Journal of Economic Behavior and Organization 196 (2022) 178–199

Contents lists available at ScienceDirect

Journal of Economic Behavior and Organization

journal homepage: www.elsevier.com/locate/jebo

On the matthew effect on individual investments in skills in arts, sports and science☆

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ARTICLE INFO

Article history:
Received 9 December 2020
Revised 27 January 2022
Accepted 7 February 2022

JEL classification:
C61
E20
I24
I26
Z11

Keywords:
Human capital accumulation
Abnormal control problem
Convex returns
Threshold
Matthew effect
Sisyphus point

ABSTRACT

This paper describes the process of capital accumulation subject to the following characteristics: (i) convex returns to (human) capital and (ii) the need to self-finance investments. Our setup is applicable to some peculiarities in the arts, sports and science, inter alia, coined the *Matthew effect* in Merton (1968) and explains, e.g., why prominent researchers get disproportional credit for their work. The potential young artist's (athlete's or scientist's) optimal strategies include quitting, or continuing and even expanding one's human capital in the respective profession. Both outcomes are separated by a threshold level in human capital. In addition, we find that it can be optimal to stay in business although consumption falls and stays at the subsistence level forever (we call this outcome a *Sisyphus point*). This possibility is also interesting from a theoretical point-of-view, as the optimal control problem may turn *abnormal*, i.e., the objective does not enter the Hamiltonian.

1. Introduction

The changes in the distribution of income and wealth, in particular the increased share of the very top of the top, i.e., the 0.00001%, since the 1980-ies (see Saez and Zucman, 2020) are of societal concern. New and emerging digital industries render the winners most of the cake due to network effects. In sports, arts and music, the returns to exceptional talent have always been high. For example, Pelé signed (as a 35-year old football player) a three-year 2.8 million dollars (14.2 million in today's dollars) contract with the New York Cosmos in 1975, making him the highest paid athlete in the world.[1] Yet,

☆ The authors thank two anonymous referees for their exceptionally helpful comments. This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors. The authors have no competing interests to declare.

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[1] As reported by ESPN (Entertainment and Sports Programming Network).

https://doi.org/10.1016/j.jebo.2022.02.008
0167-2681/

even afterwards the salaries of top athletes have only increased, e.g., Messi, and Ronaldo with annual earnings above 100 million euros in 2020, compared with their peers in the past: The above mentioned contract for Pelé amounts to 'meager' 4.7 million 2020-dollars per annum and he has earned for sure much less during his high time when playing at home in Brazil; in Germany, Uwe Seeler earned during the sixties 7,200 euro per annum (roughly 62,000 in today's euros), Franz Beckenbauer during the seventies 350,000 euros (1.75 million in today's euros).[2] Similar figures are reported in the arts and music (but the latter hampered by piracy and free streaming services based on advertisements).

One explanation for this developement is globalization, which reduces the returns to local talents,[3] combined with the *Matthew effect*: For to every one who has will more be given, and he will have abundance; but from him who has not, even what he has will be taken away, *Matthew 25:29*. This point was first addressed by the sociologist Robert K. Merton (1968), the 'father of the economist', who pursued the question why eminent scientists get disproportionately more credit for their contributions, while relatively unknowns get disproportionately little. A famous example from the academic discourse in economics is the familiar Solow-Swan model (Solow, 1956 and Swan, 1956).[4] Recently, Clarivate Web of Science (2020) writes in its executive summary that ... *of the world's scientists and social scientists, Clarivate Highly Cited Researchers truly are one in 1000.* This is in line with Max Weber's observation more than a hundred years ago that an academic career is a lottery with presumably even reduced odds today.

In this article we suggest a microeconomic underpinning of the Matthew effect and the economic consequences drawing on a variation of the Ramsey model of optimal investment with the following special features: (i) Increasing returns to capital, which address the earnings of an exceptional talent or celebrity often coined the Matthew effect,(ii) decreasing returns to investment expenditures and (iii) no debt at any point in time. From these assumptions, complex and interesting dynamics result, including multiple equilibria separated by a threshold. A particularly interesting and novel result is the possibility of a threshold of the *Sisyphus* type, i.e., the optimality of an outcome at the boundary of zero consumption, which simultaneously separates an interior equilibrium from an attractor to the origin (quitting the business, art, sport, science etc.). This explains, e.g., why a very high fraction of researchers have none or very few publications with none or at best very few citations. Moreover, this outcome leads to the mathematically interesting and, in the context of intertemporal optimization in economics, (very) rare but necessary concern about the normality of an optimal control problem (see Halkin, 1974 and El-Hodiri, 2012). Therefore, this framework is of formal interest but more importantly, it allows for a number of interesting interpretations and complex dynamics and thresholds linked to activities subject to the Matthew effect observed in the areas of arts, sports, science, and also in other related businesses.

This paper starts with the model (Section 2), which is complemented by economic interpretations (Section 3) before its analysis, theoretical and numerical (Section 4). The economic consequences of the results in Section 4 are addressed in Section 5. The paper also contributes to theory as it analyzes and shows how one can treat abnormality in optimal control problems. This part is, however, relegated to an Appendix due to its mathematical complexity.

