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THE DYNAMICS OF TWO DEMOGRAPHIC MODELS OF URBANIZATION

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Preface

Roughly 1.6 billion people, 40 percent of the world's population, live in urban areas today. At the beginning of the last century, the urban population of the world totaled only 25 million. According to recent United Nations estimates, about 3.1 billion people, twice today's urban population, will be living in urban areas by the year 2000.

Rapid rates of urban demographic and economic growth increase the difficulties of providing a population with adequate supplies of food, energy, employment, social services and infrastructure. The investment needed just to maintain present standards in many rapidly urbanizing countries calls for a doubling or tripling of institutional plant within the next 25 years.

Scholars and policy-makers often disagree when it comes to evaluating the desirability of current rapid rates of urban growth in many parts of the globe. Some see this trend as fostering national processes of socioeconomic development, particularly in the poorer and rapidly urbanizing countries of the Third World; whereas others believe the consequences to be largely undesirable and argue that such urban growth should be slowed down.

Professor Nathan Keyfitz of Harvard University spent the month of May this year collaborating with HSS scholars in their research on migration, urbanization and development. During his stay, he formulated a model of the urbanization process that stimulated a number of us. In particular, Jacques Ledent responded by writing a series of three papers dealing with extensions of the Keyfitz model. This paper, the first of the series, focuses on the dynamics of Keyfitz's model and contrasts them with those of an alternative formulation, in which gross migration flows out of both rural and urban regions are explicitly considered.

A list of related papers in the Population, Resources and Growth Series appears at the end of this publication.

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Abstract

This paper is the first of a series of three which attempt to shed some light on the urbanization phenomenon, using two alternative models of rural-urban population growth. One is a model recently proposed by Keyfitz (1978) in which migration is viewed as a net flow out of the rural region; the other is a continuous two-region version of the model of population growth and distribution developed by Rogers (1968), in which the gross migration flows out of both rural and urban regions are explicity considered.

This first paper examines and compares the dynamics of these alternative models on the basis of constant rates of natural increase and migration. It demonstrates the simplicity of their properties, especially in the case of the second model which also appears more realistic due to its symmetrical treatment of the rural and urban regions. In addition, it suggests the ability of both models to give some insights into the relative rates of natural increase and inmigration in accounting for urban population growth.

This potential will be made evident in the next two papers of this series, in which the factors of urban population growth will be examined under a regime of constant rates of natural increase and migration (second paper) as well as under a regime of varying rates (third paper).

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INTRODUCTION

In order to shed some light on the relative roles of natural increase and migration in the growth of cities, Keyfitz (1978) proposed a continuous model in which he viewed migration between the rural and urban regions as a net flow out of the rural region. This model allowed Keyfitz to address the problem of the sources of urban population growth, which he examined in the case of constant parameters.

To be sure, his assumption of constant rates of natural increase (in both rural and urban regions) and of a constant rural net outmigration rate is a very harsh one. However, no matter how crude such an assumption might be, it has its raison d'être as it answers a question of general interest in mathematical demography: what would happen if rates remained constant?

Keyfitz focused on the relative evolution of the two sources of urban growth--natural increase and net inmigration--over a period of time. But, by dating the time at which natural increase begins to exceed inmigration, the relevance of this model to the urbanization phenomenon has been somewhat neglected. We will show that a particular weakness of the Keyfitz model is its consideration of migration as a net flow from the rural to the urban region and the problems this may create when examining the long-term evolution of a rural-urban population system.

Thus, an obvious alternative to the use of this model, is the application of the interregional components-of-change model developed by Rogers (1968). We will show that such a model, which allows the explicit consideration of migration flows out of both rural and urban regions, has simpler and more realistic properties than Keyfitz's model.

In brief, the purpose of this paper is to examine and compare the dynamics of the two alternative models in view of a reconsideration of the problem originally addressed by Keyfitz, which we will carry out in two subsequent papers: the one will deal with the case of constant rates and the other with the case of varying rates.

The present paper consists of two parts. Part One presents the dynamics of Keyfitz's model and Part Two those of the Rogers model. In both cases, indications of the circumstances in which the model can be used as a model of urbanization are given, and an illustration involving its application to two polar situations (USSR and India) is also shown.

I. ANALYSIS OF KEYFITZ'S MODEL

Basically, Keyfitz (1978) considers a population system divided into two regions, urban and rural, which exhibit constant rates of natural increase, denoted by u and r respectively. In addition, he supposes a net outmigration rate from rural to urban taken as a constant fraction m of the rural population (m is supposed strictly positive).

Specification and Solution of the Model

Then, the equations describing the population growth of the rural and urban regions are:

$$\frac{dP_r(t)}{dt} = (r-m)\ P_r(t) \tag{1a}$$

and

$$\frac{dP_u(t)}{dt} = u\ P_u(t) + m\ P_r(t) \tag{1b}$$

in which $P_r(t)$ and $P_u(t)$ are the populations at time t of the rural and urban regions respectively.

The solution of this system is simply:

$$P_r(t) = \frac{P(0)}{1 + S(0)}\ e^{(r-m)t} \tag{2a}$$

$$P_u(t) = \frac{P(0)}{1 + S(0)} \left[\frac{m}{r-m-u}\ e^{(r-m)t} + \left(S(0) - \frac{m}{r-m-u}\right) e^{ut}\right] \tag{2b)*$$

*If r-m-u = 0, (2b) is to be replaced by:

$$P_u(t) = \frac{P(0)S(0)}{1 + S(0)}\ e^{(r-m)t} \tag{2b'}$$

in which $S(0)$ is the ratio of the urban to rural population in the observation period dated $t = 0$ and $P(0)$ the total population of the system in that period $[= P_r(0) + P_u(0)]$.

The first question that the availability of this solution raises is the one of knowing when $P_r(t)$ and $P_u(t)$ take on positive values over the time continuum $]-\infty, +\infty[$.

Formula (2a) clearly shows that $P_r(t)$ is positive for any value of t whereas formula (2b) indicates that $P_u(t)$ can be negative under certain conditions. Using (2b), we can express the condition that $P_u(t) \geq 0$ as

$$\frac{m\ e^{(r-m)t} - [m - S(0)(r-m-u)]\ e^{ut}}{r-m-u} \geq 0 \quad . \tag{3}$$

On multiplying both sides of (3), by $\frac{r-m-u}{me^{ut}}$, the above condition becomes

$$e^{(r-m-u)t} \geq 1 - \frac{r-m-u}{m} S(0) \qquad \text{if } r-m-u > 0 \tag{4a}$$

or

$$e^{(r-m-u)t} \leq 1 - \frac{r-m-u}{m} S(0) \qquad \text{if } r-m-u < 0 \quad . \tag{4b}$$

Consequently:

a) if $1 - \frac{r-m-u}{m} S(0) \leq 0$, an inequality which requires $r-m-u > 0$, (4a) always holds.

b) if $1 - \frac{r-m-u}{m} S(0) > 0$, both (4a) and (4b) reduce to the same condition

$$t_D = - \frac{1}{u+m+r} \ln \left(1 + \frac{u+m-r}{m} S(0)\right) \quad , \tag{5}$$

where t_D is necessarily negative.

To summarize,

a) if $r \geq u + m \frac{1+S(0)}{S(0)}$, $P_u(t)$ can vary over the whole time continuum.

b) if $r < u + m \frac{1+S(0)}{S(0)}$, $P_u(t)$ can only vary for $t \in [t_D, + \infty[$.

The knowledge of the solution (2a) - (2b) and of the time interval on which it is defined permits one to study the dynamics of the model (1a) - (1b). The analysis is started with a focus on the evolution of the relative sizes of the urban and rural populations.

Relative Growth of the Urban and Rural Populations

Let S(t) be the ratio of the urban to rural populations. An analytical expression can be obtained by dividing (2b) by (2a):

$$S(t) = \frac{m}{r-m-u} + \left(S(0) - \frac{m}{r-m-u}\right) e^{(u+m-r)t} \quad . \qquad (6)^{*}$$

Differentiating (6) with respect to time leads to:

$$\frac{dS(t)}{dt} = S(0)(u+m-r) + m \quad e^{(u+m-r)t} \quad . \qquad (7)$$

It is obvious that:

a) if $r < u + m \frac{1+s(0)}{S(0)}$, $\frac{dS(t)}{dt}$ is positive. Since in this case $P_u(t)$ is defined for $t \geq t_D$, it follows that S(t) monotonically increases from zero (for $t=t_D$) to $+\infty$ (if $r < u + m$) or $\frac{m}{r-m-u}$ (if $r > u + m$) as $t \to +\infty$.

b) if $r > u + m \frac{1+S(0)}{S(0)}$, $\frac{dS(t)}{dt}$ is negative. Since in this case $P_u(t)$ is defined over the whole time continuum, S(t) monotonically decreases from $+\infty$ (for $t \to -\infty$) to $\frac{m}{r-m-u}$ (for $t \to +\infty$)

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*If $r-m-u = 0$, S(t) is calculated by dividing (2b') by (2a) and we obtain:

$$S(t) = S(0) \qquad (6')$$

c) if $r = u + m \frac{1+S(0)}{S(0)}$, $\frac{dS(t)}{dt}$ is equal to zero and $S(t)$ remains constant over the whole time continuum.

