqrt{2\pi}}\frac{\partial}{\partial x_i}F(x,y) + \rho \cdot \frac{\partial}{\partial x_i}(\ell n(1-\alpha) - \sum_{j=1}^{n} x_j) = 0 \quad , \qquad i = 1,\ldots,n ,$$

$$\ell n(1-\alpha) - \sum_{j=1}^{n} x_j = 0 \quad ; \tag{3-17}$$

$$-\frac{\partial}{\partial y_i}F(x,y) + \lambda(M - y_n - \sum_{j=1}^{n-1}(1-a_j)y_j) = 0 \quad , \qquad i = 1,\ldots,n ,$$

$$M - (y_n + \sum_{j=1}^{n-1}(1-a_j)\,y_j) = 0 \quad . \tag{3-18}$$

If one puts

$$\rho = e^{x_1^* + \frac{U^2(e^{x_1^*})}{2}} \cdot Q(U(e^{x_1^*}) - \frac{y_1^*}{\sigma_1}) \quad ,$$

one sees immediately that (x^*,y^*) and ρ solve system (3-17) since (x^*,y^*) solves system (3-16). Since $x_1^* < 0$ it is $Q(U(e^{x_1^*}) - \frac{y_1^*}{\sigma_1}) > 0$ by definition of Q, hence $\rho > 0$. From system (3-17) follows for all $i = 1,\ldots,n$

$$Q(U(e^{x_i^*}) - \frac{y_i^*}{\sigma_i}) = \rho \cdot e^{-(x_i^* + \frac{U^2(e^{x_i^*})}{2}} \quad . \tag{3-19}$$

Thus, system (3-18) is solved by (x^*,y^*) and $\lambda \geq 0$ if the following system of equations

$$\frac{e^{-(x_i + \frac{U^2(e^{x_i})}{2})}}{\sigma_i(1 - a_i)} \cdot \rho - \lambda = 0 \quad , \qquad i = 1,\ldots,n-1 ,$$

$$\frac{e^{-(x_n + \frac{U^2(e^{x_n})}{2})}}{\sigma_n} \cdot \rho - \lambda = 0 \tag{3-20}$$

is solved by x^* and $\lambda \geq 0$. This can be seen however, from system (3-15) by putting

$$\lambda = \rho \cdot \frac{e^{-(x_1^* + \frac{U^2(e^{x_1^*})}{2})}}{\sigma_i(1 - a_1)} \quad .$$

Having shown that (x^*,y^*) and $\rho \geq 0, \lambda \geq 0$ solve systems (3-17) and (3-18), we can apply Theorem 3.5. For this purpose we define

$$G_1(x) := F(x,y^*) \quad , \qquad g_1(x) := \ln(1 - \alpha) - \sum_{j=1}^{n} x_j \ .$$

Since $y^* \varepsilon \mathbb{R}^n_+$, G_1 is convex on X_t by Theorem 3.7. Furthermore. since (x^*,y^*) solves system (3-17) with $\rho \geq 0$, from Theorem 3.5 it follows that x^* is a minimum of G_1 on $\overset{\circ}{X}_t \cap X$. Because of the continuity of $F(\cdot,y^*)$ on X_t one therefore obtains

$$F(x^*,y^*) \leq F(x,y^*) \text{ for } x \varepsilon X_t \cap X = X \quad . \tag{3-21}$$

Let us now define

$$G_2(y) := - F(x^*,y) \quad , \qquad g_2(y) := M - y_n - \sum_{j=1}^{n-1} (1 - a_j)y_j \quad .$$

Since $-F(x^*,\cdot)$ is convex on $\mathbb{R}^n$ because of Theorem 3.8 and since (x^*,y^*) solves system (3-18) with $\lambda > 0$, it follows from Theorem 3.5 that y^* is a minimum of G_2 on $\mathbb{R}^n \cap Y$, therefore

$$F(x^*,y^*) \geq F(x^*,y) \text{ for } y \in Y \quad . \tag{3-22}$$

According to definition 3.1 one sees from (3-21) and (3-22) that (x^*,y^*) is a saddlepoint of F on $X \times Y$. ∎

With the help of the following Lemma 3.11 we shall prove Theorem 3.12 which states that the system of equations (3-15) and (3-16) can be solved in $\mathring{X}_t \times \mathring{\mathbb{R}}^n_+$.

