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COST ALLOCATION AND DEMAND REVELATION IN PUBLIC ENTERPRISES

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ABSTRACT

A classic problem in both the public economics and game theory literature is how to allocate the joint costs of a public enterprise equitably among the customers. Traditional normative solutions, like Ramsey pricing or the Shapley value, have the distinct disadvantage of requiring full information about demand, which in practice may not be known. This paper describes a simple noncooperative bidding mechanism that discovers the efficient set to serve and simultaneously allocates costs. Consumers bid to be served and the game regulator offers to serve that coalition that maximizes net surplus. It is shown that a Nash equilibrium -- indeed a strong Nash equilibrium -- for this noncooperative game always exists, no matter what the cost function, and the resulting set of consumers served is economically efficient. The resulting allocations constitute a normative solution concept for cooperative games that is apparently new and generalizes the core in a natural way. The principal application is to determine prices and output levels for a regulated public enterprise with incomplete information about demand.

COST ALLOCATION AND DEMAND REVELATION IN PUBLIC ENTERPRISES*

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1. MOTIVATION

The aim of this paper is to describe a simple method for allocating joint costs in cooperative or public enterprises by a noncooperative bidding mechanism.

Traditional approaches to the cost allocation problem suffer from the difficulty that, implicitly or explicitly, demands -- and hence the optimal scale of production -- are assumed to be known. For example, the literature on public utility pricing and optimal taxation concentrates on the following situation: there are large homogeneous markets, demands are known, and the object is to find prices and outputs that maximize some objective like consumer surplus, subject to a zero-profit constraint (Ramsey, 1927; Manne, 1952; Baumol and Bradford, 1970). Recently, cooperative game theory has begun to be applied to problems in cost allocation and regulated pricing; see for example Sharkey, 1974; Faulhaber, 1975; Sharkey and Telser, 1978; Zajac, 1978; Young, Okada and Hashimoto, 1980).
These methods usually take the set of goods to be produced as given, and seek to allocate the full costs of producing these goods among the consumers according to normative and strategic

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considerations. The methods are able to cope with more general cost functions and small, differentiated markets but implicitly they assume it is known which goods should be produced and what consumers are willing to pay for them.

The approach we take here differs from these in that neither demands nor the optimal level of production are assumed known. Rather, they are "discovered" by use of a noncooperative bidding mechanism. The method is in much the same spirit as some of the recent literature on the design of incentives for public goods (see Green and Laffont, 1979; Laffont, 1979) and indeed is similar to what Green and Laffont call a "direct revelation mechanism". But there is an important difference: in these other mechanisms the cost allocations are typically incorporated into the description of the alternatives themselves, but the selection mechanisms are insensitive to which particular way of allocating the costs is chosen.

In the present approach an explicit cost allocation emerges as an equilibrium of a competitive bidding process. Typically there is no dominant strategy solution, so the outcome may not reveal consumers' true demands; nevertheless it reveals them partially -- enough to ensure that the level of production is efficient. Since this is the essential point of knowing the demands in the first place, little is lost. An equilibrium outcome in pure strategies always exists and, we will show, has a very natural interpretation in terms of generalized marginal cost pricing.

However, as in the public goods problem, there may be a surplus generated by the bidding process, i.e. the consumers may bid more than the total cost of serving them. In one sense, this surplus can be viewed as the price of eliciting information, but in some cases it is a deeper phenomenon that is connected with the structure of the cost function itself and might be called "structural surplus". Indeed it may correspond to nothing more than ordinary producer's surplus in a competitive market (see Example 2 below).

Finally it is worth emphasizing that this is meant to be a partial equilibrium model and we assume away any income effects by unabashedly describing consumers' utility in terms of willingness-to-pay. The chief application of the model is in situations where the costs of goods and services of a public or cooperative enterprise must be allocated among a well-defined (usually small) number of potential customers and their willingness-to-pay is not known.

2. PROBLEM FORMULATION

A very convenient way to describe the cost allocation problem in considerable generality is by means of the characteristic function concept in cooperative game theory. Let $N = \{1,2,...,n\}$ denote a finite set of *players* who are potential customers of goods or services provided by a cooperative or public enterprise (such as a water, electricity, or telephone company). For simplicity, we assume that each customer is either served or not at some targeted level; in other words each customer is identified with a specific consumption bundle. Under suitable assumptions the model can be generalized readily to allow for different levels of consumption and nondiscrete goods. We also allow price discrimination but the model can be readily modified to accommodate nondiscriminatory solutions.

Let $c(S)$ be the *cost* of serving the customers in the set S, for every subset $S \subseteq N$. By convention we take $c(\phi) = 0$. The function c is the enterprise's *joint cost function.*

It is frequently the case that such an enterprise enjoys increasing returns to scale. One way of expressing this condition is to say that the cost function is *subadditive*, that is, $c(S) + c(T) \geqq c(S \cup T)$ whenever S and T are disjoint. In other words the single enterprise can produce S and T at least as cheaply as two enterprises could produce S and T separately. In fact this "joint production" assumption is sometimes taken to be a definition of a *natural monopoly* in the public utility literature (Faulhaber, 1975; Zajac, 1978). However this assumption is not needed in much of what follows, and will only be noted when necessary.

Let b_i be the *benefit*, or *willingness-to-pay*, of customer i to be included in the enterprise, and define the surplus *value* of any coalition S to be its members' total willingness-to-pay, net of costs:

$$(1) \qquad v(S) = \sum_S b_i - c(S) \qquad \text{for all } S \subseteq N \ .$$

The Pareto ideal is to serve some set S having maximum surplus value; any such set S is said to be *efficient*. A significant problem, in the absence of information on demand, is to determine which set, or sets, are efficient.

