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ON THE FORMAL EQUIVALENCE OF SOME SIMPLE FACILITY LOCATION MODELS

Giorgio Leonardi

February 1980
WP-80-21

To be presented at the Workshop on 'Location and Distribution Management' at the European Institute for Advanced Studies in Management, Brussels, March 20-21, 1980.

International Institute for Applied Systems Analysis
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FOREWORD

The public provision of urban facilities and services often takes the form of a few central supply points serving a large number of spatially dispersed demand points: for example, hospitals, schools, libraries, and emergency services such as fire and police. A fundamental characteristic of such systems is the spatial separation between suppliers and consumers. No market signals exist to identify efficient and inefficient geographical arrangements, thus the location problem is one that arises in both East and West, in planned and in market economies.

This problem is being studied at IIASA by the Normative Location Modeling Task, which started in 1979. The expected results of this Task are comprehensive state-of-the-art survey on current theories and applications, an established network of international contacts among scholars and institutions in different countries, a framework for comparison, unification, and generalization of existing approaches, as well as the formulation of new problems and approaches in the field of optimal location.

This paper reports on some of the first exploratory findings in the direction of a unified framework. It presents a way of generalizing both the usual allocation rules and objective functions in standard location models, and proposes a set of alternative mathematical programming formulations.

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ABSTRACT

This paper shows the equivalence among some different formulations of a simple location-allocation problem. A feature shared by all the formulations is an allocation subproblem (the distribution of users among the facilities) based on spatial-interaction theory (gravity models). The initial mathematical programming formulation, useless for computation, is shown to be equivalent to some much simpler mathematical programs, built up by suitably widening the feasible region.

Finally, a duality relationship is shown to hold between the location models where accessibility is maximized (having the location and size of the facilities as control variables) and those where travel cost is minimized (having the allocation of users to the facilities as control variables). The last ones are shown to tend to the usual location-allocation models with linear cost function as the distance decay effect increases.

CONTENTS

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INTRODUCTION

The existing literature on location-allocation models is almost entirely concerned with systems where the nearest-facility allocation rule holds. This rule is indeed the optimal one for plant and warehouse location problems, where both the costs of establishing the facilities and the transport costs must be paid by the same decision maker (the producer), and goods have to be delivered from the facilties (plants or warehouses) to the demand locations.

On the other hand, the allocation to the nearest facility is not generally accepted as a sound behavioral assumption in many service location problems, where no delivery to demand locations takes place, but rather the users have to travel from their place of residence to the available facilities. If the allocation decision is left to the users, then it is highly improbable that they will all choose the nearest facilities. Some empirical evidence would rather suggest that smoother behavioral models, like the ones developed in spatial interaction theory (sometimes called "gravity models"), give a better description of reality.

The interest on spatial-interaction based location-allocation models seems to be increasing (see Wilson, 1976; Coelho and

Wilson, 1976; Hodgson, 1978; Leonardi, 1978; Beaumont, 1979). Although most of the proposed models seem quite different, paths towards unification have been shown recently (Coelho and Wilson, 1977; Harris and Wilson, 1978; MacGill and Wilson, 1979). These notes are written in the same spirit, and a further step towards unifying results is proposed.

In what follows, the formal equivalence is stated among some different formulations of a class of simple location problems. By formal equivalence it is meant that:

1. Every location problem belonging to the class considered here can be formulated in terms of any one of the equivalent models.
2. The optimal solution is the same for all the equivalent models, i.e., solving one of them implies solving all of them.

Some of the equivalent models vary in the form assumed by the objective function. These forms are usually given different interpretations in the available literature, ranging from entropy (Wilson, 1970 and 1974), to what some authors call "locational surplus" or "consumers' surplus" (Neuberger, 1971; Coelho and Wilson, 1976; Williams, 1976; Williams and Senior, 1977) to accessibility (Hansen, 1959; Weibull, 1976; Leonardi, 1978; Williams and Senior, 1978). Here no attempt will be made to go deeply into the economic, physical, and methaphysical interpretations of the above concepts. Users' benefit measures based on accessibility and spatial discount concepts are preferred, because of their easily understandable physical meaning. However, from a formal point of view, it is possible to start from any one of the objective functions considered.