Alternatively, we can look at the function $\alpha(t)$ expressing the part of the population which is urban. It is obvious that:

$$\alpha(t) = \frac{S(t)}{1+S(t)} \tag{8}$$

Differentiating (8) with respect to time, leads to:

$$d\alpha(t) = \frac{dS(t)}{[1+S(t)]^2} \tag{9}$$

a relation which says that $\frac{d\alpha(t)}{dt}$ and $\frac{dS(t)}{dt}$ have the same sign, i.e., that the functions $\alpha(t)$ and $S(t)$ vary in the same manner.

a) if $r < u + m \frac{1+S(0)}{S(0)}$, $\alpha(t)$ monotonically increases from zero (for $t = t_D$) to 1 (if $r < u + m$) or $\frac{m}{r-u}$ (if $r > u + m$) as $t \to + \infty$.

b) if $r > u + m \frac{1+S(0)}{S(0)}$, $\alpha(t)$ monotonically decreases from 1 (for $t \to - \infty$) to $\frac{m}{r-u}$ (for $t \to + \infty$).

c) if $r = u + m \frac{1+S(0)}{S(0)}$, $\alpha(t)$ remains constant $\left(= \frac{S(0)}{1+S(0)}\right)$.

The Sources of Population Growth in the Urban Region

The net inmigration rate of the urban region is simply obtained by remarking that the flow out of the rural region is equal to the flow into the urban region:

$$m_u(t) = m \frac{P_r(t)}{P_u(t)} = \frac{m}{S(t)} \tag{10}$$

It follows that the variations of $m_u(t)$ are inversely proportional to those of $S(t)$, i.e.,

a) if $r < u + m \frac{1+S(0)}{S(0)}$, $m_u(t)$ monotonically decreases from $+\infty$ (for $t = t_D$) to zero (if $r < u + m$) or $r-m-u$ (if $r > u + m$).

b) if $r > u + m \frac{1+S(0)}{S(0)}$, $m_u(t)$ monotonically increases from zero (for $t \rightarrow -\infty$) to $r-m-u$ (if $t \rightarrow +\infty$).

c) if $r = u + m \frac{1+S(0)}{S(0)}$, $m_u(t)$ remains constant $\left(= \frac{m}{S(0)}\right)$.

Now, to determine which part of the increase of $P_u(t)$ is due to migration and which part to natural increase, one can examine the variations of the ratio $R(t)$ of net inmigration to natural increase in the urban region. Indeed, if $u \neq 0$,

$$R(t) = \frac{m_u(t)}{u} \tag{11}$$

i.e., after substituting (10) for $m_u(t)$

$$R(t) = \frac{m}{uS(t)} \quad . \tag{12a}$$

Alternatively, one can write (12a) as:

$$S(t) = \frac{m}{uR(t)} \quad . \tag{12b}$$

Differentiating (12a) yields:

$$\frac{dR(t)}{dt} = \frac{m}{uS^2(t)} \frac{dS(t)}{dt} \quad ,$$

which shows that R(t) and S(t) vary in the same direction if $u < 0$ and in opposite directions if $u > 0$. Let us first suppose that the urban rate of natural increase is positive; then:

a) if $r < u + m\,\frac{1+S(0)}{S(0)}$, R(t) monotonically decreases from $+\infty$ (for $t = t_D$) to zero (if $r < u + m$) or $\frac{r-m-u}{u}$ (if $r > u + m$).

The growth of the urban population initially provided by inmigration from the rural region (since the net inmigration rate is initially infinite), tends to be more and more the consequence of natural increase whose relative importance increases monotonically.

b) if $r > u + m\,\frac{1+S(0)}{S(0)}$, R(t) monotonically increases from zero (for $t \to -\infty$) to $\frac{r-m-u}{u}$ (for $t \to +\infty$).

Hence, by contrast, net inmigration tends to increase its share vis-a-vis natural increase in accounting for urban growth.

c) if $r = u + m \frac{1+S(0)}{S(0)}$, R(t) remains constant $= \frac{m}{uS(0)}$

In the alternative case that u is negative, the above conclusions, according to the relative positions of r and $u + m \frac{1+S(0)}{S(0)}$ must be exchanged.

The variations of S(t), $\alpha(t)$, $m_u(t)$ and R(t), according to the values taken by the parameters of the model, are diagrammatically summarized in Figure 1.

Evolution of the Rural and Urban Populations

Let us examine first the variations of the rural population. It is clear from (1a) that

a) if $r < m$, $\frac{dP_r(t)}{dt}$ is negative and thus $P_r(t)$ monotonically decreases tending toward zero as $t \to +\infty$.*

b) if $r > m$, $\frac{dP_r(t)}{dt}$ is positive and thus $P_r(t)$ monotonically increases tending toward $+\infty$ as $t \to +\infty$.*

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*If $r < u + m \frac{1+S(0)}{S(0)}$, the initial value of $P_r(t)$ is $P_r(t_D) = \frac{P(0)\left(1 + \frac{u+m-r}{m} S(0)\right)^{\frac{m-r}{m+u-r}}}{1+S(0)}$ while, if $r \geq u + m \frac{1+S(0)}{S(0)}$, $P_r(t)$ either decreases from $+\infty$ (if $r < m$) or increases from zero (if $r > m$).

Case			$t \to$
I. $r > u + m\,\frac{1+S(0)}{S(0)}$	t		$-\infty \quad\quad +\infty$
	$S(t)$		$+\infty \longrightarrow \frac{m}{r-m-u}$
	$\alpha(t)$		$+\infty \longrightarrow \frac{m}{r-u}$
	$m_u(t)$		$0 \longrightarrow r-m-u$
	$R(t)$	$u > 0$	$0 \longrightarrow \frac{r-m-u}{u}$
		$u < 0$	$0 \longrightarrow \frac{r-m-u}{u}$
II. $r = u + m\,\frac{1+S(0)}{S(0)}$ or $r = u + m$	t		$-\infty \longleftrightarrow +\infty$
	$S(t)$		$S(t) \longleftrightarrow S(0)$
	$\alpha(t)$		$\frac{S(0)}{1+S(0)} \longleftrightarrow \frac{S(0)}{1+S(0)}$
	$m_u(t)$		$\frac{m}{S(0)} \longleftrightarrow \frac{m}{S(0)}$
	$R(t)$		$\frac{m}{uS(0)} \longleftrightarrow \frac{m}{uS(0)}$
III. $r < u + m\,\frac{1+S(0)}{S(0)}$ $(r \neq u + m)$	t		$t_D \quad\quad +\infty$
	$S(t)$		$0 \longrightarrow \begin{cases} +\infty & \text{if } r < u + m \\ \frac{m}{r-m-u} & \text{if } r > u + m \end{cases}$
	$\alpha(t)$		$0 \longrightarrow \begin{cases} 1 & \text{if } r < u + m \\ \frac{m}{r-u} & \text{if } r > u + m \end{cases}$
	$m_u(t)$		$+\infty \longrightarrow \begin{cases} 0 & \text{if } r < u + m \\ r-m-u & \text{if } r > u + m \end{cases}$
	$R(t)$	$u > 0$	$+\infty \longrightarrow \begin{cases} 0 & \text{if } r < u + m \\ \frac{r-m-u}{u} & \text{if } r > u + m \end{cases}$
		$u < 0$	$-\infty \longrightarrow \begin{cases} 0 & \text{if } r < u + m \\ \frac{r-m-u}{u} & \text{if } r > u + m \end{cases}$

$S(t)$ = Ratio of urban to rural population
$\alpha(t)$ = Part of the population which is urban
$m_u(t)$ = Net inmigration rate of the urban population
$R(t)$ = Ratio of net inmigration to natural increase for the urban region.

Figure 1. The Keyfitz model: the variations of $S(t)$, $\alpha(t)$, $m_u(t)$ and $R(t)$

c) if $r = m$, $\frac{dP_r(t)}{dt}$ is equal to zero and thus $P_r(t)$ remains constant, i.e.,

$$P_r(t) = P_r(0) = \frac{P(0)}{1+S(0)}$$

for all $t \in]-\infty, +\infty[$.

By contrast, the variations of the urban population are more complex to obtain. From (1b), it follows that $\frac{dP_u(t)}{dt}$ is non-negative if

$$u\,S(t) + m \geq 0 \quad . \tag{13}$$

To determine the sign of $\frac{dP_u(t)}{dt}$, we thus examine the variations of $u\,S(t) + m$. Two main cases can be distinguished according to the sing of u. Let us suppose first that the natural increase of the urban region is non-negative ($u \geq 0$). Then, $u\,S(t) + m$ varies in the same way as $S(t)$.

a) if $r < u + m \frac{1+S(0)}{S(0)}$, $u\,S(t) + m$ increases from an initial positive value (equal to m).

b) if $r > u + m \frac{1+S(0)}{S(0)}$, $u\,S(t) + m$ decreases from $+\infty$ (for $t \to -\infty$) to $\frac{m(r-m)}{r-m-u}$ (for $t \to +\infty$), a positive value.

c) if $r = u + m \frac{1+S(0)}{S(0)}$, $u\,S(t) + m$ is constant.