<u>Lemma 3.11</u> . Let $h_1,\ldots,h_n$ be continuous and strictly monotonous increasing functions defined on $[0,\infty)$ with

$$\lim_{x\to\infty} h_i(x) = \infty \quad \text{for } i = 1,\ldots,n \ .$$

and let the following system of equations be given:

$$h_i(x_i) - h_{i-1}(x_{i-1}) = 0 \quad , \qquad i = 2,\ldots,n \quad , \tag{3-23a}$$

$$\sum_{j=1}^{n} \delta_j x_j = C \quad , \qquad \delta_j > 0 \text{ for } j = 1,\ldots,n \quad . \tag{3-23b}$$

Let $x_i = 0$ for $i = 1,\ldots,n$ be a solution of the homogeneous form (i.e. $C = 0$) of system (3-23). Then for every $C > 0$ there exists a unique solution $x_i > 0$ for $i = 1,\ldots,n$ of system (3-23).

Proof. Let us define

$$g_i := h_i^{-1} \circ h_{i-1} \quad \text{for } i = 2,\ldots,n , \qquad (3\text{-}24)$$

where h_i^{-1} denotes the inverse of h_i. From the properties of the h_i it follows that the g_i are continuous and strictly monotonous increasing functions defined on $[0,\infty)$ with

$$\lim_{x \to \infty} g_i(x) = \infty \quad \text{for } i = 2,\ldots,n ,$$

and furthermore, because of the assumption that $x_i = 0$ for $i = 1,\ldots,n$ solve the homogeneous system (3-23)

$$g_i(0) = 0 \quad \text{for } i = 2,\ldots,n .$$

Evidently for every $x_{i-1} \in [0,\infty)$, $i = 2,\ldots,n$ the system

$$g_i(x_{i-1}) = x_i \quad \text{for } i = 2,\ldots,n , \qquad (3\text{-}25)$$

is equivalent to the system (3-23a).

Let us define now

$$f_i := g_i \circ \cdots \circ g_2 \quad \text{for } i = 2,\ldots,n .$$

From the properties of the g_i it follows that the f_i are continuous and strictly monotonous increasing on $[0,\infty)$ with

$$f_i(0) = 0 \quad \text{and} \quad \lim_{x\to\infty} f_i(x) = \infty \quad \text{for } i = 2,\dots,n \;.$$

Evidently for every $x_i \varepsilon [0,\infty)$ the system

$$f_i(x_1) = x_i \quad \text{for } i = 2,\dots,n \;, \tag{3-26}$$

is equivalent to system (3-25) and therefore to system (3-23a). Since the function $\sum_{i=2}^{n} \delta_i f_i(x_1)$ is a continuous and strictly monotonous increasing function on $[0,\infty)$ with

$$\sum_{i=2}^{n} \delta_i f_i(0) = 0 \quad , \quad \lim_{x\to\infty}(\delta_1 x + \sum_{i=2}^{n} \delta_i f_i(x)) = \infty$$

by reasons of continuity there exists a unique solution $x_1 > 0$ of the equation

$$\sum_{i=2}^{n} \delta_i f_i(x_1) = C - \delta_1 x_1$$

for every $c > 0$ and therefore, there exists a unique solution of the system (3-26) together with (3-23b) which completes the proof. ■

Theorem 3.12. F has a saddlepoint $(x^*,y^*) \varepsilon \overset{0}{X}_t \times \overset{0}{\mathbb{R}}{}^n_+$ on X x Y which solves the systems of equations (3-15) and (3-16).

Proof. Let us regard system (3-15) which we write with $z_i : = -x_i$ in the following form:

$$\frac{e^{z_i} - \frac{U^2(e^{-z_i})}{2}}{\sigma_i \cdot (1 - a_i)} - \frac{e^{z_{i-1}} - \frac{U^2(e^{-z_{i-1}})}{2}}{\sigma_{i-1} \cdot (1 - a_{i-1})} = 0 \quad \text{for } i = 2,\ldots,n-1 ,$$

$$\frac{e^{z_n} - \frac{U^2(e^{-z_n})}{2}}{\sigma_n} - \frac{e^{z_{n-1}} - \frac{U^2(e^{-z_{n-1}})}{2}}{\sigma_{n-1} \cdot (1 - a_{n-1})} = 0 \qquad (3\text{-}15')$$

