PROOF FOR A CASE WHERE DISCOUNTING

ADVANCES THE DOOMSDAY

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Proof for a Case where Discounting Advances the Doomsday [1,2]

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In a previous paper (Koopmans [1]) I considered some problems of " optimal " consumption $\hat{r}_t$ over time of an exhaustible resource of known finite total availability R. In one of the cases studied, consumption of a minimum amount of the resource is assumed to be essential to human life, in such a way that all life ceases upon its exhaustion at time T. Assuming a constant population until that time, and denoting by $\underline{r}$ the positive minimum consumption level needed for survival of that population, the survival period T is constrained by

$$0 < T \leq R/\underline{r} \equiv \overline{T}. \qquad \qquad (1)$$

Here equality $(T = \overline{T})$ can be attained only by consuming at the minimum level $(r_t = \underline{r})$ at all times, $0 \leq t \leq \overline{T}$.

However, optimality is defined in terms of maximization of the integral over time of discounted future utility levels,

$$V(\rho, T, (r_t)) \equiv \int_0^T e^{-\rho t} v(r_t) dt, \qquad \qquad ...(2)$$

where ρ is a discount rate, $\rho \geq 0$, applied in continuous time to the utility flow $v(r_t)$ arising at any time t from a consumption flow r_t of the resource. The utility flow function $v(r)$ is defined for $r \geq \underline{r}$, is twice continuously differentiable and satisfies

$$v'(r) > 0, \qquad \qquad ...(3a)$$

$$v''(r) < 0 \text{ for } r > \underline{r}, \qquad \qquad ...(3b)$$

$$v(\underline{r}) = 0, \qquad \qquad ...(3c)$$

$$\lim_{r \to \underline{r}} v'(r) = \infty. \qquad \qquad ...(3d)$$

That is, $v(r)$ is (a) strictly increasing and (b) strictly concave. The stipulation (c) anchors the utility scale. Some such anchoring, though not necessarily the given one, is needed whenever population size is a decision variable. The last requirement (d) simplifies a step in the proof, and can be secured if needed by a distortion of $v(r)$ in a neighbourhood of $\underline{r}$ that does not affect the solution.

The paper referred to gives an intuitive argument for the following

Theorem. *For each $\rho \geqq 0$ there exists a unique optimal path $r_t = \hat{r}_t$, $0 \leqq t \leqq \hat{T}_\rho$, maximizing* (2) *subject to*

$$r_t \text{ is a continuous function on } [0, T], \qquad \qquad ...(4a)$$

$$\int_0^T r_t dt \leqq R, \; r_t \geqq \underline{r}, \; 0 \leqq t \leqq T. \qquad \qquad ...(4b)$$

For $\rho = 0$, the optimal path $(\hat{r}_t \mid 0 \leqq t \leqq \hat{T}_0)$ is defined by

$$\hat{r}_t = \hat{r}, \text{ a constant, for } 0 \leqq t \leqq \hat{T}_0, \qquad \qquad ...(5a)$$

$$v(\hat{r}) = \hat{r}v'(\hat{r}), \qquad \qquad ...(5b)$$

$$\hat{r}\hat{T}_0 = R. \qquad \qquad ...(5c)$$

For $\rho > 0$ it is defined by

$$e^{-\rho t}v'(\hat{r}_t) = e^{-\rho T_\rho}v'(\hat{r}), \quad 0 \leqq t \leqq \hat{T}_\rho, \quad \hat{r} \text{ as in } (5b), \qquad \qquad ...(6a)$$

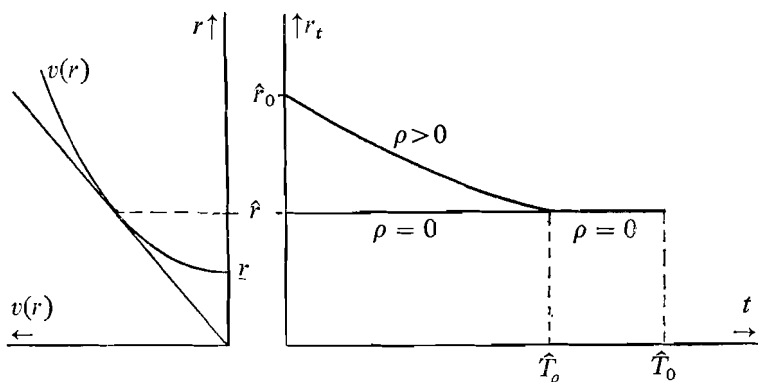$$\int_0^{T_\rho} \hat{r}_t dt = R. \qquad \qquad ...(6b)$$