Thus, if the natural increase of the urban region is non-negative, $\frac{dP_u(t)}{dt}$ is positive and the population of the urban region increases. Note that $P_u(t)$ originates from zero (for $t = t_D$ if $r < u + m \frac{1+S(0)}{S(0)}$ or for $t \to -\infty$ if $r > u + m \frac{1+S(0)}{S(0)}$*) and becomes infinitely positive as $t \to +\infty$.

Now, let us suppose that the natural increase of the urban region is negative ($u < 0$). Then $u\,S(t) + m$ varies in the opposite direction of $S(t)$. Consequently,

a) if $r < u + m$, $u\,S(t) + m$ decreases from m (for $t = t_D$) to $-\infty$ (for $t \to +\infty$). There exists a value t_u, solution of $u\,S(t) + m = 0$, such that $u\,S(t) + m$ is positive for $t < t_u$ and negative for $t > t_u$. It is simple to establist that:

$$t_u = \frac{1}{u+m-r} \ln \left(\frac{m(r-m)}{u[m-S(0)\ (r-m-u)]} \right) . \tag{14}$$

*In the latter case, if the natural increase of the urban region is zero, $P_u(t)$ increases from $\frac{P(0)\left(S(0) - \frac{m}{r-m}\right)}{1+S(0)}$ (for $t \to -\infty$).

b) if $r + m < r < u + m \frac{1+S(0)}{S(0)}$, $u\,S(t) + m$ decreases from m (for $t = t_D$) to $\frac{m(r-m)}{r-m-u}$ (for $t \rightarrow +\infty$). It follows that if $r > m$, $u\,S(t) + m$ is always positive. Whereas if $r < m$, $u\,S(t) + m$ is positive for $t < t_u$ and negative for $t > t_u$ where t_u is also given by (14).

c) if $r > u + m \frac{1+S(0)}{S(0)}$, $u\,S(t)$ increases from $-\infty$ (for $t \rightarrow -\infty$) to $\frac{r(r-m)}{r-m-u}$ (for $t \rightarrow +\infty$). It follows that if $r < m$, $u\,S(t) + m$ is always negative. Whereas if $r > m$, $u\,S(t) + m$ is negative for $t < t_u$ and positive for $t > t_u$ where t_u is given by (14).

Consequently, if the rate of natural increase in the urban region is negative, the sign of $\frac{dP_u(t)}{dt}$ may change at most once over the interval on which $P_u(t)$ is defined. Four alternative situations can be encountered:

a) if $u + m < r < u + m \frac{1+S(0)}{S(0)}$ and $r > m$, $P_u(t)$ monotonically increases.

b) if $r > u + m \frac{1+S(0)}{S(0)}$ and $r < m$, it monotonically increases.

c) if $r < u + m$ or $u + m < r < u + m \frac{1+S(0)}{S(0)}$ and $r < m$, it increases and then decreases.

d) if $r > u + m \frac{1+S(0)}{S(0)}$ and $r > m$, it decreases and then increases.*

The particular cases $r = u + m \frac{1+S(0)}{S(0)}$ and $r = u + m$ which involve simple expressions of $P_u(t)$ whose variations are obvious remains to be studied.

The above discussion of the variations of $P_r(t)$ and $P_u(t)$ according to the values of the parameters of the model, is diagrammatically summarized in Figure 2.

Use of the Model as a Model of Urbanization

The model defined by (1a) - (1b) has been defined by Keyfitz (1978) to deal with the urbanization phenomenon. In such a context, it is most likely that both the rural and urban natural increase rates be positive and that r be less than u + m. Then, the model is of the third type defined earlier $\left(r < u + m \frac{1+S(0)}{S(0)}\right)$ with the urban population increasing over a time interval $[t_D, +\infty[$.

Figure 3 indicates the variations of the main functions of the model, which are all monotonic. The direction of variations for all these functions is uniquely defined, except for the rural population which monotonically increases or decreases according to the relative magnitudes of r and m.

*In this case, the limiting values of $P_u(t)$ are the following: $P_u(t)$ is initially equal to zero for $t = t_D$ (if $r < u + m \frac{1+S(0)}{S(0)}$) or infinitely positive for $t \to -\infty$ (if $r > u + m \frac{1+S(0)}{S(0)}$). As $t \to +\infty$, $P_u(t)$ tends toward zero except if $u + m < r < u + m \frac{1+S(0)}{S(0)}$ and $r \geq m$ in which case $P_u(t)$ tends toward $-\frac{m\, P(0)}{u(1+S(0))}$

I. $r > u + m\,\frac{1+S(0)}{S(0)}$

	t	$-\infty$	t_u	$+\infty$
$P_r(t)$	(a) $r > m$	0	$\nearrow$	$+\infty$
	(b) $r = m$	$\frac{P(0)}{1+S(0)}$	$\longleftrightarrow$	$\frac{P(0)}{1+S(0)}$
	(c) $r < m$	$+\infty$	$\searrow$	0
$P_u(t)$	(a) $u \geq 0$	$\frac{S(0) - \frac{m}{r-m}P(0)}{1+S(0)}$ if $u = 0$; 0 if $u > 0$	$\nearrow$	$+\infty$
	(b) $u < 0$, $r \geq m$	$+\infty$	$\searrow$	0
	(c) $u < 0$, $r < m$	$+\infty$	$\searrow$ $P_u(t_u)$ $\nearrow$	$+\infty$

II. $r = u + m\,\frac{1+S(0)}{S(0)}$ or $r = u + m$

	t	$-\infty$		$+\infty$
$P_r(t)$	(a) $r > m$	0	$\nearrow$	$+\infty$
	(b) $r = m$	$\frac{P(0)}{1+S(0)}$	$\longleftrightarrow$	$\frac{P(0)}{1+S(0)}$
	(c) $r < m$	$+\infty$	$\searrow$	0
$P_u(t)$	(a) $r > m$	0	$\nearrow$	$+\infty$
	(b) $r = m$	$\frac{P(0)S(0)}{1+S(0)}$	$\longleftrightarrow$	$\frac{P(0)S(0)}{1+S(0)}$
	(c) $r < m$	$+\infty$	$\searrow$	0

IIIa. $u + m < r < u + m\,\frac{1+S(0)}{S(0)}$

	t	t_D	t_u	$+\infty$
$P_r(t)$	(a) $r > m$	$P_r(t_D)$	$\nearrow$	$+\infty$
	(b) $r = m$	$\frac{P(0)}{1+S(0)}$	$\longleftrightarrow$	$\frac{P(0)}{1+S(0)}$
	(c) $r < m$	$P_r(t_D)$	$\searrow$	0
$P_u(t)$	(a) $u \geq 0$	0	$\nearrow$	$+\infty$
	(b) $u < 0$, $r = m$	0	$\nearrow$	$+\infty$
	(c) $u < 0$, $r > m$	0	$\nearrow$	$-\frac{mP(0)}{u\,1+S(0)}$
	(d) $u < 0$, $r \leq m$	0	$\nearrow$ $P_u(t_u)$ $\searrow$	0

IIIb. $r < u + m$

	t	t_D	t_u	$+\infty$
$P_r(t)$	(a) $r > m$	$P_r(t_D)$	$\nearrow$	$+\infty$
	(b) $r = m$	$\frac{P(0)}{1+S(0)}$	$\longleftrightarrow$	$\frac{P(0)}{1+S(0)}$
	(c) $r < m$	$P_r(t_D)$	$\searrow$	0
$P_u(t)$	(a) $u \geq 0$	0	$\nearrow$	$+\infty$
	(b) $u \leq 0$	0	$\nearrow$ $P_u(t_u)$ $\searrow$	0

Figure 2. The Keyfitz model: the variations of $P_u(t)$ and $P_r(t)$

	t	t_D		$+\infty$
$P_r(t)$	(a) $r > m$	$P_r(t_D)$	$\nearrow$	$+\infty$
	(b) $r = m$	$\frac{P(0)}{1+S(0)}$	$\longrightarrow$	$\frac{P(0)}{1+S(0)}$
	(c) $r < m$	$P_r(t_D)$	$\searrow$	0
$P_u(t)$		0	$\nearrow$	$+\infty$
$S(t)$		0	$\nearrow$	$+\infty$
$\alpha(t)$		0	$\nearrow$	1
$m_u(t)$		$+\infty$	$\searrow$	0
$R(t)$		$+\infty$	$\searrow$	0

Figure 3. The Keyfitz model as a model of urbanization: the variations of the model's functions

To summarize, when the model is used as a model of urbanization, the system appears as one which:

a) originates with its whole population concentrated in the rural region and,

b) presents a regional distribution such that the ratio of urban to rural population is equal to $S(0)$, after a period of time equal to $-t_D$ where t_D is given by (5).

Numerical Illustration: Application to USSR and India

The application of the above model to the USSR and India demonstrates the polar subcases stemming from the two possible directions of variations for the rural population.