$$\sum_{i=1}^{n} z_i = -\ln(1 - \alpha) \quad .$$

Let us define

$$h_i(z_i) := \frac{e^{z_i} - \frac{U^2(e^{-z_i})}{2}}{\sigma_i(1 - a_i)} \quad \text{for } i = 2,\ldots,n-1 ,$$

$$h_n(z_n) := \frac{e^{z_n} - \frac{U^2(e^{-z_n})}{2}}{\sigma_n}$$

$$h_i(0) := 0 \quad \text{for } i = 1,\ldots,n ,$$

$$\delta_i := 1 \quad \text{for } i = 1,\ldots,n ,$$

$$C := -\ln(1 - \alpha) \quad .$$

Since the function $z - \frac{1}{2} U^2(e^{-z})$ is strictly monotonous increasing on $[0,\infty)$ and since

$$\lim_{z\to\infty} (z - \tfrac{1}{2} U^2(e^{-z})) = \infty$$

(both statements are proven in the appendix) it follows that the h_i are strictly monotonous increasing on $[0,\infty)$ with

$$\lim_{z\to\infty} h_i(z) = \infty \quad \text{for } i = 1,\ldots,n \quad .$$

Therefore, as trivially $z_i = 0$ for $i = 1,\ldots,n$ solves the homogeneous system, the application of Lemma 3.11 immediately gives the result that there exists a set $z_i^* > 0$ for $i = 1,\ldots,n$ which solves system (3-15'). Hence $x_i^* := -z_i^*$, $i = 1,\ldots,n$ solves system (3-15).

It remains to be shown that $x_i^* > \ln(1 - \alpha)$ for $i = 1,\ldots,n$. However, this is evident as the assumption $x_i^* < \ln(1 - \alpha)$, equivalent with the assumption $z_i^* > -\ln(1 - \alpha)$, together with $z_i^* > 0$ would lead to the contradiction $\sum_{i=1} z_i^* > -\ln(1 - \alpha)$. Let us consider now system (3-16). We define

$$h_i(y_i) := e^{x_i^* + \frac{U^2(e^{x_i^*})}{2}} \cdot Q(U(e^{x_i^*}) - \frac{y_i}{\sigma_i}) \qquad \text{for } i = 1,\ldots,n ,$$

$$\delta_i := 1 - a_i \qquad \text{for } i = 1,\ldots,n ,$$

$$\delta_n := 1$$

$$C := M .$$

Because of Lemma 3.6 the h_i are strictly monotonous increasing on $[0,\infty)$. As proven in the appendix, it is

$$\lim_{y \to \infty} Q(-y) = \infty ,$$

hence, since $U(e^{x_i^*}) < \infty$ for $x_i^* < 0$, we have

$$\lim_{y \to \infty} h_i(y) = \infty \text{ for } i = 1,\ldots,n .$$

Furthermore, it is

$$h_i(0) = e^{x_i^* + \frac{U^2(e^{x_i^*})}{2}} \cdot Q(U(e^{x_i^*})) = \frac{1}{\sqrt{2\pi}} .$$

Hence, $y_i = 0$ for $i = 1,\ldots,n$ solves the homogeneous form of system (3-16). Therefore, the application of Lemma 3.11 gives the result that there exist y_i^*: $y_i^* > 0$ for $i = 1,\ldots,n$ which solve system (3-16). As $x^* := (x_1^*,\ldots,x_n^*)$ and $y^* := (y_1^*,\ldots,y_n^*)$ fulfill the assumptions of Theorem 3.10, (x^*,y^*) is a saddlepoint of F on X x Y. ■

Theorem 3.13. F has a uniquely determined saddlepoint on X x Y.

Proof. Because of Theorem 3.12 only the uniqueness of the saddlepoint remains to be shown. Let $(x^*,y^*)\varepsilon X \times Y$ be the saddlepoint of F according to Theorem 3.10, therefore, $y_i^* > 0$ for $i = 1,\ldots,n$. Let $(x',y')\varepsilon X \times Y$ be another saddlepoint of F. Because of Theorem 3.2 (x',y^*) is a saddlepoint of F, too. Since $y^*\varepsilon \mathbb{R}_+^n$ is convex on X therefore,

$$F(x' + \lambda(x^* - x'),y^*) \leq \lambda F(x^*,y^*) + (1 - \lambda)F(x',y^*) \ \forall \lambda\varepsilon[0,1).$$