FIGURE 1

Figure 1 illustrates the solution. For $\rho = 0$, (6) implies (5), and consumption of the resource is constant during survival. Its optimal level $\hat{r}$ is obtained in (5b, c) by balancing the number of years of survival against the constant level of utility flow that the total resource stock makes possible during survival. Since $\hat{r} > \underline{r}$, the optimum survival period $\hat{T}_0$ is shorter than the maximum $\bar{T}$ defined by (1).

For $\rho > 0$, the optimal path $\hat{r}_t$ follows a declining curve given by (6a), which starts from a level $\hat{r}_0$ such that, when resource exhaustion brings life to a stop at time $t = \hat{T}_\rho$, the level $\hat{r}_{T_\rho} = \hat{r}$ is just reached. Since the decline is steeper when ρ is larger, the survival period is shorter, the larger is ρ—which explains the title of this note.

The intuitive argument already referred to gives insight into the theorem; the following proof establishes its validity.

Proof. We first consider paths optimal under the added constraint of some arbitrarily fixed value $T = T^*$ of T satisfying $0 < T^* < \bar{T}$. Assume that such a " T^*-optimal " path r_t^* exists and that

$$r_t^* \geqq \underline{r} + \delta \text{ for } 0 \leqq t \leqq T^* \text{ and some } \delta > 0. \qquad \qquad ...(7)$$

Then, if s_t is a continuous function defined for $0 \leqq t \leqq T^*$ such that

$$| s_t | \leqq \delta, \quad \int_0^{T^*} s_t dt = 0, \qquad \qquad \ldots(8)$$

the path

$$r_t = r_t^* + \varepsilon s_t, \quad 0 \leqq t \leqq T^*, \qquad \qquad \ldots(9)$$

is T^*-feasible for $| \varepsilon | \leqq 1$ and satisfies

$$V(\rho, T^*, (r_t)) - V(\rho, T^*, (r_t^*)) = \int_0^{T^*} e^{-\rho t}(v(r_t) - v(r_t^*)) dt \qquad \ldots(10a)$$

$$= \varepsilon \int_0^{T^*} e^{-\rho t} v'(r_t^*) s_t dt + R(\varepsilon), \qquad \ldots(10b)$$

where the remainder $R(\varepsilon)$ is of second order in ε. It is therefore a necessary condition for the T^*-optimality of r_t^* that

$$p_t \equiv e^{-\rho t} v'(r_t^*) = \text{constant} = e^{-\rho T^*} v'(r_{T^*}^*), \text{ say}, \qquad \ldots(11)$$

because, if we had $p_{t'} \neq p_{t''}$, $0 \leqq t', t'' \leqq T^*$, we could by choosing s_t of one sign in a neighbourhood in $[0, T^*]$ of t', s_t of the opposite sign in one of t'' and zero elsewhere while preserving (8) make the last member of (10) positive for some ε with $| \varepsilon | \leqq 1$.

In the light of (3a, b), (11) justifies our assumption that r_t^* is a continuous function of t. We now find that r_t^* is constant for $\rho = 0$, strictly decreasing for $\rho > 0$. Given $r_{T^*}^*$, say, the solution r_t^* of (11) is uniquely determined, and, for each t, r_t^* is a strictly increasing differentiable function of the given $r_{T^*}^*$. Also, by (3d),

$$\lim_{r_{T^*}^* \to \underline{r}} \int_0^{T^*} r_t^* dt = \int_0^{T^*} \underline{r} dt = T^* \underline{r} < \bar{T} \underline{r} = R,$$

whereas, for sufficiently large $r_{T^*}^*$,

$$\int_0^{T^*} r_t^* dt > R.$$

Therefore there is a unique number $\alpha^* > \underline{r}$ such that the unique solution r_t^* of (11) with $r_{T^*}^* = \alpha^*$ satisfies

$$\int_0^{T^*} r_t^* dt = R. \qquad \qquad \ldots(12)$$

From here on r_t^* will denote that path for the chosen T^*. Note that this path satisfies (7)