The USSR Case

Rogers (1976) reports that the urban population of the USSR was growing at an annual rate of approximately 2.5 percent during the early 1970s. This rate was the sum of a rate of natural increase of 0.9 percent and a net migration rate of 1.6 percent. At the same time, the rural population was declining at an annual rate of 1.1 percent which was the sum of a rate of natural increase of 1.0 percent and a net migration rate of -2.1 percent. Then, in this system,

$$u = 0.009 \quad ; \quad r = 0.010 \quad \text{and} \quad m = 0.021$$

while the ratio of urban to rural population, equal to the ratio of the rural net outmigration rate to the urban net inmigration rate is,

$$S(0) = \frac{0.021}{0.016} = 1.3125 \qquad .$$

This corresponds to a ratio (0) equal to 0.5676, i.e., 56.76 percent of the whole population lives in urban areas. From the above parameter values, we easily observe that r which is equal to 0.010 is smaller than u + m which takes on the value of 0.030. Then the rural-urban system of the USSR follows the pattern described in Figure 3.

The system evolves from an initial state--in which the whole population is concentrated in the rural region--occurring $(-t_D)$

years before the observation period:

$$t_D = \frac{1}{0.02} \ln 2.25 = -40.5 \text{ years}$$

the urban population monotonically increases (with growth rate $u = 0.9$ percent) while the rural population monotonically decreases (with growth rate $r-m = -1.1$ percent)* and eventually vanishes. In other words, the population of the system tends to become entirely urban.

The Indian Case

We provide another illustration for India, observed in the late sixties, for which data can be found in Rogers and Willekens (1976).

$$u = 0.20 \quad ; \quad r = 0.002 \quad \text{and} \quad m = 0.005 \quad .$$

Noting that the urban net inmigration rate is 0.017, this yields a ratio $S(0)$ equal to $\frac{0.005}{0.017}$, i.e., 0.294 (which corresponds to an urban population accounting for 22.73 percent of the whole population).

From these parameter values, we immediately observe that r, which is equal to 0.022, is smaller than $u + m$ which takes on the value of 0.025. Then, the rural-urban system of India follows the pattern described in Figure 3.

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*The size of the observed rural population is 0.64 times its initial size

$$\frac{P_r(0)}{P_r(t_D)} = (2.25)^{-0.555} = 0.64$$

The system evolves from an initial state--in which the whole population is concentrated in the rural region--occurring $(-t_D)$ years before the observation period:

$$t_D = - \frac{1}{0.003} \ln 1.1764 = -54.2 \text{ years}$$

Then the rural and urban populations monotonically increase tending toward + with growth rates which are respectively $r - m = 1.8$ percent and $u = 2$ percent.* But, the ratio $S(t)$ of urban to rural population also monotonically increases, which corresponds to a *relative* concentration of the population in the urban region.

II. ANALYSIS OF THE ROGERS TWO-REGION MODEL

As an alternative to the assumption of a constant net migration rate out of the rural region, we can introduce a more symmetric hypothesis, e.g., constant gross migration rates out of the rural and urban regions denoted by o_r and o_u respectively. The ensuing model is a two-region components-of-change model (Rogers 1968), applied to a rural-urban population system.

Specification and General Solution of the Model

Instead of the discrete setting put forth by Rogers, a continuous setting--analogous to the one underlying Keyfitz's model--may be used to describe the population growth of the rural and

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*The size of the observed rural population is 2.51 times its initial size:

$$\frac{P_r(0)}{P_r(t_D)} = (1.764)^{5.6666} = 2.51$$

and urban regions.* The corresponding equations are respectively:

$$\frac{dP_r(t)}{dt} = (r-o_r)\ P_r(t) + o_u\ P_u(t) \tag{15a}$$

and

$$\frac{dP_u(t)}{dt} = o_r\ P_r(t) + (u-o_u)\ P_u(t) \tag{15b}$$

The above equations can then be rewritten more compactly as:

$$\left\{\frac{dP(t)}{dt}\right\} = \underset{\sim}{M}\ \{P(t)\} \tag{16}$$

in which $\{P(t)\}$ is a two-element vector whose first element is $P_r(t)$ and whose second element is $P_u(t)$, $\left\{\frac{dP(t)}{dt}\right\}$ is the first derivative of $\{P(t)\}$ with respect to time, and $\underset{\sim}{M}$ is a two by two matrix.

$$\underset{\sim}{M} = \begin{bmatrix} r-o_r & o_u \\ o_r & u-o_u \end{bmatrix} \tag{17}$$

*The discrete version of the model which indeed yields similar results is briefly examined in the appendix.

The solution of (16) is immediately obtained as:

$$\{P(t)\} = a_1 \, e^{x_1 t} \, \{w_1\} + a_2 \, e^{x_2 t} \, \{w_2\} \tag{18}$$

in which x_1 and x_2 are the two eigenvalues of $\underset{\sim}{M}$ and $\{w_1\}$ and $\{w_2\}$ the corresponding right eigenvectors.

The eigenvalues of $\underset{\sim}{M}$ are the two roots of

$$[x - (r-o_r)]\,[x - (u-o_u)] - o_r \, o_u = 0 \tag{19}$$

which can be rewritten as:

$$x^2 - (r-o_r + u-o_u) \, x + (r-o_r)(u-o_u) - o_r \, o_u = 0 \tag{20}$$

The discriminant of (20) which is equal to

$$\Delta = (r-o_r + u-o_u)^2 - 4(r-o_r)(u-o_u) + 4 \, o_r \, o_u$$

can be rewritten as:

$$\Delta = [r-o_r - (u-o_u)]^2 + 4 \, o_r \, o_u \quad ,$$

an expression that is always positive.

Thus, the two real eigenvalues x_1 and x_2, solutions of (20) are always positive.

The Solution of the Model in Terms of the Eigenvalues

If one suppose that the second element of $\{w_1\}$ and $\{w_2\}$ are different from zero, $\{w_1\}$ and $\{w_2\}$ can be written as:

$$\{w_1\} = \begin{Bmatrix} x_1 - (u-o_u) \\ o_r \end{Bmatrix} \quad ; \tag{21a}$$

and

$$\{w_2\} = \begin{Bmatrix} x_2 - (u-o_u) \\ o_r \end{Bmatrix} \tag{21b}$$

Substituting (21a) and (21b) into (18) and setting $t = 0$, leads to two linear equations in a_1 and a_2

$$P_r(0) = \frac{P(0)}{1+S(0)} = a_1 \left[x_1 - (u-o_u)\right] + a_2 \left[x_2 - (u-o_u)\right] \tag{22a}$$

$$P_u(0) = \frac{S(0)P(0)}{1+S(0)} = (a_1 + a_2)\, o_r \quad . \tag{22b}$$

The solution to this system is readily obtained as:

$$a_1 = \frac{S(0)\left[(u-o_u) - x_2\right] + o_r}{\left[1+S(0)\right] o_r \,(x_1 - x_2)} \tag{23a}$$

and

$$a_2 = - \frac{S(0)\left[(u-o_u) - x_1\right] + o_r}{\left[1+S(0)\right] o_r \ (x_1 - x_2)} \qquad (23b)$$

Substituting (23a) and (23b) into (18) and observing that

$$\left[(u-o_u) - x_1\right]\left[(u-o_u) - x_2\right] = - o_r \ o_u \quad *$$

yields:

$$\left\{P(t)\right\} = \frac{P(0)}{\left[1+S(0)\right](x_1 - x_2)} \left[e^{x_1 t} \left\{ \begin{array}{c} S(0)o_u + x_1 - (u-o_u) \\ S(0)\left[(u-o_u) - x_2\right] + o_r \end{array} \right\} \right.$$

$$\left. - \ e^{x_2 t} \left\{ \begin{array}{c} S(0)o_u + x_2 - (u-o_u) \\ S(0)\left[(u-o_u) - x_1\right] + o_r \end{array} \right\} \right] \qquad (24)$$

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* $\left[(u-o_u) - x_1\right]\left[(u-o_u) - x_2\right] = (u-o_u)^2 - (u-o_u)(x_1-x_2) + x_1 \ x_2$. Since the sum of x_1 and x_2 is $u - o_u + r - o_r$ and their product $(r-o_r)(u-o_u) - o_r \ o_u$ (see equation 20), this expression then reduces to $- o_r \ o_u$.

Thus, the rural and urban populations at time t can be written as:

$$P_r(t) = A\, e^{x_1 t} - B\, e^{x_2 t} \tag{25a}$$

$$P_u(t) = C\, e^{x_1 t} - D\, e^{x_2 t} \tag{25b}$$

in which:

$$A = \frac{P(0)\,\left[S(0)\, o_u + x_1 - (u-o_u)\right]}{\left[1+S(0)\right]\,(x_1 - x_2)} \quad ; \tag{26a}$$

$$B = \frac{P(0)\,\left[S(0)\, o_u + x_2 - (u-o_u)\right]}{\left[1+S(0)\right]\,(x_1 - x_2)} \quad ; \tag{26b}$$

$$C = \frac{P(0)\left[S(0)\,\left[(u-o_u) - x_2\right] + o_r\right]}{\left[1+S(0)\right]\,(x_1 - x_2)} \quad ; \tag{26c}$$

$$D = \frac{P(0)\left[S(0)\,\left[(u-o_u) - x_1\right] + o_r\right]}{\left[1+S(0)\right]\,(x_1 - x_2)} \quad . \tag{26d}$$

Note that $\frac{C}{A}$ is the ratio of the two elements of $\left\{w_1\right\}$ and $\frac{D}{B}$ the ratio of the two elements of $\left\{w_2\right\}$. Then from (21a) and (21b) we have:

$$\frac{C}{A} = \frac{o_r}{x_1 - (u-o_u)} \tag{27a}$$

$$\frac{D}{B} = \frac{o_r}{x_2 - (u-o_u)} \tag{27b}$$

or, alternatively by substituting (19) into these two last equations:

$$\frac{C}{A} = \frac{x_1 - (r-o_r)}{o_u} \quad ; \tag{28a}$$

$$\frac{D}{B} = \frac{x_2 - (r-o_r)}{o_u} \quad ; \tag{28b}$$

Let us suppose that x_1 is the largest of the two roots. When setting x equal to $(u-o_u)$ in (20), one obtains a negative left hand side. Thus $(o-o_u)$ is between the two roots of (20), i.e.,

$$x_2 \leq u - o_u \leq x_1 \quad * \tag{29}$$

An immediate consequence of this inequality is that A and C are positive (see relations 25a and 26b): B and D however have no definite sign but necessarily have opposite signs (see relation 27b), i.e., BD < 0 or B = D = 0.