In cooperative game theory, knowledge of the efficient set is typically taken for granted. Further, the value of any coalition is usually defined to be the maximum value among all of its subcoalitions. This gives the characteristic function

$$(2) \qquad \bar{v}(S) = \max_{T \subseteq S} v(T) \geqq 0 \qquad \text{for all } S \subseteq N \ .$$

By definition, $\bar{v}(N)$ is the maximum surplus obtainable by any coalition. The usual approach is then to apply some normative solution concept like the core, the nucleolus, or the Shapley value to allocate the benefits, x_i (equivalently the costs p_i, the connection being that $x_i = b_i - p_i$). For example, the core of $\bar{v}$ is the set of all vectors $\underset{\sim}{x} = (x_1,\ldots,x_n)$ satisfying

$$(3) \qquad \sum_N x_i = \bar{v}(N) \qquad \textit{(zero-profit)}$$

and

$$(4) \qquad \sum_S x_i \geqq \bar{v}(S) \qquad \textit{(group-rationality)} \ .^*$$

When voluntary agreement is required the group-rationality principle seems quite compelling. It also has another interesting interpretation in the context of public utility pricing: if prices of a regulated monopoly are set such that $\sum_S p_i - c(S) > 0$ for some subset of products S (which implies that (4) is violated) there is a risk that another firm could underbid these prices and still make a profit. Hence one reason for choosing a solution in the core is that it prevents competitive entry (Faulhaber, 1975; Panzar and Willig, 1977).

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*A special case of this set-up is the problem of providing a single public good at fixed cost with exclusion possible. Then the cost function has the form $c(\phi) = 0$ and $c(S) = c \geqq 0$ for all $S \neq \phi$. If $b_i \geqq 0$ for all i then the efficient set is N and the surplus game has a core.

In the public utilities literature the classical approach to cost allocation is *Ramsey pricing*, which asks for prices and quantities that maximize consumer surplus subject to a profit constraint, usually zero-profit. In the present framework this has a simple enough interpretation: find prices $p_i \leqq b_i$ and a set S such that $\sum_S p_i = c(S)$ and $\sum_S b_i - \sum_S p_i$ is maximized. In other words, find an efficient set and divide its costs in any way consistent with willingness-to-pay. (Ramsey pricing becomes more interesting under nondiscriminatory pricing.)

In practice, the difficulty with both of these approaches is that only i may know b_i. The problem for the regulator is how to elicit sufficient information about the b_i's to implement an efficient decision, and simultaneously cover costs. We now describe a mechanism for achieving this.

3. THE NONCOOPERATIVE BIDDING MECHANISM

Let each player submit a sealed bid naming the amount, p_i, that he would be willing to pay to be served (this may not equal his true willingness-to-pay). If included, he pays p_i and his net payoff is $b_i - p_i$; if not his payoff is zero*. On the basis of the bids submitted, the regulator or auctioneer determines a set S that maximizes $\sum_S p_i - c(S)$ and announces this set. (Note that $\sum_S p_i \geqq c(S)$ since the empty set has zero surplus; indeed, the empty set may be the one announced.) The players may then revise their bids. The regulator terminates the process after some predetermined (but undisclosed) number of rounds or by using some convergence criterion. The last-announced set S^* is the definitive outcome: the excluded players have no recourse, and the enterprise serves the players in S^* at the prices last bid. Since $\sum_{S^*} p_i \geqq c(S^*)$, all costs are allocated and in some cases a surplus may remain. The existence of an end surplus may be associated with the price of eliciting information, but in some situations it also has a

*This assumes that i cannot provide the good or service for himself at cost c(i). If he can, then his payoff would be $\max\{b_i - c(i), 0\}$ even when he is excluded from the enterprise. The equilibrium payoffs in this case turn out to be exactly the same as in the case treated in the text (provided c is subadditive) because in the latter case the equilibrium prices must satisfy $p_i \leqq \min\{b_i, c(i)\}$ (see Young, 1980).

structural meaning. In particular, if $\overline{v}$ has no core, then the existence of a surplus may be interpreted instead as the price of achieving a stable outcome. The possibility of a surplus is well-known in other treatments of the public goods problem (Green and Laffont, 1979).

As a practical matter, the surplus might be taken in the form of a lump-sum tax. Another possibility is that it be redistributed to the players served in proportion to their final bids. In theory this could lead to players distorting their willingness-to-pay by overbidding, but as a practical matter it seems unlikely that much distortion would result because the players do not know the others' bids (indeed may not even know the cost function), hence do not know how much is available to be redistributed.

Formally a *mechanism* is defined as a function $g(p_1,...,p_n) = S$ that tells which set is served as a function of the bids submitted. g can also be interpreted as a supply function, since it gives quantities produced as a function of prices. g is a *surplus-maximizing* mechanism if

$$(5) \quad g(p_1,...,p_n) = S \quad \text{implies} \quad \sum_S p_i - c(S) \geqq \sum_{S'} p_i - c(S') \quad \text{for all } S' \subseteq N .$$

Note that the surplus-maximizing g is uniquely defined, except when two or more sets have exactly the same surplus with respect to $\underset{\sim}{p}$. Note also that $g(\underset{\sim}{p})$ may be the empty set, and that the surplus of $g(\underset{\sim}{p})$ is always nonnegative.

The *cost allocation game* associated with a mechanism g is the normal-form, noncooperative game ϕ defined as follows:

$$\phi_i(\underset{\sim}{p}) = \begin{cases} b_i - p_i & \text{if } i \in g(\underset{\sim}{p}) \\ 0 & \text{if } i \notin g(\underset{\sim}{p}) . \end{cases}$$

A highly desirable equilibrium concept for such a game is a strengthening of the Nash equilibrium due to Aumann (1959), known as a strong equilibrium. A *strong equilibrium* has the property that no player, or group of players, can simultaneously change their strategies and each do better (barring side-payments). This concept has sometimes been interpreted as a cooperative equilibrium notion, since it means that no coordinated action by a coalition of players can improve the payoffs of all members of the coalition. But it is also an important noncooperative equilibrium concept, since it means that no uncoordinated, but simultaneous, groping by the players will be reinforced. Any observer of simultaneous auctions will recognize this as an important condition for stability. Unfortunately, not many games possess such equilibria.