1. STATEMENT OF THE PROBLEM AND BASIC ASSUMPTIONS

The following problem of distributing facilities among zones will be considered: Given a set of places of residence for the users, and a set of possible locations for the facilities, determine the size (possibly zero) of the facility in each location, in order to maximize a measure of benefit for the users and to meet some physical and/or economic constraints.

It will be assumed that:

1. the locational decision maker controls the size, or capacity, of the facilities;
2. the users make trips from their places of residence, or demand points, to the facilites;
3. the trips made by the users from the demand points to the facilities are generated by a doubly-constrained spatial interaction model with exponential distance decay (see Wilson, 1971)[1];
4. the benefit accruing to the users from a given distribution of the facilities over the zones is a measure of the accessibility of the users to the facilities (Hansen, 1959; Leonardi, 1973; Weibull, 1976; Leonardi, 1979); the spatial discount factor used in the accessibility measure is equal to the distance decay rate appearing in the spatial interaction model;
5. no special assumption is made on the physical and economic constraints on the sizes of the facilities, except that their set of feasible solutions is compact.

In order to state assumptions 3-5 in mathematical terms, the following definitions are needed:

- H is the set of subscripts labeling the zones which are places of residence for the users; the number of zones in this set is n;
- L is the set of subscripts labeling the zones which are possible locations for the facilities; the number of zones in this set is m;
- P_i is the number of users (the demand) living in zone i, $i \in H$;
- x_j is the size (capacity) of the facility in zone j, $j \in L$;
- X is the vector whose components are x_j, $j \in L$;
- C_{ij} is the cost of a trip from zone i to zone j, $i \in H$, $j \in L$;

β is a spatial discount rate (also, a distance decay rate);

Γ is the set of feasible solutions for the physical and economic constraints, so that for a feasible X it must be $X \varepsilon \Gamma$;

S_{ij} is the number of users living in zone i and choosing the facility in zone j, $i \varepsilon H$, $j \varepsilon L$.

According to assumption 3, vectors of weights $W = (w_1, \ldots, w_m)$ exist such that:

$$S_{ij} = P_i \frac{w_j e^{-\beta C_{ij}}}{\sum_{j \varepsilon L} w_j e^{-\beta C_{ij}}} , \qquad i \varepsilon H, \quad j \varepsilon L \tag{1}$$

and

$$\sum_{i \varepsilon H} S_{ij} = x_j , \qquad j \varepsilon L \tag{2}$$

Loosely speaking, the weights w_j can be looked at as a measure of the attractiveness of the facilities, as perceived by the users. [For a more technical treatise on the relationships between the weights and decay functions observed in the spatial interaction behavior and the weights and discount factors by means of which each user builds up its own utility function, see Smith (1976).] They are defined up to a multiplicative constant, since if W is any solution to equations (2), then also αW is, for any $\alpha > 0$. In order to eliminate this arbitrariness, the vector W is usually required to be normalized. The most common normalization rule is an additive one, such as:

$$\sum_{j \varepsilon L} w_j = 1$$

However, in the models and generalizations of the following sections, it will be useful to have a multiplicative normalization rule like:

$$\prod_{j \in L} w_j^{\theta_j} = 1 \quad , \quad \text{where } \theta_j = x_j / \sum_{j \in L} x_j \quad , \quad j \in L \qquad (3)$$

By means of (3), a constraint is placed on the geometric mean of the components of W, weighted with the relative sizes of the facilities. Given any vector $Y = (y_1, \ldots, y_m)$ which solves (2), a vector W satisfying (3) can be constructed by means of the formula:

$$w_j = \alpha y_j \quad , \qquad j \in L$$

where

$$\alpha = \left(\prod_{j \in L} y_j^{\theta_j} \right)^{-1}$$

and clearly W is a solution to (2).

It will be stressed again that the normalization rule (3) has been chosen for convenience only, since it will simplify some results in the following sections. However, the choice of a normalization rule is quite arbitrary, as long as users do not perceive it [the normalizing factor disappears in (1), independently of the method used to build it]. Therefore, rules other than (3) could be employed, if needed.