To prove the unique T^*-optimality of r_t^*, let r_t be any T^*-feasible path such that $r_{t_0} \neq r_{t_0}^*$ for some $t_0 \in [0, T]$. Then, by the continuity of r_t, r_t^*, $r_t \neq r_t^*$ for all t in some neighbourhood τ of t_0 in $[0, T^*]$. By (3b), for all $t \in [0, T^*]$,

$$v(r_t) - v(r_t^*) \begin{bmatrix} < \\ \leqq \end{bmatrix} (r_t - r_t^*) v'(r_t^*) \text{ for } t \in \begin{bmatrix} \tau \\ \tau^* \end{bmatrix}, \qquad \ldots(13)$$

where $\tau^* = [0, T^*] - \tau$. Therefore, we have from (10a), (11), (4b) with $T = T^*$, and (12) that

$$V(\rho, T^*, (r_t)) - V(\rho, T^*, (r_t^*)) = \left(\int_\tau + \int_{\tau^*} \right) e^{-\rho t}(v(r_t) - v(r_t^*)) dt$$

$$< \int_0^{T^*} (r_t - r_t^*) e^{-\rho t} v'(r_t^*) dt$$

$$= e^{-\rho T^*} v'(r_{T^*}^*) \int_0^{T^*} (r_t - r_t^*) dt \leqq 0.$$

Hence r_t^* is uniquely T^*-optimal.

We now make T^* a variable, writing T instead of T^* and r_t^T instead of r_t^*. Note that, for each t, $0 \leqq t < \bar{T}$, r_t^T is a differentiable function of T for $t \leqq T < \bar{T}$. Therefore

$$V_T \equiv V(\rho, T, (r_t^T)) = \int_0^T e^{-\rho t} v(r_t^T) dt$$

is a differentiable function of T for $0 \leqq T < \bar{T}$, and

$$\frac{dV_T}{dT} = e^{-\rho T} v(r_T^T) + \int_0^T e^{-\rho t} v'(r_t^T) \frac{dr_t^T}{dT} dt$$

$$= e^{-\rho T} v(r_T^T) + e^{-\rho T} v'(r_T^T) \int_0^T \frac{dr_t^T}{dT} dt$$

by (11). But, by (12),

$$0 = \frac{dR}{dT} = r_T^T + \int_0^T \frac{dr_t^T}{dT} dt.$$

Therefore,

$$e^{\rho T} \frac{dV_T}{dT} = v(r_T^T) - r_T^T v'(r_T^T).$$

But then, from (5b), since $d(v(r) - rv'(r))/dr = -rv''(r) > 0$ for $r > \underline{r} > 0$, by (3b),

$$\frac{dV_T}{dT} \begin{bmatrix} < \\ = \\ > \end{bmatrix} 0 \quad \text{for} \quad r_T^T \begin{bmatrix} < \\ = \\ > \end{bmatrix} \hat{r}.$$

Finally, since $0 < T < T' < \bar{T}$ implies $r_{T'}^{T'} \leqq r_T^{T'} < r_T^T$,

$$\frac{dV_T}{dT} \begin{bmatrix} < \\ = \\ > \end{bmatrix} 0 \quad \text{for} \quad T \begin{bmatrix} > \\ = \\ < \end{bmatrix} \hat{T}_\rho.$$

Thus, V_T reaches its unique maximum for that value $\hat{T}_\rho$ of T for which $r_T^T = \hat{r}$.

This establishes the second part of the theorem. The first part follows by specialization when $\rho = 0$.

REFERENCE

[1]   Koopmans, T. C. " Some Observations on ' Optimal ' Economic Growth and Exhaustible Resources ", in Bos, Linnemann and de Wolff, Eds., *Economic Structure and Development*, essays in honours of Jan Tinbergen (North-Holland Publishing Co., 1973), pp. 239-255.