The first problem that the availability of the solution (25a) - (25b) raises is the question of knowing when $P_r(t)$ and $P_u(t)$ take on positive values over the time continuum $]-\infty, +\infty[$.

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*Generally, the inequalities contained in (29) are strict inequalities. An equality between $(u-o_u)$ and one of the roots x_1 and x_2 can occur only if $o_u = 0$.

Formulas (25a) and (25b) indicate that $P_r(t)$ and $P_u(t)$ are strictly psotive over the whole time continuum only if $B = D = 0$. In the general case, only one of the two populations is positive for all values of t--the rural population if $B < 0$, (i.e., $D > 0$) the urban population if $B > 0$ (i.e., $D < 0$)--while the other one is negative for all values of t less than a fixed value.

a) if $B < 0$, then the urban population is positive for $t > t_D$ where

$$t_D = \frac{1}{x_1 - x_2} \ln \frac{D}{C} \quad . \tag{30}$$

Recalling (26c) and (26d) leads to

$$t_D = \frac{1}{x_1 - x_2} \ln \left(\frac{S(0) \left[(u-o_u) - x_1 \right] + o_r}{S(0) \left[(u-o_u) - x_2 \right] + o_r} \right) , \tag{31}$$

an expression which shows that t_D is necessarily negative: $x_1 > x_2$ imples that $D < C$ and thus $\ln \frac{D}{C} < \ln 1 = 0$.

b) If $B > 0$, then the rural population is positive for $t > t_C$, where

$$t_C = \frac{1}{x_1 - x_2} \ln \frac{B}{A} \tag{32}$$

Recalling (26a) and (26b) leads to

$$t_c = \frac{1}{x_1 - x_2} \ln \left(\frac{S(0)\ o_u + x_2 - (u-o_u)}{S(0)\ o_u + x_1 - (u-o_u)} \right) , \tag{33}$$

an expression which shows that t_c is necessarily negative: $x_1 > x_2$ implies that $B < A$ and thus $\ln \frac{B}{A} < \ln 1 = 0$.

Long-term Behavior of the Model

It is a well known fact (Rogers 1968) that the interregional components-of-change model (16) if subjected to the unchanging schedule of fertility, mortality and mobility that is defined by $\underset{\sim}{M}$ in (17), ultimately will increase at a constant stable rate and will assume an unchanging stable distribution.

From (18), one can see that, if t tends toward + ∞, $\{P(t)\}$ is equivalent to its first component $a_1 e^{x_1 t} \{w_1\}$, which suggests that the stable distribution of the total population is given by the eigenvector corresponding to this dominant eigenvalue.

Relative Growth of the Urban and Rural Populations

The ratio S(t) of the urban to rural population is obtained by dividing (25b) by (25a):

$$S(t) = \frac{C\ e^{x_1 t} - D\ e^{x_2 t}}{A\ e^{x_1 t} - B\ e^{x_2 t}} \tag{34}$$

Dividing both numerator and denominator by $e^{x_2 t}$ and differentiating with respect to time leads to:

$$\frac{dS(t)}{dt} = \frac{(AD-BC)(x_1-x_2)\ e^{(x_1-x_2)t}}{\left\{e^{(x_1-x_2)t}\left[S(0)\ o_u + x_1 - (u-o_u)\right] - \left[S(0)\ o_u + x_2 - (u-o_u)\right]\right\}^2} \quad (35)$$

Since $x_1 > x_2$, it follows that $\frac{dS(t)}{dt}$ has the sign of AD-BC. Because of the restrictions concerning B and D, three alternative cases are possible:

a) $D > 0$ and $B < 0$, i.e., $AD-BC > 0$, then $\frac{dS(t)}{dt}$ is positive,

b) $D = 0$ and $B = 0$, i.e., $AD-BC = 0$, then $\frac{dS(t)}{dt}$ is equal to zero,

c) $D < 0$ and $B > 0$, i.e., $AD-BC < 0$, then $\frac{dS(t)}{dt}$ is negative.

Recalling (26a) through (26d), we can express AD-BC in terms of the parameters of the model and state that $\frac{dS(t)}{dt}$ has the sign of

$$\left[S(0)\ o_u + x_1 - (u-o_u)\right]\left[(S(0)\ [(u-o_u) - x_1] + o_r\right]$$

$$-\left[S(0)\ o_u + x_2 - (u-o_u)\right]\left[(S(0)\ [(u-o_u) - x_2] + o_r\right]$$

Carrying out the multiplications included in this last quantity, rearranging terms, and factoring leads to:

$$(x_1-x_2)\left[-S(0)(x_1+x_2) - S(0)^2 o_u + 2S(0)(u-o_u) + o_r\right] .$$

After substituting $\sum = r - o_r + u - o_u$ for (x_1+x_2), we thus obtain that $\frac{dS(t)}{dt}$ has the sign of

$$f\big(S(0)\big) = - S(0)^2 o_u + S(0)\left[(u-o_u) - (r-o_r)\right] + o_r \quad . \qquad (36)$$

First we note that if there exists no outmigration from the urban area, i.e., $o_u = 0$ we obtain again the conclusions of Section I: $\frac{dS(t)}{dt}$ has the sign of $S(0)[u + o_r - r] + o_r$*.

In the general case $(o_u \neq 0)$, the polynomial $f\big(S(0)\big)$ has a discriminant:

$$\Delta = \left[(u-o_u) - (r-o_r)\right]^2 + 4\, o_r\, o_u$$

which is positive and thus admits two real roots; since the product of these roots is $-\frac{o_r}{o_u}$ and thus negative, these roots have opposite signs.

*The assumption of no urban outmigration leaves us with a gross migration flow out of the rural region, which formally appears no different from the net migration case considered by Keyfitz.

Then,

a) if $S(0)$ is less than the positive root $S_m(0)$ of $f\,S(0) = 0$, i.e.,

$$S(0) < S_m(0) = \frac{(u-o_u) - (r-o_r)+\sqrt{\left[(u-o_u)(r-o_r)\right]^2 + 4\,o_r\,o_u}}{2\,o_u} \tag{37}$$

AD-BC is positive (i.e., $D > 0$ and $B < 0$). Therefore $\frac{dS(t)}{dt}$ is positive and $S(t)$ monotonically increases from zero (for $t = t_D$).

b) if $S(0) = S_m(0)$, AD-BC = 0 (i.e., $D = B = 0$) and thus $\frac{dS(t)}{dt}$ is equal to zero, i.e., $S(t)$ is constant.

c) if $S(0) > S_m(0)$, AD-BC is negative (i.e., $D < 0$ and $B > 0$). Thus $\frac{dS(t)}{dt}$ is negative and $S(t)$ monotonically decreases from $+\infty$ (for $t = t_c$).

As t tends towards $+\infty$, both numerator and denominator are equivalent to their terms in $e^{x_1 t}$ and thus $S(t)$ tends toward

$$S(\infty) = \frac{C}{A} = \frac{S(0)\left[(u-o_u) - x_2\right] - o_r}{S(0)\,o_u + x_1 - (u-o_u)} \quad . \tag{38}$$

Alternative equivalent expressions of (38) can be obtained by using (27a) and (28a):

$$S(\infty) = \frac{o_r}{x_1 - (u-o_u)} = \frac{x_1 - (r-o_r)}{o_u}\ * \qquad (39)$$

Alternatively, we can look at the function $\alpha(t)$ expressing the part of the population which is urban. From (9) we know that $\alpha(t)$ and $S(t)$ vary in the same direction.

a) if $S(0) < S_m(0)$, $\alpha(t)$ monotonically increases from zero (for $t = t_D$) toward $\alpha(\infty) = \frac{S(\infty)}{1+S(\infty)}$. Recalling (39), we have

$$\alpha(\infty) = \frac{o_r}{x_1 - u + o_u + o_r} \qquad (40)$$

b) if $S(0) = S_m(0)$, $\alpha(t)$ is constant

c) if $S(0) > S_m(0)$, $\alpha(t)$ monotonically decreases from $+\infty$ (for $t = t_D$) toward $\alpha(\infty)$, also given by (4).

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* Comparing $S(\infty) = \frac{x_1 - (r-o_r)}{o_u}$ with $S_m(0)$ given by (37) reveals that $S_m(0) = S(\infty)$.