By means of equations (1), (2), and (3) a one-to-one mapping between the X vectors and the W vectors is established, provided the needed existence and uniqueness requirements are met [equivalence of equations (1), (2), and (3) to a biproportional adjustment problem can be seen, so that these requirements are actually met under very general conditions, see Deming and Stephan (1940), Sinkhorn (1967), Ireland and Kullback (1968), Bacharach (1970), and many others].

The notations $W(X)$ (for the vector) and $w_j(X)$ (for each component) will be used for this mapping. Substitution into equation (1) yields

$$S_{ij} = P_i \frac{w_j(X)\, e^{-\beta C_{ij}}}{\sum_{j \varepsilon L} w_j(X)\, e^{-\beta C_{ij}}} \quad , \quad i \,\varepsilon\, H, \quad j \,\varepsilon\, L, \qquad (4)$$

By means of (4) the dependence of the flow variables, S_{ij}, on the control vector X is evidenced.

The accessibility measure from i, consistent with (1), or (4), is defined as:

$$\Phi_i = \sum_{j \varepsilon L} w_j\, e^{-\beta C_{ij}} \quad , \qquad i \,\varepsilon\, H \qquad (5)$$

and this is the sum of the relative weights w_j, exponentially discounted over space with discount rate β.

In order to construct an overall accessibility measure, a composition rule is needed. Let $I \subseteq L$ be a subset of zones; I will be called an <u>aggregation</u> of zones.[2] The aggregation I can be treated as a new zone, replacing all $j \,\varepsilon\, I$. Two definitions are needed:

- $\bar{w}_I$ is the relative weight, or measure of attractiveness, for the aggregation I;
- $\bar{C}_{iI}$ is the cost of a trip from a zone $i \,\varepsilon\, H$ to the aggregation I.

According to the doubly-constrained spatial interaction model, in order to assure the consistency between the S_{ij} and w_j variables <u>before</u> and <u>after</u> the aggregation has been introduced, the following equations must be satisfied:

$$\bar{w}_I = \prod_{j \varepsilon I} w_j^{\bar{\theta}_j} \quad , \quad \text{where } \bar{\theta}_j = \theta_j / \sum_{j \varepsilon I} \theta_j \quad , \quad j \,\varepsilon\, I \qquad (6)$$

$$P_i \frac{\sum\limits_{j\varepsilon I} w_j\, e^{-\beta C_{ij}}}{\sum\limits_{j\varepsilon I} w_j\, e^{-\beta C_{ij}} + \sum\limits_{j\varepsilon L-I} w_j\, e^{-\beta C_{ij}}} = \tag{7}$$

$$= P_i \frac{\bar{w}_I\, e^{-\beta \bar{C}_{iI}}}{\bar{w}_I\, e^{-\beta \bar{C}_{iI}} + \sum\limits_{j\varepsilon L-I} w_j\, e^{-\beta C_{ij}}}$$

Equation (6) assures that the normalizing condition (3) is met. Substitution of (6) into (7) yields for $\bar{C}_{iI}$ the solution:

$$\bar{C}_{iI} = -\frac{1}{\beta} \log \sum_{j\varepsilon I} \frac{w_j}{\prod\limits_{j\varepsilon I} w_j^{\bar{\theta}_j}}\, e^{-\beta C_{ij}} \quad , \qquad i \varepsilon H \tag{8}$$

This is a well known result. It states that the right-hand side of (8) is the only composition rule assuring consistent aggregations in exponentially decaying spatial interactions (see Wilson, 1974; Batty, 1976; Williams, 1977). If, as a limit, I = L, substitution in (8) and use of (3), (5), and (6) yield

$$\bar{C}_{iL} = -\frac{1}{\beta} \log \Phi_i \quad , \qquad i \varepsilon H \tag{9}$$