Theorem 1. *For any cost function* c *there exists a surplus-maximizing mechanism* g *such that the cost allocation game has a strong equilibrium. Moreover, for any such equilibrium* $\underset{\sim}{p}$ *the set produced,* $g(\underset{\sim}{p})$*, is efficient.*

Some choices of g, i.e. some ways of breaking ties in the surplus-maximizing mechanism, do not produce an equilibrium. In theory, this means that the auctioneer may have to "probe" different ways of breaking ties if ties occur; in practice of course exact ties are unlikely to arise.

We next illustrate the theorem by several examples. The proof is given in the Appendix.

Example 1.

Let 1, 2, and 3 be three towns that can be connected to a common municipal water system. The capacity needed to supply each town is assumed to be predetermined, the decision being only whether to connect or not. The costs of connection are as follows:

$$c(\phi) = 0$$

$$\begin{array}{llllll} c(1) & = & 32 & c(12) & = & 50 \\ c(2) & = & 35 & c(13) & = & 56 \\ c(3) & = & 40 & c(23) & = & 60 \end{array}$$

$$c(123) = 86$$

Consider for example the prices $p_1 = 30$, $p_2 = 25$, $p_3 = 30$. With these prices, {1,2} is uniquely the most profitable set. If this set is announced as the tentative outcome, player 3 may try to raise his price, and players 1 and 2 may try to lower theirs. It may be checked that there is only one strong equilibrium: namely, $p_1^* = 26$, $p_2^* = 30$, $p_3^* = 36$. At $\underset{\sim}{p}^*$, each of the sets {1,2}, {1,3}, {2,3} and {1,2,3} has a surplus of 6 units and equilibrium is achieved with any surplus-maximizing mechanism g that breaks the tie in favor of the set {1,2,3} i.e. such that $g(\underset{\sim}{p}^*) = \{1,2,3\}$. Then no player can lower his bid without being excluded, and no one has an incentive to raise his bid. It is also interesting to note that p_i^* is just the marginal cost of serving i, that is, the difference between c(N) and c(N-i), for sll i. This result holds generally for 3-person, superadditive games without cores (Young, 1980).

Example 2.

Consider six towns of equal size that can be supplied with water from a common source. The common cost c(s) of developing the source depends only on the number s of towns served, as shown in Figure 1. In addition there may be a fixed connection for each town, depending on its distance from the source. We assume that these direct costs must be paid by each town separately if connected, so do not enter into the cost allocation problem explicitly. Net of direct connection costs let the *demands* for being supplied from the common source be given by $\underset{\sim}{b} = (18,18,15,14,12,8)$; as shown in Figure 2. Figure 2 also shows for each number of participants s the marginal cost c(s) - c(s-1) of serving the s^{th} participant.

The efficient number of towns to serve is five, since beyond that the marginal cost exceeds the benefits. The strong equilibrium price occurs, roughly speaking, where the demand curve

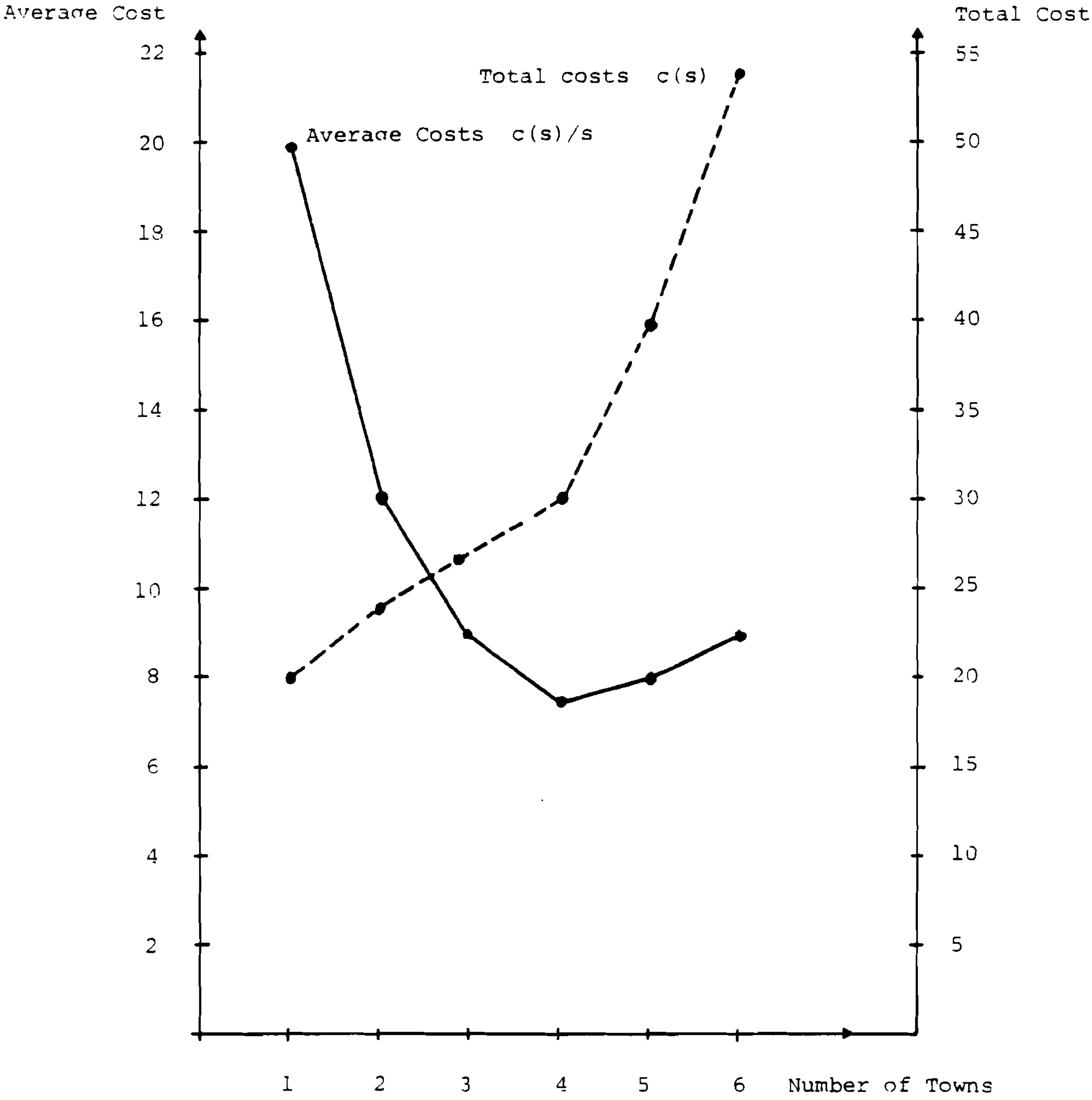

FIGURE 1. TOTAL AND AVERAGE COMMON COSTS OF SERVING s TOWNS.