Obviously equality must hold. This implies

$$\frac{d^2}{d\lambda^2} F(x' + \lambda(x^* - x'),y^*) = 0 \quad .$$

But as $y_i^* > 0$ for $i = 1,\ldots,n$ because of (3-13) this can be the case only if $x_i' = x_i^*$ for $i = 1,\ldots,n$ hence, $x' = x^*$. Because of Theorem 3.2 (x^*,y') is a saddlepoint of F, too. Therefore, because of the concavity of $F(x^*,\cdot)$ on Y we have

$$F(x^*,y' + \lambda(y^* - y')) \geq \lambda F(x^*,y') + (1 - \lambda)F(x^*,y') \ \forall \lambda\varepsilon[0,1).$$

Since equality must hold we have

$$0 = \frac{d^2}{d\lambda^2} F(x^*, y' + \lambda(y^* - y')) = \sum_{i=1}^{n} \left(\frac{y_i^* - y_i'}{\sigma_i}\right)^2 Q'(U(e^{x_i^*}) - \frac{y_i' + (y_i^* - y_i')}{\sigma_i}) \quad .$$

Since $U(e^{x_i^*}) < \infty$ for $x_i^* < 0$, because of Lemma 3.6 we have

$$Q(U(e^{x_i^*}) - \frac{y_i' + \lambda(y_i^* - y_i')}{\sigma_i}) < 0$$

and therefore, $y_i^* - y_i' = 0$ for $i = 1,\ldots,n$ which completes the proof. ■

As it can be seen easily from Theorem 3.12 the x-coordinate of the saddlepoint depends only on α but not on M whereas the y-coordinate depends on both α and M. This property of the saddlepoint is important for the applications. Theorem 3.12 and Lemma 3.11 provide a simple method for the numerical calculation of the saddlepoint. Because of Theorem 3.12 we have to solve only two systems of equations of the type used in Lemma 3.11. Let x^* be the solution of system (3-23). Because of Lemma (3.11) we have $x_1^* \varepsilon (0,C)$ for $C > 0$. We define $x_L^{(1)} := 0$ and $x_R^{(1)} := C$ and choose

$$x_1^{(1)} = \frac{1}{2} (x_L^{(1)} + x_R^{(1)}) \quad .$$

Then the $x_i^{(1)}$, $i = 2,\ldots,n$ are calculated by consecutively solving the equations

$$h_i(x_i^{(1)}) - h_{i-1}(x_{i-1}^{(1)}) = 0 \quad \text{for } i = 2,\ldots,n \quad .$$

If $\sum_{i=1}^{n} \delta_i x_i^{(1)} > C$ we put $x_R^{(2)} = x_1^{(1)}$, if $\sum_{i=1}^{n} \delta_i x_i^{(1)} < C$ we put $x_L^{(2)} = x_1^{(1)}$ and start the algorithm again with

$$x_1^{(2)} = \frac{1}{2} (x_L^{(2)} + x_R^{(2)}) \quad .$$

It can be seen easily that $\{x^{(i)}\}$ converges to x^*.

3.4 Treatment of a Special Case

In the following we will analyze the question of a saddlepoint of F on $X \times Y'$ where Y' had been given by (3-5):

$$Y' = \{y = (y_1,\ldots,y_n) \in \mathbb{R}^n,\ A^{-1}(y_1,\ldots,y_n)^t \geq (0,\ldots,0)^t,$$

$$y_n + \sum_{i=1}^{n-1} (1 - a_i) y_i = M\} \quad ,$$

i.e. the case $M_i \geq 0$ for $i = 1,\ldots,n$ (see eqs. (2-8)). As Y' is convex, closed and bounded, F has according to Theorem 3.9 a saddlepoint (x',y') on $X \times Y'$. As $y_i' > 0$ for $i = 1,\ldots,n$ must hold which can easily be seen, one can show with the same arguments as used in the proof of Theorem 3.13 that the saddlepoint (x',y') is unique. If for the saddlepoint (x^*,y^*) of F on $X \times Y'$ holds $y^* \in Y'$ then evidently one has $(x',y') = (x^*,y^*)$. Since $Y \supsetneq Y'$, this must not be the case, in fact, there are counter-examples.

In the following we will establish a sufficient condition for y^* to be an element of Y'; thus, in case this condition is fulfilled (x^*,y^*) is a saddlepoint of F on $X \times Y'$.