If the right-hand term of (8) is looked at as a special averaging operator applied to the travel costs the result in (9) may be called the average cost of a trip, as perceived by a user living in $i \varepsilon H$. The sum of the average costs over all users is given by

$$\bar{C} = \sum_{i\varepsilon H} P_i\, \bar{C}_{iL} = -\frac{1}{\beta} \sum_{i\varepsilon H} P_i \log \Phi_i \tag{10}$$

and this quantity is minimized if the function

$$\Psi = \sum_{i \varepsilon H} P_i \log \Phi_i \tag{11}$$

is maximized. This function (Leonardi, 1973; 1978) is a very intuitive measure of total benefit, since it is the sum of the logarithms of the accessibilities, weighted with the corresponding demands. The average discount factor corresponding to (10) can be computed as

$$e^{-\beta \bar{C}/P} = \exp \frac{1}{P} \sum_{i \varepsilon H} P_i \log \Phi_i = \prod_{i \varepsilon H} \Phi_i^{P_i/P} \tag{12}$$

where

$$P = \sum_{i \varepsilon H} P_i$$

The right-hand side of (12) is the <u>geometric mean of the accessibilities</u> from each zone. Maximization of the function Ψ, as defined by (11), is thus equivalent to maximization of the geometric mean of accessibilities.

2. THE OPTIMIZATION MODELS

According to the assumptions, definitions, and results of Section 1., the optimal location problem can be formulated as a mathematical program as follows

$$\max_X \sum_{i \varepsilon H} P_i \log \sum_{j \varepsilon L} w_j(X)\, e^{-\beta C_{ij}} \tag{13}$$

subject to

$$\sum_{j \varepsilon L} x_j = P \tag{14}$$

$$X \in \Gamma \tag{15}$$

where

$$P = \sum_{i \in H} P_i \quad \text{is the total demand.} \tag{16}$$

The objective function (13) is the total accessibility measure defined by (11), and it has to be maximized. The dependence on the vector X is made explicit, so that the functions $w_j(X)$ are determined by equations (1), (2), and (3). The choice of X is subject to constraint (14), which requires a total capacity equal to the total demand, and to the physical and economic constraints (15).

The formulation given by (13)-(15) is surely the closest one to the real problem, but is not the best one for computation. The mapping W(X) cannot be expressed in closed form, and the values of W can be computed only numerically, by solving equations (1), (2), and (3). Other formulations are therefore needed, which solve problem (13)-(15) and require less computational effort. Indeed, it will be shown that many very simple formulations equivalent to (13)-(15) exist. The resulting Location Models (LM) are listed below, and discussed in Section 3.

<u>LM1</u>

$$\max_{X,W} \sum_{i \in H} P_i \log \sum_{j \in L} w_j e^{-\beta C_{ij}} \tag{17}$$

subject to

$$D_j(W) = x_j \quad , \qquad j \in L \tag{18}$$

$$\sum_{j \in L} x_j \log w_j = 0 \tag{19}$$

$$\sum_{j \in L} x_j = P \tag{20}$$

$$X \varepsilon \Gamma \tag{21}$$

where the functions $D_j(W)$ are defined as

$$D_j(W) = \sum_{i \varepsilon H} P_i \frac{w_j e^{-\beta C_{ij}}}{\sum_{j \varepsilon L} w_j e^{-\beta C_{ij}}} , \qquad j \varepsilon L \tag{22}$$

LM2

$$\max_{X,W} \sum_{i \varepsilon H} P_i \log \sum_{j \varepsilon L} w_j e^{-\beta C_{ij}} \tag{23}$$

subject to

$$\sum_{j \varepsilon L} x_j \log w_j = 0 \tag{24}$$

$$\sum_{j \varepsilon L} x_j = P \tag{25}$$

$$X \varepsilon \Gamma \tag{26}$$

LM3

$$\max_{X,W} \sum_{i \varepsilon H} P_i \log \sum_{j \varepsilon L} \frac{w_j}{\prod_{j \varepsilon L} w_j^{x_j/P}} e^{-\beta C_{ij}} \tag{27}$$

subject to

$$X \varepsilon \Gamma \tag{28}$$

or

$$\max_{X,W} \sum_{i \varepsilon H} P_i \log \sum_{j \varepsilon L} w_j e^{-\beta C_{ij}} - \sum_{j \varepsilon L} x_j \log w_j \tag{29}$$

subject to

$$X \in \Gamma \tag{30}$$

<u>LM4</u>

$$\min_{S} \sum_{i \in H} \sum_{j \in L} S_{ij}(\log S_{ij} + \beta C_{ij}) \tag{31}$$

subject to

$$\sum_{j \in L} S_{ij} = P_i \quad , \qquad i \in H \tag{32}$$

$$\sum_{i \in H} S_{ij} = x_j \quad , \qquad j \in L \tag{33}$$

$$X \in \Gamma \tag{34}$$

where

$S = (S_{ij})$ is the matrix whose elements are the trips for each origin-destination pair, as defined in Section 1.