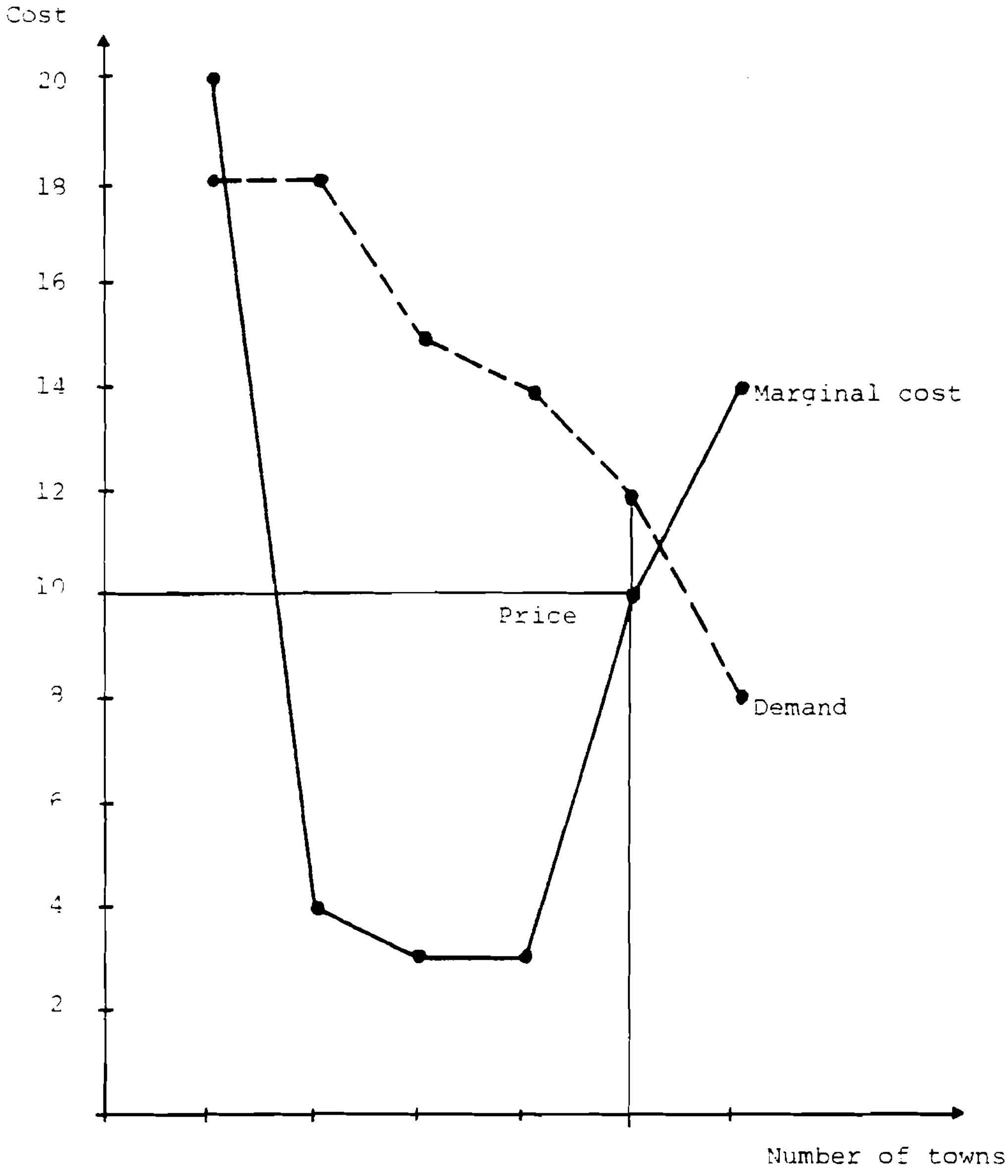
Cost
20
18
16
14
12
10
8
6
4
2
Marginal cost
Price
Demand
Number of towns

FIGURE 2

"crosses" the marginal cost curve. In particular, let each of the five towns served be charged a uniform price of 10, which is the marginal cost of serving the fifth town. At these prices total revenue is 50, joint costs are 40, and producer's surplus is 10. However the surplus from serving any four towns is also 10, which means that if any of the five players served lower their prices, at least one of them will be excluded from all maximum surplus sets. By this reasoning we find that a uniform price of 10 for the towns served is a strong equilibrium, and indeed it is the unique one. (The price of the excluded town can be anything up to 10).

The example shows why it is necessary to consider strong equilibria instead of simple Nash equilibria: if all towns offer a price of zero, then no town can *unilaterally* increase its offer and do better, since the cost of serving a single town is 20, while the maximum willingness to pay is only 18. However, the natural tendency of all the excluded towns will be to raise their prices -- probably simultaneously -- so we would not expect the bidding to become stuck at such an outcome, even assuming non-cooperation.

In this kind of example the equilibrium price need not necessarily equal the marginal *cost* of serving the last town. Thus, if the sixth town's willingness-to-pay were 11 instead of 8, then the unique equilibrium price would also be 11 (but the sixth town would still not be served).

These examples are analogous to that of a firm with a single, divisible product operating in a competitive market: if the firm does not control prices, it will produce at the efficient point where demand equals marginal cost. Moreover if this occurs where average costs are rising, i.e. where marginal cost exceeds average costs, then, in the short run at least, the firm will realize a surplus (Figure 3), which could be termed "structural surplus".