Theorem 3.14. Let the following inequalities be fulfilled:

(i) $\sigma_i \cdot (1 - a_i) \geq \sigma_{i-1} \cdot (1 - a_{i-1})$ for $i = 2,\ldots,n$,

$$\sigma_n \geq \sigma_{i-1} \cdot (1 - a_{n-1}) ,$$

(ii) $\sigma_i \geq \sigma_{i-1} \cdot a_{i-1}$ for $i = 2,\ldots,n$.

Then it is $y^* \varepsilon Y'$.

Proof. Let us assume $y^* \notin Y'$. Then, using eq. (2-8), there exists an index $i > 1$ with $M_i^* < 0$. Because of Theorem 3.12 y_{i-1}^* and $y_i^* = M_i^* + a_{i-1} \cdot y_{i-1}^*$ fulfill the equation

$$e^{x_i^*} + \frac{U^2(e^{x_i^*})}{2} \cdot Q(U(e^{x_i^*}) - \frac{M_i^* + a_{i-1}y_{i-1}^*}{\sigma_i}) = e^{x_{i-1}^*} + \frac{U^2(e^{x_{i-1}^*})}{2}$$

$$\cdot Q(U(e^{x_{i-1}^*}) - \frac{y_{i-1}^*}{\sigma_{i-1}}) .$$

Since $M_i^* < 0$ and $Q' < 0$ (Lemma 3.6) we have

$$e^{x_i^*} + \frac{U^2(e^{x_i^*})}{2} \cdot Q(U(e^{x_i^*}) - \frac{a_{i-1} \cdot y_{i-1}^*}{\sigma_i}) > e^{x_{i-1}^*} + \frac{U^2(e^{x_{i-1}^*})}{2}$$

$$\cdot Q(U(e^{x_{i-1}^*}) - \frac{y_{i-1}^*}{\sigma_{i-1}}) . \qquad (3\text{-}26)$$

Because of assumption (i) and $y_{i-1}^* > 0$ we have

$$\frac{a_{i-1}}{\sigma_i} \cdot y_{i-1}^* \leq \frac{y_{i-1}}{\sigma_{i-1}} . \qquad (3\text{-}27)$$

Since x_{i-1}^* and x_i^* solve the equation

$$\frac{e^{-(x_i + \frac{U^2(e^{x_i})}{2})}}{\sigma_i \cdot (1 - a_i)} = \frac{e^{-(x_{i-1} + \frac{U^2(e^{x_{i-1}})}{2})}}{\sigma_{i-1} \cdot (1 - a_{i-1})}$$

we have

$$x_i \leq x_{i-1} \tag{3-28}$$

because of assumption (ii) and the fact that $-x - \frac{1}{2} U^2(e^x)$ is strictly monotonously decreasing on $(-\infty,0)$ which is proven in the appendix.

However, as the function

$$e^{x + \frac{U^2(e^x)}{2}} \cdot Q(U(e^x) - \frac{y}{\sigma_i})$$

is monotonously increasing in y since $Q' < 0$ and as it is monotonously increasing in x (see 3-13), inequalities (3-27) and (3-28) yield a contradiction to inequality (3-26). Therefore, the assumption $y^* \notin Y$ is wrong. ■

Let us consider the special case

$$\text{var } I_i = \sigma_I^2 \text{ for } i = 0,1,\ldots,n \text{ and var } D_i = \sigma_D^2 \text{ for } i = 1,\ldots,n \quad .$$

Then for n = 2 the assumptions for Theorem 3.14 are fulfilled as with (2-11b) and (2-5) we have

$$\sigma_2 \geq \sigma_1 \cdot (1 - a_1) \text{ and } \sigma_2 \geq \sigma_1 \cdot a_1 \quad .$$

For n > 2 however, the assumptions for Theorem 3.14 are not fulfilled: as we have

$$a_{1,2} = \frac{\sigma_I^2}{\sigma_{1,2}^2} \quad ,$$

assumption (i) of Theorem 3.14 is not fulfilled if $\sigma_2 < \sigma_1$. Now we have according to eq. (2-6)

$$\text{var } S_1 < \sigma_I^2 \quad .$$

Therefore

$$\sigma_2^2 = \text{var } S_1 + \sigma_D^2 + \sigma_I^2 < \sigma_I^2 + \sigma_D^2 + \sigma_I^2 = \sigma_1^2 \quad .$$

4. Concluding Remark

The material accountability problem treated in this paper has been formulated as a two-person zero-sum inspection game, the payoff of which was the overall probability of detection for the sequence of inventory periods under consideration. This may be considered to be consistent with the intuitive criterion of optimization that any disappearance or diversion of even small amounts of material should be detected with a probability of detection as high as possible.