3. THE EQUIVALENCE RESULTS

It will now be shown that problems LM1, LM2, LM3, and LM4 are all equivalent to problem (13)-(15).

3.1 Equivalence for LM1

The equivalence for LM1 to problem (13)-(15) is seen at once. Indeed, LM1 is the same as (13)-(15), except for the way it accounts for the mapping W(X). While in (13)-(15) the mapping is introduced explicitly in the objective function, in LM1 the

W vector is added to the list of decision variables, and equations (2) and (3) are introduced as constraints [see constraints (18) and (19)[3]].

3.2 Equivalence for LM2

Problem LM1 is almost as difficult to solve as problem (13)-(15), due to the nonlinearity and nonseparability of the functions defined by (22). The difficulty in handling constraints such as (18) in optimization problems has been noticed by many authors (e.g. Wilson, 1976; Coelho and Wilson, 1976; Hodgson, 1978). In problem LM2 constraint (18) has been eliminated, so that LM2 has a wider feasible region than the one of LM1, and includes it. On the other hand, the Lagrange optimality conditions for W in LM2 are

$$\frac{\partial \Psi}{\partial w_j} - \nu x_j / w_j = 0 \quad , \qquad j \varepsilon L \qquad (35)$$

where

ν is a Lagrange multiplier

Ψ is the objective function (23), as defined in (11).

Derivation of (23) and substitution from (22) yield

$$\frac{\partial \Psi}{\partial w_j} = \sum_{i \varepsilon H} P_i \frac{e^{-\beta C_{ij}}}{\sum_{j \varepsilon L} w_j e^{-\beta C_{ij}}} = D_j(W)/w_j \quad , \qquad j \varepsilon L \qquad (36)$$

Substitution of the left-hand term of (36) into (35), summation over j and substitution from (25) give for ν the value $\nu = 1$. The final form assumed by (35) is thus

$$D_j(W) = x_j \quad , \qquad j \varepsilon L \qquad (37)$$

and it is identical to (18).

It can be concluded that the optimal solution for LM2 is a feasible solution for LM1. Since LM2 is less constrained than LM1, its solution is optimal for LM1 too, i.e., LM1 and LM2 are equivalent.

3.3 Equivalence for LM3

Constraints (19) and (24) force the optimal W for LM1 and LM2 to be unique. This is rather artificial, however, since according to (1) users perceive only relative values, i.e., they do not distinguish between W and αW, for any $\alpha > 0$. Problem LM3 takes this into account, by means of the objective function (27), which is invariant under multiplication of W by a scalar. This property is accomplished by incorporating the normalization (3) in the objective function, instead of keeping it as a constraint.

It will be shown that the optimal solution to LM2 is also one of the optimal solutions to LM3. Since W is unconstrained in LM3, the derivatives of (27), or (29), with respect to w_j, $j \in L$, must vanish at the optimal points. This implies

$$D_j(W) = x_j \quad , \qquad j \in L \qquad (38)$$

where $D_j(W)$ are the functions defined by (22). But equations (38) are the same as (37), and they are satisfied by the solution of LM2. Let W be this solution, then the whole set of solutions to LM3 is given by αW, $\alpha > 0$. It follows that LM2 and LM3 are equivalent.