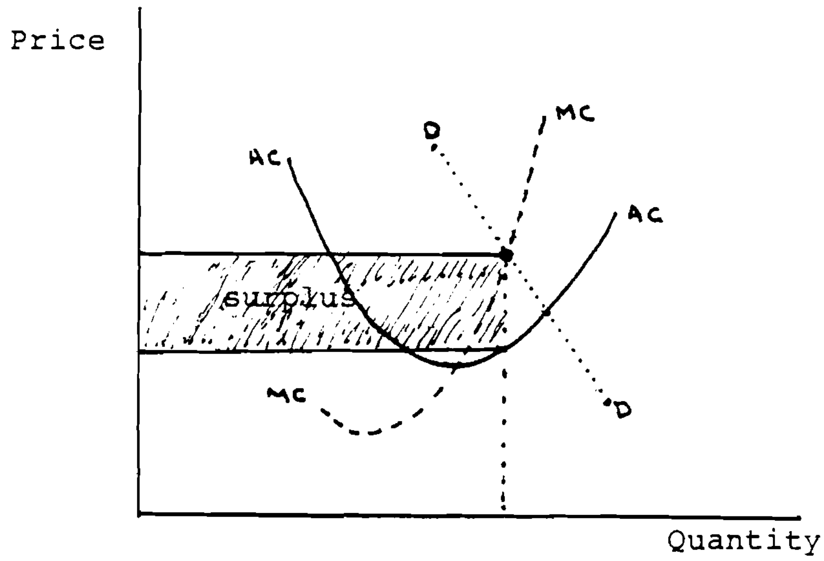

FIGURE 3

However, if demand falls in the area where average costs are declining, as in Figure 4, then for the firm to break even prices must be set above marginal costs. In fact a non-uniform price is *necessary* to achieve efficiency. The firm can just break even by charging a single price up to point A and then price-discriminating down the demand curve to the efficient point B. The point A is determined such that the area under PABQ equals the total cost of supplying the quantity Q. This phenomenon is illustrated for the following variation of Example 2.

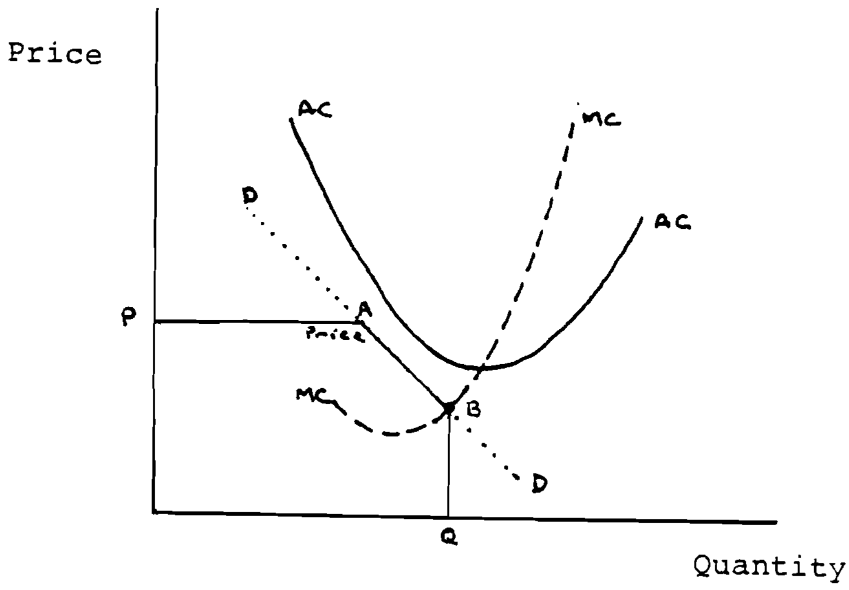

FIGURE 4

Example 3.

Let the cost function be the same as before but let the net demands be (18,11,8,6,4,3). Then it is efficient to serve only four towns, and there is a whole class of strong equilibrium price vectors, corresponding to the fact that the core of the game $\overline{v}$ is nonempty. Producer's surplus is zero in every case. One equilibrium price vector, (8,8,8,6,-,-) is shown in Figure 5.