There is, however, another criteron: any disappearance or diversion of material should be detected as early as possible. As in the framework of our model the detection time is determined by the length of an inventory period--only at the end of an inventory period can a statement be made. According to the second criterion one would like to have the greatest possible number of inventory periods per reference time.

It is clear that for economical reasons only a limited number of physical inventories can be performed per reference time. In fact, in nuclear fabrication and reprocessing plants not more than two to four physical inventories will be performed for safeguards purposes. However, apart from the economical point of view, the criterion of early detection--i.e. large number of inventory periods per reference time--

may conflict with the criterion of high probability of detection for the reference time. There are cases where an increasing number of physical inventories leads to a decrease of the probability of detection; the question of an "optimal" combination of both criteria thus arises.

This problem is subject to further investigation.

Appendix: Properties of the Gaussian Distribution Function

Lemma. Let $\Phi(x)$ be the Gaussian Distribution Function, $U(x)$ its inverse and

$$Q(x) := \frac{\Phi'(x)}{\Phi(x)} \quad .$$

Then the following statements are valid:

a) $Q(x) > -x \quad \text{for } x < 0$

b) $x - \frac{1}{2}U^2(e^{-x})$ is strictly monotonously increasing on $[0,\infty)$

c) $\lim\limits_{x\to\infty} (x - \frac{1}{2}U^2(e^{-x})) = \infty \quad .$

Proof

a) For $R(x)$, defined by eq. (3-10), the inequality

$$R(x) < \frac{1}{x} \quad \text{for } x > 0$$

is valid (cf. [6]). Since $Q(x) = \frac{1}{R(-x)}$ the proof is completed.

b) With the substitution $x = -\ln y$ we have to show

$$\frac{d}{dy}\left(-\ln y - \frac{1}{2}U^2(y)\right) < 0 \quad \text{for } y \in (0,\infty) \quad .$$

As we have

$$\frac{d}{dy}\left(-\ln y - \frac{1}{2}U^2(y)\right) = -\frac{1}{y} - \sqrt{2\pi}\, U(y) e^{\frac{U^2(y)}{2}} \quad ,$$

the inequality is valid for $y \geq \frac{1}{2}$. Therefore, in the following we assume $y < \frac{1}{2}$. By substituting $z = U(y)$ we have $z < 0$, and it remains to be shown that

$$\frac{1}{\Phi(z)} > -\sqrt{2\pi} \cdot z \cdot e^{\frac{z^2}{2}} \quad \text{for } z < 0 \quad .$$

This inequality however, is equivalent to the inequality

$$Q(z) > -z \quad \text{for } z < 0$$

which has been proven in a).

c) As $x - \frac{1}{2}U^2(e^{-x})$ is monotonous we have either $\lim_{x\to\infty} (x - \frac{1}{2}U^2(e^{-x})) = \infty$ or $\lim_{x\to\infty} (x - \frac{1}{2}U^2(e^{-x})) = :k < \infty$.

In the following we assume the second relation to be valid and show that this will lead to a contradiction.

We have

$$x - \frac{1}{2}U^2(e^{-x}) \leq k \quad \text{for } x \in \mathbb{R} \; .$$

Therefore

$$-\sqrt{2(x-k)} \geq U(e^{-x}) \quad \text{for } x \geq \ln 2$$

or

$$\Phi(-\sqrt{2(x-k)}) \geq e^{-x} \quad \text{for } x \geq \ln 2 \; .$$

This is equivalent to

$$\frac{e^k \cdot \Phi(-\sqrt{2(x-k)})}{\sqrt{2\pi}} \geq \frac{e^{-\frac{1}{2}(-\sqrt{2(x-k)})^2}}{\sqrt{2\pi}} \quad \text{for } x \geq \ln 2$$

or

$$\frac{e^k}{\sqrt{2\pi}} \geq Q(-\sqrt{2(x-k)}) \quad \text{for } x \geq \ln 2 \; .$$

This however, contradicts (a). ■

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