3.4 Equivalence for LM4

Problem LM4 is formulated in terms of the flow variables matrix S, rather than in terms of the attraction weights W. It is well known (see Coelho and Wilson, 1977, for instance) that the minimization of a function like (31), subject to constraints like (32) and (33) implies that the S_{ij} are given by (1), if the components of the W vector are assigned the values

$$w_j = e^{-\mu_j} \quad , \qquad j \in L \qquad (39)$$

where the μ_j are the Lagrange multipliers of constraints (33). It is readily seen from (1) and (22) that constraints (33) can be rewritten as

$$D_j(W) = x_j \quad , \qquad j \in L \qquad (40)$$

where W is defined by (39). Comparison of (40) with (38) shows that the weights computed by (39) are optimal for LM3 and, by introducing suitable multiplicative factors, for LM1 and LM2 as well.

Since the W vector defined by (39) depends on the dual variables μ_j, problems LM1, LM2, and LM3 are different versions of the dual problem corresponding to LM4 (here only duality between S and W is considered, while X is kept constant). The duality relationship between problem LM4 and problems LM1, LM2, LM3 is another well known result in spatial interaction theory (Wilson and Senior, 1974; Nijkamp and Paelink, 1974; Evans, 1976; Champernowne et al, 1976).

4. DISCUSSION

A brief discussion is worthwhile on the relative advantages, or disadvantages, of models LM1, LM2, LM3, LM4. Model LM1 is useless for computation, and will be excluded from now on. Models LM2 and LM3 are very similar. However, LM2 is better for computation, since its solution is forced to be unique by the normalizing constraint (24). On the other hand, LM3 is theoretically the soundest, since it uses an accessibility measure which is independent from the absolute value of the w_j. Both LM2 and LM3 have less variables than LM4. This makes them better than LM4 for computation (indeed, the solution to LM4 must always be expressed in terms of the dual variables - the balancing factors in the biproportionality method). Model LM4 has an objective function with no intuitive physical meaning [although many efforts have been carried out to give it one, see Neuberger (1971), Coelho and Wilson (1976), Williams (1977), Coelho and Williams (1977)], while accessibility can be easily interpreted. Formulation LM4 has a theoretical advantage on LM2 and LM3, however. In LM4 the relationship between the models discussed in this paper and the classical location-allocation models in discrete space (Balinski, 1961; Efroymson and Ray, 1966; ReVelle and Swain, 1970; Scott, 1971; Erlenkotter, 1978) can be seen immediately. Indeed, if β is very large, i.e., the spatial discount effect is very strong, the users will tend to choose only the nearest facility. The terms containing $\log S_{ij}$ can thus be neglected and, if Γ is suitably defined, a linear-integer location-allocation model is obtained. In other words, classical location-allocation models are included in LM4 as limiting cases. This property is quite analogous to the limiting relationship existing between the doubly-constrained spatial interaction models and the linear programming transportation problem (Evans, 1973; Wilson and Senior, 1974). By means of the equivalence relations stated in Section 3, it can be concluded that all the models discussed in this paper are generalizations of a location-allocation problem.

5. AN EXAMPLE

Let the set Γ be defined as

$$\Gamma = \left\{ X : X \geq 0 \quad , \quad \sum_{j \in L} f_j(x_j) = R \right\} \tag{41}$$

where

R is a total given budget

$f_j(x_j)$ is the cost for establishing a facility of capacity x_j in location j, $j \in L$

The functions f_j are further assumed to have the general form

$$f_j(x) = \begin{cases} 0 \quad , & \text{if } x = 0 \\ a_j + b_j x \quad , & \text{if } x > 0 \end{cases} \quad , \quad j \in L \tag{42}$$

where

a_j is a fixed charge, independent of the size x

b_j is a unit cost

Due to the fixed charges a_j, economies of scale are introduced, and usually some x_j will be zero in the optimal solution. Let the boolean variables be introduced

$$\delta_j = \begin{cases} 0 \quad , & \text{if } x_j = 0 \\ 1 \quad , & \text{if } x_j > 0 \end{cases} \quad , \quad j \in L \tag{43}$$

For every combination of δ_j values, the notation

$$J = \{ j : \delta_j = 1 \}$$

will be used for the set of chosen locations.