The Sources of Population Growth in the Urban Region

The net migration rate of the urban region is simply:

$$m_u(t) = o_r \frac{P_r(t)}{P_u(t)} - o_u$$

$$= \frac{o_r}{S(t)} - o_u \qquad (41)^*$$

The urban net inmigration rate thus experiences variations opposite to those of $S(t)$:

a) if $S(0) < S_m(0)$, $m_u(t)$ monotonically decreases from $+\infty$ (for $t = t_D$) toward $m_u(\infty) = \frac{o_r}{S(\infty)} - o_u$, i.e., by substituting (39)

$$m_u(\infty) = x_1 - u \qquad (44)$$

---

*Note that the net inmigration of the rural population is

$$m_r(t) = o_u \frac{P_u(t)}{P_r(t)} - o_r = o_u S(t) - o_r \qquad (42)$$

Its variations are similar to those of $S(t)$. As $t \to +\infty$, it tends toward

$$m_r(\infty) = o_u S(\infty) - o_r = x_1 - r \qquad (43)$$

b) if $S(0) = S_m(0)$, $m_u(t)$ is constant.

c) if $S(0) > S_m(0)$, $m_u(t)$ monotonically increases from $-o_u$ (for $t = t_D$) to $m_u(\infty)$, also given by (44).

Now, to determine which part of the increase of $P_u(t)$ is due to migration and which part to natural increase, one can examine the variations of the ratio $R(t)$ of urban net inmigration to urban natural increase, given by (11).

Substituting (41) into (11) yields:

$$R(t) = \frac{o_r}{uS(t)} - \frac{o_u}{u} \quad , \tag{45a}$$

or alternatively

$$S(t) = \frac{o_r}{uR(t) + o_u} \quad . \tag{45b}$$

Differentiating (45a) with respect to time leads to:

$$\frac{dR(t)}{dt} = - \frac{o_r}{uS^2(t)} \frac{dS(t)}{dt} \quad , \tag{46}$$

an equation showing that $R(t)$ and $S(t)$ vary in the same direction if $u < 0$, and in opposite directions if $u > 0$.

Let us suppose first, that the urban rate of natural increase is positive. Then:

a) if $S(0) < S_m(0)$, $R(t)$ monotonically decreases from $+\infty$ to $R(\infty) = \frac{m_u(\infty)}{u}$, i.e.,

$$R(\infty) = \frac{x_1 - u}{u} \tag{47}$$

b) if $S(0) = S_m(0)$, $R(t)$ is constant.

c) if $S(0) > S_m(0)$, $R(t)$ monotonically increases from $-\frac{o_u}{u}$ to $R(\infty)$ given by (47).

Thus:

a) if $S(0) < S_m(0)$, the growth of the urban population, initially provided by inmigration from the rural region (since the net inmigration rate is initially infinite), tends to be more and more the consequence of natural increase, whose relative importance monotonically increases.

b) if $S(0) > S_m(0)$, net inmigration tends to increase its share vis-a-vis natural increase in accounting for urban growth.

In the alternative case of $u < 0$, the above conclusions, according to relative positions of $S(0)$ and $S_m(0)$, must be exchanged.

Finally, the variations of $S(t)$, $\alpha(t)$, $m_u(t)$ and $R(t)$, according to the values taken by the parameters of the model, are diagrammatically summarized in Figure 4.

Evolution of the Rural and Urban Populations

From (15a) and (15b) it is clear that

$$\frac{dP_r(t)}{dt} \text{ is } \underset{(\text{negative})}{\text{positive}} \text{ if } P_u(t) \underset{(<)}{>} \frac{o_r - r}{o_u} P_r(t) \text{ ,}$$

i.e.,

$$S(t) \underset{(<)}{>} \frac{o_r - r}{o_u}$$

and that

$$\frac{dP_u(t)}{dt} \text{ is } \underset{(\text{negative})}{\text{positive}} \text{ if } P_r(t) \underset{(<)}{>} \frac{o_u - u}{o_r} P_u(t) \text{ ,}$$

i.e.,

$$\frac{1}{S(t)} \underset{(<)}{>} \frac{o_u - u}{o_r}$$

Combining this finding with the earlier results on the variations of $S(t)$ affirms that

a) if $S(0) < S_m(0)$, $\frac{dP_r(t)}{dt}$ is positive for all $t \in \,]t_D, +\ [$ when the smallest value of $S(t)$ over that range (i.e., zero) is larger than $\frac{o_r}{o_u}$, which is possible when $r > o_r$.

I. $S(0) > S_m(0)$	t	t_c		$+\infty$
	$S(t)$	$+\infty$	$\longrightarrow$	$\frac{x_1 - (r-o_r)}{o_u}$
	$\alpha(t)$	1	$\longrightarrow$	$\frac{o_r}{x_1 - u + o_u - o_r}$
	$m_u(t)$	0	$\longrightarrow$	$x_1 - u$
	$R(t)$ $u > 0$	0	$\longrightarrow$	$\frac{x_1 - u}{u}$
	$R(t)$ $u < 0$	0	$\longrightarrow$	$\frac{x_1 - u}{u}$
II. $S(0) = S_m(0)$	t	$-\infty$		$+\infty$
	$S(t)$	$S(0)$	$\longleftrightarrow$	$S(0)$
	$\alpha(t)$	$\frac{S(0)}{1+S(0)}$	$\longleftrightarrow$	$\frac{S(0)}{1+S(0)}$
	$m_u(t)$	$-o_u + \frac{o_r}{S(0)}$	$\longleftrightarrow$	$-o_u + \frac{o_r}{S(0)}$
	$R(t)$	$\frac{o_r - o_u S(0)}{uS(0)}$	$\longleftrightarrow$	$\frac{o_r - o_u S(0)}{uS(0)}$
III. $S(0) < S_m(0)$	t	t_D		$+\infty$
	$S(t)$	0	$\longrightarrow$	$\frac{x_1 - (r-o_r)}{o_u}$
	$\alpha(t)$	0	$\longrightarrow$	$\frac{o_r}{x_1 - u + o_u - o_r}$
	$m_u(t)$	$+\infty$	$\longrightarrow$	$x_1 - u$
	$R(t)$ $u > 0$	$+\infty$	$\longrightarrow$	$\frac{x_1 - u}{u}$
	$R(t)$ $u < 0$	$-\infty$	$\longrightarrow$	$\frac{x_1 - u}{u}$

$S(t)$ = Ratio of Urban to Rural Population
(t) = Part of the population which is Urban
$m_u(t)$ = Net inmigration rate of the Urban population
$R(t)$ = Ratio of net inmigration to natural increase for the Urban region.

Figure 4. The Rogers model: the variations of $S(t)$, $\alpha(t)$, $m_u(t)$ and $R(t)$.

By contrast, if $r < o_r$, $\frac{dP_r(t)}{dt}$ is negative for all $t \in]t_D, t_r[$ in which t_r is the solution of $S(t) = \frac{o_r - r}{o_u}$ and positive for all $t > t_r$.*

It can be established that

$$t_r = \frac{1}{x_1 - x_2} \ \ln \left(\frac{Do_u - B(o_r - r)}{Co_u - A(o_r - r)} \right) \tag{48}$$

Substituting (28a) and (28b) into the last expression leads to:

$$t_r = \frac{1}{x_1 - x_2} \ \ln \left(\frac{Bx_2}{Ax_1} \right)$$

$$= \frac{1}{x_1 - x_2} \ \ln \left(\frac{\left[S(0) \ o_u + x_2 - (u - o_u) \right] x_2}{\left[S(0) \ o_u + x_1 - (u - o_u) \right] x_1} \right) \tag{49}$$

a) It is clear that t_r is negative if $S(0) > \frac{o_r - r}{o_u}$, and positive if $S(0) < \frac{o_r - r}{o_u}$.

b) if $S(0) > S_m(0)$, $\frac{dP_r(t)}{dt}$ is positive for all $t \in]t_c, + \infty[$ if the smallest value of $\frac{1}{S(t)}$ over that range

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*Note that t_r exists because $\frac{o_r - r}{o_u} < S(\infty) = \frac{x_1}{o_u} + \frac{o_r - r}{o_u}$. Since $B < 0$, from (49) we find that x_2 is negative, which can also be seen from the fact that $x_2 = u - o_u + r - o_r - x_1$ where $u - o_u - x_1 < 0$ and $r - o_r < 0$.

(i.e., zero) is larger than $\frac{o_u - u}{o_r}$, i.e., when $u > o_u$.

By contrast, if $u < o_u$, $\frac{dP_r(t)}{dt}$ is negative for all $t \in\,]t_c, t_u]$ where t_u is the solution of $S(t) = \frac{o_r}{o_u - u}$, and positive for all $t > t_u$.*

It can be shown that:

$$t_u = \frac{1}{x_1 - x_2} \ln \left(\frac{D(o_u - u) - Bo_r}{C(o_u - u) - Ao_r} \right) \tag{50}$$

Substituting (27a) and (27b) in the last expression leads to

$$t_u = \frac{1}{x_1 - x_2} \ln \left(\frac{Dx_2}{Cx_1} \right)$$

$$= \frac{1}{x_1 - x_2} \ln \left(\frac{\left[S(0)\left[(u-o_u) - x_1\right] + o_r\right] x_2}{\left[S(0)\left[(u-o_u) - x_2\right] + o_r\right] x_1} \right) \tag{51}$$

a quantity which is negative if $S(0) > \frac{o_r}{o_u - u}$ and positive if $S(0) < \frac{o_r}{o_u - u}$.