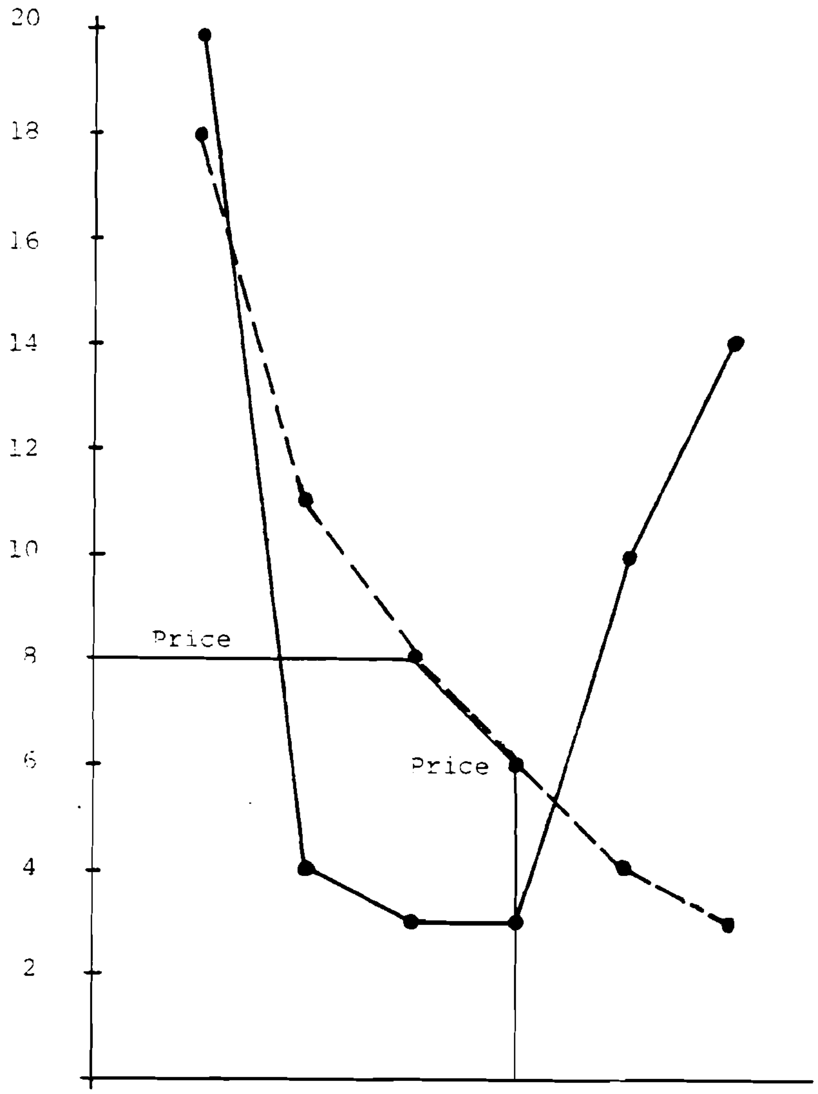

FIGURE 5

4. EQUILIBRIUM AND MARGINAL VALUE

The preceding examples suggest that a connection exists between equilibrium solutions to the cost allocation game, and the value of including or excluding more participants at the margin. In this section we make this idea precise, and show that it is closely related to the core of the cooperative game defined in Section 2 and to the concept of subsidy-free pricing treated in the public utilities literature.

A well-established idea in this literature is that every participant in a public enterprise should be charged *at least* the marginal cost of including him. This principle is sometimes called the "incremental cost test" (Alexander, 1912; Ransmeier, 1942; Faulhaber, 1975). Assuming for the moment that N is the efficient set to serve, this condition says that prices should satisfy

$$(6) \qquad p_i \geqq c(N) - c(N-i) \qquad \text{for all } i \quad .$$

It is natural (as suggested by Faulhaber, 1975) to generalize this principle to groups of participants and require further that

$$(7) \qquad \sum_S p_i \geqq c(N) - c(N-S) \qquad \text{for all subsets } S \subseteq N \quad .$$

We call these the principles of *individual* and *group marginal cost coverage*, respectively.

Another way of expressing these conditions is in terms of the surplus value function v(S). For example, (7) is equivalent to

$$\sum_S p_i \geqq (c(N) - \sum_N b_i) + \sum_S b_i + (\sum_{N-S} b_i - c(N-S)) \quad \text{for all } S \subseteq N,$$

that is

$$\sum_S x_i = \sum_S (b_i - p_i) \leqq v(N) - v(N-S) \qquad \text{for all } S \subseteq N \quad .$$

More generally, if N is not efficient we require that

$$(8) \qquad \sum_S x_i \leqq \bar{v}(N) - \bar{v}(N-S) = \max_{T \subseteq N} v(T) - \max_{T \cap S = \phi} v(T) \quad \text{for all } S \subseteq N \ .$$

This condition says simply that the surplus enjoyed by the set S should not exceed the difference between the maximum surplus obtainable with S and the maximum surplus obtainable without S. It is called the *group marginality principle*. If it is not satisfied for some group S, that is, if $\sum_S x_i > \bar{v}(N) - \bar{v}(N-S)$, then the group S is being *subsidized*. Thus, another way of interpreting condition (8) is that it implies a price structure which is *subsidy-free*, to use Faulhaber's term (1975).

Suppose that we require in addition that the enterprise be zero-profit, i.e. that total revenues equal total costs. This is the same as requiring that $\sum_N x_i = \bar{v}(N)$. Then it is easy to see that (8) is equivalent to the group rationality condition (4) defining the core, because $\sum_N x_i = \bar{v}(N)$ implies that $\sum_{N-S} x_i \leqq \bar{v}(N) - \bar{v}(N-(N-S))$ if and only if $\sum_S x_i \geqq \bar{v}(S)$. Unfortunately, the core may be empty in perfectly reasonable cases (e.g. Examples 1 and 2; see also Panzar and Willig (1977) for a more detailed analysis of this possibility). However, if we are willing to relax the zero-profit constraint to one of break-even or better -- i.e., to nondeficit -- then it is perfectly possible to have a subsidy-free price structure, even though there may still be no core.* This point seems to have been largely overlooked in the literature on regulated pricing, which has concentrated mainly on the zero-profit hypothesis.

If we simply require that total revenues *cover* total costs (equivalently, that $\sum_N x_i \leqq \bar{v}(N)$), then we can certainly satisfy (8) because all inequalities on $\underset{\sim}{x}$ run in the same direction. This allows, for example, the possibility of zero consumer surplus, which corresponds to the case of perfect price discrimination ($p_i = b_i$ for all i). In many applications this may not be a reasonable solution -- nor feasible, if consumers' willingness-to-pay is unknown. A more reasonable idea is to ask

*Because of the equivalence noted above when a zero-profit constraint is imposed, it has become customary to use the term subsidy-free synonymously with core solutions. In the present context we prefer to distinguish between the idea of subsidy-free, as expressed by (8), and the group-rationality condition defining the core, which may also be interpreted as entry preventing.

what is the maximum consumer surplus, in the Pareto sense, that is consistent with non-subsidization? The answer -- which brings us full circle -- is precisely the equilibrium payoffs that result from the cost allocation game defined in the previous section.

More precisely, a vector $(x_1,\ldots,x_n) \geqq \underset{\sim}{0}$ is said to be a *marginal value* for the game $\overline{v}$ if it is subsidy-free and is not dominated by any vector that is also subsidy-free (Young, 1979).*

<u>*Theorem 2*</u>. *For any cost function* c *and demands* $\underset{\sim}{b}$, $\underset{\sim}{x}$ *is an equilibrium payoff vector for some surplus-maximizing mechanism* g *if and only if* $\underset{\sim}{x}$ *is a marginal value for the associated surplus game* $\overline{v}(S) = \max\limits_{T\subseteq S} (\sum\limits_{T} b_i - c(T))$.