With reference to LM3, the following optimal location problem can be formulated

$$\max_{J,X,W} \sum_{i \in H} P_i \log \sum_{j \in J} w_j e^{-\beta C_{ij}} - \sum_{j \in J} x_j \log w_j \tag{44}$$

subject to

$$\sum_{j\varepsilon J} (a_j + b_j x_j) = R \tag{45}$$

$$J \subseteq L \tag{46}$$

Problem (44)-(45) looks a bit different from the ones considered so far, because of the combinatorial problem inplied by the choice of the subset J. However, if J is kept fixed, the general form LM3 still applies for the resulting subproblem. It can be rewritten as

$$\max_{X,W} \sum_{i\varepsilon H} P_i \log \sum_{j\varepsilon J} w_j e^{-\beta C_{ij}} - \sum_{j\varepsilon J} x_j \log w_j \tag{47}$$

subject to

$$\sum_{j\varepsilon J} b_j x_j = \bar{R} \tag{48}$$

where

$$\bar{R} = R - \sum_{j\varepsilon J} a_j$$

The optimality conditions for W are given by (38). The optimality conditions for X are

$$-\log w_j - \lambda b_j = 0 \quad , \qquad j \varepsilon J \tag{49}$$

where

λ is the Lagrange multiplier associated with constraint (48)

From (49) it follows that

$$w_j = e^{-\lambda b_j} \quad , \qquad j \in J \tag{50}$$

The unknown multiplier λ can be computed as the root of the equation

$$\sum_{j \in J} b_j D_j(e^{-\lambda b_j}) = \bar{R} \tag{51}$$

where $D_j(\cdot)$ are the functions defined by (22). When (50) is substituted in the objective function (47), and the subset J is allowed to vary again, the following purely combinatorial problem is obtained

$$\max_{J \subseteq L} \sum_{i \in H} P_i \log \sum_{j \in J} e^{-(\beta C_{ij} + \lambda b_j)} + \lambda R - \lambda \sum_{j \in J} a_j$$

This problem can also be expressed in terms of the boolean variables defined in (43)

$$\max_{\Delta} \sum_{i \in H} P_i \log \sum_{j \in L} \delta_j e^{-(\beta C_{ij} + \lambda b_j)} + \lambda R - \lambda \sum_{j \in L} \delta_j a_j \tag{52}$$

where

$$\Delta = (\delta_1, \ldots, \delta_m)$$

It must be stressed that problem (52) is more difficult than it looks at first sight, because the λ multiplier is not a constant, but a function of Δ

The equivalent formulation of (44)-(46) in the form LM4 is

$$\min_{J,S} \sum_{i \in H} \sum_{j \in J} S_{ij}(\log S_{ij} + \beta C_{ij}) \tag{53}$$

subject to

$$\sum_{j\varepsilon J} S_{ij} = P_i \quad , \qquad i\ \varepsilon\ H \tag{54}$$

$$\sum_{j\varepsilon J} (a_j + b_j \sum_{i\varepsilon H} S_{ij}) = R \tag{55}$$

$$J \subseteq L \tag{56}$$

Problem (53)-(56) can be restated in terms of the boolean variables defined by (43):

$$\min_{\Delta,S} \sum_{i\varepsilon H} \sum_{j\varepsilon L} S_{ij}(\log S_{ij} + \beta C_{ij}) \tag{57}$$

subject to

$$\sum_{j\varepsilon L} S_{ij} = P_i \qquad i\ \varepsilon\ H \tag{58}$$

$$\sum_{i\varepsilon H} S_{ij} \leq \delta_j M \qquad j\ \varepsilon\ L \tag{59}$$

$$\sum_{j\varepsilon L} a_j \delta_j + \sum_{j\varepsilon L} b_j \sum_{i\varepsilon H} S_{ij} = R \tag{60}$$

$$\delta_j = 0,1 \qquad j\ \varepsilon\ L \tag{61}$$

where

M is a very large number

Constraints (58)-(61) are the usual ones for an investment-constrained location problem (ReVelle and Rogeski, 1970; Hansen and Kaufman, 1976; Bigman and ReVelle, 1979). Indeed, problem (57)-(61) is a plant location problem with a nonlinear objective function, the nonlinearity arising from the presence of $\log S_{ij}$.