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*Note that t_u exists since $\frac{o_u - u}{o_r}$ is lesser than $\frac{1}{S(\infty)} = \frac{x_1}{o_r} + \frac{o_u - u}{o_r}$ [as obtained from (39)]. Since $D < 0$, from (51) we find that x_2 is negative, which can also be seen from the fact that $x_2 = u - o_u + r - o_r + x_1$ where $r - o_r - x_1 < 0$ and $u - o_u > 0$.

We must still determine the variations of $P_u(t)$ when $S(0) < S_m(0)$ and of $P_r(t)$ when $S(0) > S_m(0)$. Let us examine the former case first. Recalling (25b) we may write $\frac{dP_r(t)}{dt}$ as

$$\frac{dP_u(t)}{dt} = e^{x_2 t}\left[Cx_1\, e^{(x_1-x_2)t} - Dx_2\right] . \tag{52}$$

Recalling expression (30), we have for all the values of t, for which $P_u(t)$ is defined (i.e., $t > t_D$)

$$e^{(x_1-x_2)t} > e^{(x_1-x_2)t_D} > \frac{D}{C} .$$

Substituting this inequality into (52), yields:

$$\frac{dP_u(t)}{dt} > e^{x_2 t}\; D(x_1-x_2) . \tag{53}$$

Since the right hand side of (53) is positive [D is positive because of the assumption $S(0) < S_m(0)$], $\frac{dP_u(t)}{dt}$ is positive over the time continuum for which $P_u(t)$ is defined.

In the same way, if $S(0) > S_m(0)$, we can show that for any $t > t_C$

$$\frac{dP_r(t)}{dt} > e^{x_2 t}\; B(x_1-x_2) \tag{54}$$

is an equality which indicates that $\frac{dP_r(t)}{dt}$ is positive over the time continuum for which $P_r(t)$ is defined.*

Thus, in all circumstances, both the rural and urban populations become infinitely large at $t \to +\infty$. In most instances, their variation is monotonic over their intervals of definition. Exceptions only occur for the urban population if $S(0) > S_m(0)$ and $u < o_u$ and for the rural population if $S(0) < S_m(0)$ and $r < o_r$: in these cases, the populations concerned first decrease until they reach a minimum for values of t equal to t_u and t_r and given by (50) and (49) respectively--and then increase indefinitely.

The above discussion of the variations of $P_r(t)$ and $P_u(t)$ according to the values of the parameters of the model is diagrammatically summarized in Figure 5.

Use of the Model as a Model of Urbanization

It is obvious from the variations of S(t) that the above model is a suitable model of urbanization only if:

$$S(0) < S_m(0) \quad ,$$

an assumption which allows S(t) to increase as time goes by. If such inequality holds, then $P_u(t)$ monotonically increase from zero (for $t = t_D$) to $+\infty$ (as $t \to +\infty$) and, if u is positive, R(t) monotonically decreases.

Figure 6 indicates the variations of the main functions of the model, which are all monotonic, except in the case of the rural population, which if $r < o_r$, may decrease first before

*In the special case $S(0) = S_m(0)$, B and D are equal to zero and $\frac{dP_r(t)}{dt}$ and $\frac{dP_u(t)}{dt}$ are both positive for any value of $t \in]-\infty,+\infty[$.

I. $\dot{S}(0) > S_m(0)$		t	t_c	t_u	$+\infty$
	$P_r(t)$		0		$+\infty$
	$P_u(t)$	(a) $u < o_u$	$P_u(t_c)$	$P_u(t_c)$	$+\infty$
		(b) $u > o_u$	$P_u(t_c)$		$+\infty$
II. $S(0) = S_m(0)$		t	$-\infty$		$+\infty$
	$P_r(t)$		0		$+\infty$
	$P_u(t)$		0		$+\infty$
III. $S(0) < S_m(0)$		t	t_D	t_r	$+\infty$
	$P_r(t)$	(a) $u < o_r$	$P_r(t_D)$	$P_r(t_r)$	$+\infty$
		(b) $u > o_r$	$P_r(t_D)$		$+\infty$
	$P_u(t)$		0		$+\infty$

$$t_c = \frac{1}{x_1 - x_2} \ln \frac{S(0)o_u + x_2 - (u-o_u)}{S(0)o_u + x_1 - (u-o_u)} \quad ; \quad t_u = \frac{1}{x_1 - x_2} \ln \frac{[S(0)(u-o_u-x_1) + o_r]x_2}{[S(0)(u-o_u-x_2) + o_r]x_1}$$

$$t_D = \frac{1}{x_1 - x_2} \ln \frac{S(0)(u-o_u-x_1) + o_r}{S(0)(u-o_u-x_2) + o_r} \quad ; \quad t_r = \frac{1}{x_1 - x_2} \ln \frac{[S(0)o_u + x_2 - (u-o_u)]x_2}{[S(0)o_u + x_1 - (u-o_u)]x_1}$$

Figure 5. The Rogers model: the variations of $P_u(t)$ and $P_r(t)$

increasing indefinitely. (Note the difference with Keyfitz's model, in which the rural population could either increase or decrease indefinitely.)

Therefore, when the model is used as a model of urbanization, the system appears as one that:

a) originates with its whole population concentrated in the rural region, and

b) presents a regional distribution such that the ratio of urban to rural population is equal to S(0) after a period of time equal to $-t_D$ where t_D is given by (31).

<u>Numerical Illustration: Application to USSR and India</u>

We now turn to the application of Rogers' model to the two actual population systems already examined in Part I.

<u>The USSR Case</u>

Let us recall that:

$$u = 0.009 \quad ; \quad r = 0.010 \quad \text{and} \quad S(0) = 1.3125 \quad .$$

As indicated in Rogers (1976), the gross urban and rural outmigration rates are respectively:

$$o_u = 0.011 \quad \text{and} \quad o_r = 0.035 \quad .$$

From these parameter values, it follows that the observed distribution S(0) which is equal to 1.3125 is smaller than

$$S_m(0) = \frac{0.023 + \sqrt{0.002069}}{0.022}$$

$$= \frac{0.023 + 0.0455}{0.022} = 3.113 \quad .$$

t	t_D		t_r		$+\infty$
$P_r(t)$ (a) $r < o_r$	$P_r(t_D)$	$\searrow$	$P_r(t_r)$	$\nearrow$	$+\infty$
$P_r(t)$ (b) $r \geq o_r$	$P_r(t_D)$		$\nearrow$		$+\infty$
$P_u(t)$	0		$\nearrow$		$+\infty$
$S(t)$	0		$\nearrow$		$\frac{x_1-(r-o_r)}{o_u}$
$\alpha(t)$	0		$\nearrow$		$\frac{o_r}{x_1-u+o_u-o_r}$
$m_u(t)$	$+\infty$		$\searrow$		x_1-u
$R(t)$	$+\infty$		$\searrow$		$\frac{x_1-u}{u}$

Figure 6. The Rogers model as a model of urbanization: the variations of the model's functions

The rural-urban system of the USSR follows the patterns described in Figure 6. It evolves from an initial concentration of the population in the rural region occurring ($-t_D$) years before the observation period.*

$$t_D = \frac{1}{0.0455} \ln \frac{0.0245}{0.0850} = -30.2 \text{ years}$$

*The calculation of the eigenvalues of $\underset{\sim}{M}$ gives:

$$x_1 = 0.00925 \qquad \text{and} \qquad x_2 = 0.03625$$

The rural population first decreases until it reaches a minimum for

$$t_r = \frac{1}{0.0455} \ln \frac{0.0007195}{0.0002371} = 24.4 \text{ years}$$

(i.e., 5.8 years before the observation period) and then increases indefinitely.* By contrast, the urban population monotonically increases tending toward $+\infty$. As $t \to +\infty$, both populations grow at the same growth rate (x_1 = .925 percent).

The ratio $S(t)$ of the urban to rural population monotonically increases from zero (for $t = t_D$) to $S(\infty) = \frac{0.0925 + 0.025}{0.011} =$ 3.113. This corresponds to an ultimate regional distribution in which 75.44 percent of the population is urban against 56.76 percent currently (and 100 percent in the long-run of the Keyfitz model).

The urban net inmigration rate decreases from $+\infty$ (for $t = t_D$) to $m_u(\infty) = 0.00925 - 0.009$, i.e., 0.025 percent, which is about 0.03 times the rate of natural increase. Thus, in the long-run, as net inmigration becomes neglibible, natural increase is preponderant in accounting for urban growth (about 97 percent).

The Indian Case

In the case of the rural-urban system of India in the late sixties, we already know that:

$$u = 0.029 \quad ; \quad r = 0.022 \quad \text{and} \quad S(0) = 0.294$$

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*The size of the observed rural population is 0.58 times its initial size.

The gross urban and rural outmigration rates are respectively $o_u = 0.010$ and $o_r = 0.007$ (Rogers and Willekens, 1976).