Moreover, $\underset{\sim}{x}$ does not depend on the choice of g, that is all surplus-maximizing g that have some equilibrium yield precisely the same set of equilibrium payoffs.

Concretely, $\underset{\sim}{x}$ is a marginal value if and only if it satisfies the following three conditions:

(9) $$\underset{\sim}{x} \geqq \underset{\sim}{0}$$

(10) $$\sum_S x_i \leqq \overline{v}(N) - \overline{v}(N-S) \quad \text{for every } S \subseteq N \quad ,$$

(11) the sets S such that equality holds in (10) cover N .

In particular, every vector in the core of $\overline{v}$ is a marginal value: every core vector is subsidy-free and is undominated because condition (10) with S = N requires that $\sum\limits_N x_i \leqq \overline{v}(N)$. Thus marginal values generalize the core in a natural way and have the distinct advantage, unlike the core, that they always exist.

If the game $\overline{v}$ is superadditive (i.e. if the cost function c is subadditive), then it may be shown that the marginal values satisfy a kind of quasi-core condition and are always individually rational -- that is, $x_i \geqq \overline{v}(i)$ for all i. Further, if $\overline{v}$ is convex (i.e. if c is concave) then the core is nonempty and the set of marginal values is identical with the core (Young, 1980).

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*A particular case are the subsidy-free vectors that maximize total consumer surplus $\sum\limits_N x_i$.

5. EXTENSIONS

The preceding describes a method for allocating joint costs when demands are not known. The merit of the method is that it is simple and easy to implement; it leads to an efficient solution in the absence of information on demand; and the resulting prices have a normative justification related to the core -- in particular, they are Pareto-optimal subject to being subsidy-free.

To help test the workability of the method, a gaming experiment was recently conducted at IIASA using actual cost data from a public enterprise in Sweden. The game involved six players and had a relatively small core. The players were given information only about their own demand. The bidding was conducted as described in Section 2 with only the "winning" set being announced at each stage. Nevertheless, within 10 rounds the bidding had converged to a solution within about 0.3% of a core solution in spite of the fact that the players did not even know the cost function.

To be truly useful, however the analysis needs to be extended to allow for nondiscriminatory pricing and for different levels of consumption by consumers. Both of these can be handled by straightforward modifications of the mechanism discussed above, the details of which will be given elsewhere. Another area of investigation is the extension of the approach to a general equilibrium framework. In view of the non-closure of the system this is likely to present difficulties. Most probably it can be done, but with very much more restrictive assumptions on the shape of the production and demand functions. This would be sacrificing a lot, however, since one of the prime motivations of this study is to provide an approach to allocation that is workable even in messy situations where there is nonconvexity, discrete products, and highly differentiated markets. In fact, conditions like these are to be expected precisely in situations where established markets do not exist and some regulatory mechanism is required.

Finally, there is the question of whether a mechanism could be designed that guarantees zero-profit without sacrificing efficiency and other desirable properties. The answer appears to be that something must be given up: either efficiency or existence of equilibrium, or at the least, convergency of the mechanism. Another possibility is to restrict the shape of the production and demand functions. For example, if in the present model the cost function is assumed to be concave, then the surplus-maximizing mechanism will give only zero-profit outcomes, because it may be shown (Young, 1980) that *all* equilibrium payoffs are in the core.

APPENDIX

Theorem 1. *For any cost function c, there exists a surplus-maximizing mechanism g such that the cost allocation game has a strong equilibrium; moreover for any such equilibrium $\underset{\sim}{p}$ the set produced, $g(\underset{\sim}{p})$, is efficient.*

Proof. Suppose that $\overline{S}$ and $\underset{\sim}{\overline{p}}$ satisfy the following conditions:

(12) $\qquad \underset{\sim}{\overline{p}} \leqq \underset{\sim}{b}$ and $\overline{p}_i = b_i$ for all $i \notin \overline{S}$,

(13) $\qquad$ for every i there exists a maximum surplus set excluding i ,

(14) $\qquad \overline{S}$ is a maximum surplus set .

We claim that if $g(\underset{\sim}{\overline{p}}) = \overline{S}$, then $\underset{\sim}{\overline{p}}$ is a strong equilibrium. Suppose instead that $\underset{\sim}{p}'$ is a change of strategies such that every player who changes does better than before. If i changes strategies, $p_i' \neq \overline{p}_i$, then we must have $p_i' < \overline{p}_i$ and $i \in g(\underset{\sim}{p}')$, because i's payoff was nonnegative under $\underset{\sim}{\overline{p}}$, so it must be positive under $\underset{\sim}{p}'$. But then, by (13), $g(\underset{\sim}{p}')$ cannot be a maximum surplus set, contradicting the definition of the mechanism g. Therefore $\underset{\sim}{\overline{p}}$ is a strong equilibrium.

Next, it is easy to see that such a pair $(\overline{S},\overline{p})$ exists. Begin with $p = b$; let $\overline{S}$ be an efficient set, which is the same as a maximum surplus set when $p = b$. If some player i is contained in every maximum surplus set, lower its price until it is not, and repeat for all players. In at most n steps a price vector $\overline{p}$ will be obtained such that: $\overline{S}$ is still a maximum surplus set under $\overline{p}$; no player is contained in every maximum surplus set; $p \leqq b$ and $\overline{p}_i = b_i$ for all $i \notin \overline{S}$. Now choose any mechanism g such that $g(\overline{p}) = \overline{S}$, and the proof of existence is completed.