When β is very large, the term $\log S_{ij}$ can be neglected and (57)-(61) reduces to

$$\min_{\Delta,S} \sum_{i\varepsilon H} \sum_{j\varepsilon L} S_{ij} C_{ij}$$

subject to

constraints (58)-(61) and to $S_{ij} \geq 0$, $i \varepsilon H$, $j \varepsilon L$

The above problem is a classical plant location problem with a linear objective function.

6. CONCLUDING REMARKS

The relationship between LM4 and the other models is a special case of a well known duality result concerning Kullback's divergence minimization and biproportional matrices. This result can be summarized as follows:

given nonnegative real constants

$$a_i, \ b_j, \ f_{ij}, \qquad \text{with} \ \sum_i a_i = \sum_j b_i$$

let

$$\Omega = \min_X \sum_{ij} x_{ij} \log \frac{x_{ij}}{f_{ij}} \tag{62}$$

subject to

$$\sum_j x_{ij} = a_i \tag{63}$$

$$\sum_i x_{ij} = b_j \tag{64}$$

and

$$\Psi = \max_{Y} \sum_{i} a_i \log \frac{\sum_{j} y_j f_{ij}}{a_i} \qquad (65)$$

subject to

$$\sum_{j} b_j \log y_j = 0 \qquad (66)$$

then these equalities hold

$$\Omega = \Psi \qquad (67)$$

$$x_{ij} = a_i \frac{y_j f_{ij}}{\Sigma_j y_j f_{ij}} \qquad (68)$$

Since (65)-(66) is the general form assumed by the accessibility maximizing problems, the above duality implies that a close relationship exists between such problems and the Kullback's divergence minimization problem (62)-(64). It must be pointed out that this is just a formal result and, while accessibility has been given a physical interpretation, no real "information" is implied in (62). Starting from the above duality, many generalizations are possible. If, for example, different kinds of facilities are introduced, the primal problem (62)-(64) assumes the form of a multiproportional adjustment problem, very closely related to the ones discussed in Willekens, Pór, and Raquillet (1979). Different constraints can be placed on the flow variables, varying from the usual ones on total demand and total capacity, to constraints on the land availability, on the capacity of the transport network links, and so on.

The link between the location models discussed in this paper and the methods for bi - or multiproportional adjustment broadens the range of application for these methods, which seem to play the role of a *factotum* in Regional Science. However, it should be recalled that the really awkward part of a location problem lies in the topology of the feasible set for the control vector (the sizes and locations of the facilities), so that the availability of multiproportionality techniques solves only a subproblem. The real difficulties lie in problems like (52), which are purely combinatorial, and for which no really efficient algorithm has been found as yet.

From the applied point of view, the most useful property of the equivalences stated in Section 3 is that they all have assumptions 1-5 of Section 1 as a starting point. Among these, assumption 3 is of special importance, since it roots the location models to an empirically observable and testable physical phenomenon, namely, doubly-constrained spatial interaction. Assumption 4 is also important, since it introduces a measure of benefit which is both intuitive and consistent with the behavioral assumption 3. Entropy and related concepts have never been used explicitly, nor taken as starting axioms. This is apparently in contrast with the more usual approach (e.g., Wilson and Senior, 1974; Coelho and Williams, 1977; Coelho and Wilson, 1977) where models related to LM4 are assumed as starting points.

However, due to the equivalence results, and more generally, to the duality relationship, no real difference exists between the two formulations. This is true for the mathematical form, although it may not be so for the interpretation of the models. The discussion on the real meaning of concepts such as entropy, locational surplus, and accessibility, leads to problems of a philosophical nature, which are outside the scope of these notes.

NOTES

1. We note that this is a behavioral assumption, not a normative rule. The spatial interaction model is assumed to arise as the pooled output of many individual choices, which are not controlled directly by the locational decision maker. Users' choices can be influenced only indirectly, by changing the size, and hence the attractiveness, of the facilities.

2. The zones belonging to I are assumed to form a connected geographical area. This is not a mathematical requirement, but it is a sensible physical assumption, if I has to be considered as an "aggregation" of smaller zones.

3. Constraint (19) is obtained by taking logarithms on both sides of equation (3), and dropping the multiplicative constant 1/P.

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