From these parameter values we find that the observed population distribution S(0) which is equal to 0.254 is smaller than

$$S_m(0) = \frac{-0.005 + \sqrt{0.000305}}{0.020}$$

$$= \frac{-0.005 + 0.0175}{0.020} = 0.625 \quad ,$$

Then, the functions of the model follow the patterns of Figure 6 with $P_r(t)$ monotonically increasing since $r > o_r$.

We find that the Indian system evolves from an initial concentration of the population in the rural region occurring $(-t_D)$ years before the observation period.*

$$t_D = \frac{1}{0.0175} \ln \frac{0.0036925}{0.008837} = -\frac{0.872}{0.0175} = -49.9 \text{ years}$$

instead of -54.2 years as given by Keyfitz's model.

Over the years the rural population monotonically increases tending toward $+\infty$ with a long-term growth rate equal to x_1 = 2.125 percent against 1.7 percent in Keyfitz's model.** The urban population follows a similar evolution tending toward $+\infty$ with the same long-term growth rate (against 2 percent in Keyfitz's model).

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*The calculation of the eigenvalues of $\underset{\sim}{M}$ leads to

$$x_1 = 0.02125 \qquad \text{and} \qquad x_2 = 0.00375$$

**The size of the observed rural population is 2.29 times its initial size.

On the other hand, the ratio S(t) of the urban to rural population monotonically increases from zero (for $t = t_D$) to $S(\infty) = \frac{0.02125 - 0.015}{0.010} = 0.625$ which corresponds to an ultimate regional distribution in which 38.46 percent of the population is urban against 22.73 percent currently (and 100 percent in the long-run of Keyfitz's model).

The urban net inmigration rate decreases from $+\infty$ (for $t = t_D$) to $m_u(\infty) = 0.02125 - 0.020$, i.e., .125 percent, which is about .06 times the rate of natural increase. Thus, in the long-run as net inmigration becomes negligible, natural increase is preponderent in accounting for urban growth (about 94 percent).

CONCLUSION

From a theoretical point of view, there are at least three main reasons that make the Rogers model more desirable than the Keyfitz model:

1) The Keyfitz model incorporates less information than the alternative model: it considers only the net migration flow between the rural and urban region, whereas Rogers's model includes the gross migration flows occurring in both directions. In this sense, the first model appears as a reduced form of the second.

2) The Keyfitz model views the net migration flow as one coming out of the rural region. Alternatively, it would be possible to propose a similar model viewing this flow as one entering the urban region (with a constant net inmigration rate). Then, it is not clear which of the two models would be preferred.

3) The Keyfitz model does not lend itself to a straightforward extension to the case of three or more regions, while Rogers's model can easily be extended if gross

migration flows out of each region are separately considered by region of destination (Rogers, 1968).

Moreover, the analysis of the dynamics of the two models with constant rates has revealed that the Rogers model posseses simpler and more realistic properties. On the one hand, the completion of this analysis was easier with the Rogers model than with the Keyfitz model because, unlike the alternative model, it does not introduce any asymmetry between the urban and rural regions. On the other hand, the Rogers model, when applied to the problem of urbanization, leads to ultimately growing populations in both rural and urban regions, while the Keyfitz model may in some circumstances lead to emptying out rural regions (see the USSR example).

The contrast between the long-term evolution of the two models, in the context of urbanization, can be easily seen in the comparison of Figures 3 and 6--which summarize the dynamics of the two models--as well as from Table 1--which recalls the main results obtained by the application of the two models to two actual cases (USSR and India).

We note that the Keyfitz model leads to a concentration of the population in the urban region: it corresponds either to an emptying out of the rural area (USSR case) or to an urban population becoming infinitely large with respect to the rural population (India case). By contrast, the Rogers model, in which the urban and rural regions eventually grow at the same rate, leads to a long-term equilibrium distribution. In general, this equilibrium is characterized by an urban region in which a large part of the population growth is caused by natural increase: net migration does not tend toward zero as in Keyfitz's model, but towards a small fraction of the total growth (about 3 and 6 percent in the cases of the USSR and India respectively).

Finally, the simple variations of the ratio R(t) of net inmigration to natural increase in the urban region suggests the ability of both models to give some insights into the relative roles of the natural increase and inmigration of urban population growth.

This potential will be made evident in the next two papers of this series in which the factors of urban population growth will be examined under a regime of constant rates (second paper) as well as under a regime of varying rates (third paper).

Table 1. Application to the USSR and India - Comparison of the main results obtained from two alternative models

	USSR		INDIA	
	Keyfitz's Model	Rogers' Model	Keyfitz's Model	Rogers' Model
t_D	- 40.5 years	- 30.2 years	- 54.2 years	- 49.9 years
$\frac{P_r(0)}{P_r(t_D)}$	0.64	0.58	2.51	2.29
$P_r(\infty)$	0	$+\infty$	$+\infty$	$+\infty$
$\frac{dP_r(\infty)}{dt}$	- 1.1 percent	0.925 percent	1.7 percent	2.125 percent
$P_u(\infty)$	$+\infty$	$+\infty$	$+\infty$	$+\infty$
$\frac{dP_u(\infty)}{dt}$	0.9 percent	0.925 percent	2 percent	2.125 percent
$S(\infty)$	$+\infty$	3.113	$+\infty$	0.625
$\alpha(\infty)$	100 percent	75.44 percent	100 percent	38.46 percent
$m_u(\infty)$	0	.025 percent	0	.125 percent
$R(\infty)$	0	.03	0	.06

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Appendix

A BRIEF NOTE ON THE DISCRETE COUNTERPARTS ON THE TWO MODELS

The two continuous models examined in this paper have discrete counterparts which permit one to determine the evolution of an urban-rural population system in case the variations of the rates of natural increase and migration are not simple.

If those rates remain constant, one can, however, derive analytically the rural and urban populations which, as shown below, appear to resemble their counterparts in the continuous model. Similar results and formulas then follow.

1. The Keyfitz Model: Discrete Version

The equations describing the evolution of the system are now:

$$P_r(t+1) - P_r(t) = (r-m)\ P_r(t) \tag{A1}$$

and

$$P_u(t+1) - P_u(t) = u\ P_u(t) + m\ P_r(t) \quad . \tag{A2}$$

The solution of this system is given by:

$$P_r(t) = \frac{P(0)}{1+S(0)}\ (1 + r-m)^t \tag{A3}$$

and

$$P_u(t) = \frac{P(0)}{1+s(0)} \left[\frac{m}{r-m-u}\ (1 + r-m)^t + \left(S(0) - \frac{m}{r-m-u} \right) (1+u)^t \right] \tag{A4}$$

From this last formula, it is clear that

a) if $r > u + m \frac{1+S(0)}{S(0)}$, t varies over the whole time continuum $]-\infty,+\infty[$.

b) if $r < u + m \frac{1+S(0)}{S(0)}$, t varies from a lower bound

$$t_D = - \frac{\ln\left(1 + \frac{u+m-r}{m} S(0)\right)}{\ln\left(\frac{1+u}{1+r-m}\right)} \tag{A5}$$

2. The Rogers Model: Discrete (Original) Version

The equations describing the evolution of the system are now now:

$$P_r(t+1) - P_r(t) = (r-o_r)\ P_r(t) + o_u\ P_u(t) \tag{A6}$$

and

$$P_u(t+1) - P_u(t) = o_r\ P_r(t) + (u-o_u)\ P_u(t) \tag{A7}$$

The above equations can then be rewritten more compactly as:

$$\{P(t+1)\} = \underset{\sim}{G}\ \{P(t)\} \tag{A8}$$

in which

$$\underset{\sim}{G} = \begin{bmatrix} 1+r-o_r & o_u \\ o_r & 1+u-o_u \end{bmatrix} \tag{A9}$$

The solution of this system is:

$$\{P(t)\} = \frac{P(0)}{[1+S(0)]\,(\lambda_1-\lambda_2)} \left[\lambda_1^t \begin{Bmatrix} S(0)o_u + \lambda_1 - (1+u-o_u) \\ S(0)\left[(1+u-o_u) - \lambda_2\right] + o_r \end{Bmatrix} \right.$$

$$\left. - \lambda_2^t \begin{Bmatrix} S(0)o_u + \lambda_2 - (1+u-o_u) \\ S(0)\left[(1+u-o_u) - \lambda_1\right] + o_r \end{Bmatrix} \right] \tag{A10}$$

in which λ_1 and λ_2 are the two eigenvalues of $\underset{\sim}{G}$.

From (A10) we have

a) if $S(0)\,o_u + \lambda_2 - (1+u-o_u) < 0$, t varies from a lower bound

$$t_D = \frac{\ln\left(\dfrac{S(0)\left[(1+u-o_u) - \lambda_1\right] + o_r}{S(0)\left[(1+u-o_u) - \lambda_2\right] + o_r}\right)}{\ln \dfrac{\lambda_1}{\lambda_2}} \tag{A11}$$

b) if $S(0)\ o_u + \lambda_2 - (1+u-o_u) > 0$, t varies from a lower bound

$$t_c = \frac{\ln\left(\dfrac{S(0)\ o_u + \lambda_2 - (1+u-o_u)}{S(0)\ o_u + \lambda_1 - (1+u-o_u)}\right)}{\ln \dfrac{\lambda_1}{\lambda_2}} \qquad \text{(A11)}$$

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