It remains to be shown that for every strong equilibrium $\overline{p}$, the set $\overline{S} = g(\overline{p})$ is efficient. For every set T and $0 \leqq \alpha \leqq 1$ define $p(\alpha,T)$ as follows:

$$p_i(\alpha,T) = \begin{cases} \alpha\overline{p}_i + (1-\alpha)b_i & \text{if } i \in T \\ \overline{p}_i & \text{if } i \notin T \end{cases}$$

Now let $T' = \{i \notin \overline{S} : \overline{p}_i \leqq b_i\}$ and suppose that for some $0<\alpha'<\frac{1}{2}$ $\overline{S}$ is not maximum surplus for $p(\alpha',T') = p'$. Let S' have maximum surplus under p'. Since $\overline{S}$ was a maximum surplus set for $\overline{p}$, and $\overline{p} \leq p'$, it must be that $S' \cap T' \neq \phi$. Letting $T" = S' \cap T'$ and $\alpha" = 2\alpha$, it follows that *every* most profitable set for $p(\alpha",T") = p"$ contains T". Therefore $T" \subseteq \overline{g}(p")$, so under $p"$ every player in T" gets a *positive* payoff, whereas under p every player in T" got zero payoff. Since p and $p"$ differ only on T", the strong equilibrium condition is contradicted.

The conclusion is that $\overline{S}$ maximizes surplus under $p(\alpha,T')$ whenever $\alpha > 0$. Hence in the limit $\overline{S}$ also maximizes surplus under $p(0,T') = p^0$. Observe that

$$p_i^0 = \begin{cases} \overline{p}_i & \text{if } i \in \overline{S} \\ \max\{\overline{p}_i, b_i\} & \text{if } i \notin \overline{S} \end{cases} .$$

Hence if we define $\overline{\overline{p}}$ by

$$(15) \qquad \bar{\bar{p}}_i = \begin{cases} \bar{p}_i & \text{if } i \in \bar{S} \\ b_i & \text{if } i \notin \bar{S} \end{cases}$$

then $\bar{\bar{p}} \leqq p^0$ and $\bar{S}$ also maximizes surplus under $\bar{\bar{p}}$. Therefore

$$\sum_{\bar{S}} \bar{\bar{p}}_i - c(\bar{S}) \geqq \sum_{S} \bar{\bar{p}}_i - c(S) \qquad \text{for all } S \quad ,$$

which by (15) implies

$$(16) \qquad \sum_{\bar{S}-S} \bar{p}_i - c(\bar{S}) \geqq \sum_{S-\bar{S}} b_i - c(S)$$

But $\bar{p}_i \leqq b_i$ for all $i \in \bar{S}$ because at equilibrium no player receives a negative payoff. Therefore (16) implies

$$\sum_{\bar{S}-S} b_i - c(\bar{S}) \geqq \sum_{S-\bar{S}} b_i - c(S) \quad ,$$

so

$$\sum_{\bar{S}} b_i - c(\bar{S}) \geqq \sum_{S} b_i - c(S) \quad ,$$

proving that $\bar{S}$ is efficient. □

<u>*Theorem 2*</u>. *Given any cost function* c *and demands* $\underset{\sim}{b}$, $\underset{\sim}{x}$ *is an equilibrium payoff vector for some surplus-maximizing mechanism* g *if and only if* $\underset{\sim}{x}$ *is a marginal value for the associated surplus game* $\bar{v} = \max_{T \subseteq S} (\sum_T b_i - c(T))$.

<u>*Proof*</u>. Suppose that $\bar{\underset{\sim}{p}}$ is a strong equilibrium, that $g(\bar{\underset{\sim}{p}}) = \bar{S}$ is the set produced, and $\underset{\sim}{x}$ is the payoff vector. We know that

$$(17) \qquad \underset{\sim}{x} \geqq \underset{\sim}{0} \quad \text{and} \quad x_i = 0 \quad \text{for all } i \notin \bar{S} \quad .$$

Second, we know from Theorem 1 that $\bar{S}$ is efficient; and since $\bar{S}$ maximizes profits under $\bar{\underset{\sim}{p}}$ we have

$$(18) \quad v(T) - \sum_T x_i = \sum_T p_i - c(T) \leqq \sum_{\bar{S}} p_i - c(\bar{S}) = v(\bar{S}) - \sum_{\bar{S}} x_i \text{ for all } T \ ,$$

and

(19) $$\sum_{\overline{S}-T} x_i \leqq v(\overline{S}) - v(T) \quad \text{for all } T \quad .$$

Using (17) it follows that if $S \cap T = \phi$,

$$\sum_{S} x_i = \sum_{S\cap\overline{S}} x_i \leqq \sum_{\overline{S}-T} x_i \leqq v(\overline{S}) - v(T) \ ,$$

hence

(20) $$\sum_{S} x_i \leqq v(\overline{S}) - \max_{T:T\cap S=\phi} v(T) = \overline{v}(N) - \overline{v}(N-S) \text{ for every } S \ .$$

This says that $\underset{\sim}{x}$ is subsidy-free. Finally, since $\underset{\sim}{\overline{p}}$ is an equilibrium, no player can raise his price and do better. This means that for every i there is a maximum surplus set T_i excluding i. For $T = T_i$, (18) and (19) hold as equalities, and hence (20) holds as an equality for some set S containing i. Therefore the payoff vector $\underset{\sim}{x}$ is Pareto-optimal subject to being subsidy-free, so it is a marginal value.

Conversely let $\underset{\sim}{x} \geqq \underset{\sim}{0}$ be Pareto-optimal subject to being subsidy-free. Then every efficient set $\overline{S}$ is a maximum-surplus set with respect to the price vector $\underset{\sim}{\overline{p}} = \underset{\sim}{b} - \underset{\sim}{x}$. Using (20) with $S = N-\overline{S}$, we deduce that $\sum_{N-\overline{S}} x_i \leqq 0$. Since $\underset{\sim}{x} \geqq \underset{\sim}{0}$, it follows that $x_i = 0$, that is, $\overline{p}_i = b_i$, for all $i \notin \overline{S}$. Finally, Pareto-optimality implies that, for every i there is a maximum-surplus set excluding i. Therefore by the first part of the proof of Theorem 1, $\underset{\sim}{\overline{p}}$ is a strong equilibrium for any surplus-maximizing g such that $g(\underset{\sim}{\overline{p}}) = \overline{S}$. Moreover, the payoff to i is $x_i = 0$ for all $i \notin \overline{S}$, and $x_i = b_i - \overline{p}_i$ for all $i \in \overline{S}$. □

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