2. Motivation

Following Merton (1968), many studies have investigated the so-called Matthew effect, both analytically and numerically. It is empirically documented in many fields, including less obvious ones like education (reading and math). The Matthew effect holds most clearly in the arts, in the past but also presently, in particular for painters, musicians and poets. Bask and Bask (2015) argue that the cumulative advantage is an intra-individual while the Matthew effect is an inter-individual phenomenon and that this difference in phenomena has consequences for the modelling of socio-economic processes. Both of these are indeed detected in data. Since it is very difficult to measure quality and thus precludes convincing empirical assessments of the magnitude of the status effect, Azoulay et al. (2014) address this problem by examining the impact of a major status-conferring prize (becoming a Howard Hughes Medical Institute Investigator) that shifts an actor's positions in a prestige ordering. They find only small and short-lived evidence of a post-appointment citation boost but prize winners are of (relatively) low status gain. The review of Perc (2014) shows that the Matthew effect, labelled as *the concept of preferential attachment*, is ubiquitous across social and natural sciences and is related to the power law, which characterizes patterns of scientific collaboration, the growth of socio-technical and biological networks, the propagation of citations, education, as well as many other aspects, like the usage of words in a given language, the size of cities and visits to web-sites.

2.1. Arts

In the arts, income disparity is often higher compared to other professions. If we take into account the ratio of young people who start learning music, art or sports in their childhood and the number of professionals who can earn a living from this profession, we observe an even larger gap. For instance, 1.7% of all American students (of higher education) are enrolled in music programs. However, for half of them music is not the major and the ability to find employment is only the

[2] See Fussball-Geld (2013).

[3] Until the seventies and eighties, many leagues restricted or even did not allow at all, the hiring of foreign players. Similar restrictions applied in many other areas, not only in sports.

[4] For which Solow got all (and still gets most of) the credit (including a Nobel prize) although Swan developed the model independently and published it around the same time (but in a less prestigeous journal).

4th top ranking factor in their motivation to enroll in such a program (according to The College Music Society, 2015). It is highly likely that only a relatively small fraction will become professional musicians making a living from their engagement. See also Salganik et al. (2006), who find in their study (a web-based experiment with 14,000 participants) of a cultural market for talents that payoffs to the top are reinforced while payoffs to the bottom are diminished – even without social influence.

2.2. Sports

Sports differ from arts by a more objective evaluation. However, this applies only at the late stage because athletes face many other problems in their early career, such as proper talent discovery, access to funds for training or social connections. The wages of the top football players in Europe are above 10 million dollars per year, but some smaller clubs in the European leagues pay much lower wages even to top players. The income of football players in the 2nd and 3rd leagues are not much different from their respective national averages. Thus, football can provide a living to only a minority of athletes even in a big football country (like the UK, Spain, Italy or Germany). At the same time millions of children start playing football (at school, in small clubs) so that many potential talents drop out already at an early stage of a potential professional career. Thurman (2016) compares rewards in different US sport leagues (NFL, MLB, NBA) highlighting the different orders of magnitude between minimal, mean and top salaries. For example, in 2015 in the NBA (442 players) the minimal annual wage was 525,000 dollars, the top was 25 million dollars (ratio $1:50$), while the mean was 4.8 million dollars. The Gini scores were close to 0.5, and even higher (0.6) for the MLB. Compared to regulated US team sports (to foster competition), the top earnings and the ratios to the averages are even higher in unregulated sports like soccer (Messi 126, Ronaldo 117 and Neymar 96 million dollars),[5] golf (Tiger Woods earned 62 million dollars in 2020),[6] car racing (Lewis Hamilton is going to earn 133 million euros for his next three attempts to win the driver's championship topping Michael Schumacher's record of seven titles),[7] and all this in a year plagued by the COVID-19 pandemic.

2.3. Science

In our field, the distribution of scientific publications and citations follows a power law, discovered by Lotka (1926) and later explained by Merton (1968, 1988) as cumulative advantage. The latter has also raised the question of what factors may lead to this cumulative effect. Some of them are subject to substantial randomness (e.g., to get a paper published or not) in the beginning of a career, but then may become self-reinforcing. While science still offers employment to a higher fraction of its graduates, there is substantial inequality in observed productivity (number of published articles).[8] What microeconomic and dynamic mechanisms can explain these observations?

3. The model

Aside from the objective to model how the Matthew effect affects individual decisions, the goal is to develop a tractable and economically meaningful model with a steady state, which is optimal and simultaneously separates the basins of attractions between high and low equilibria, *and* which is characterized by zero consumption. We call such a threshold one of the Sisyphus type because of continuing an activity, yet, enduring at the subsistence level, i.e., zero consumption. The presence of such a point creates an attractor to the origin (i.e., quitting arts, science, etc.), which explains why a high fraction of researchers only has 0 or 1 publication and 0 or 1 citation, to use the example from the work of Merton (1968), or why many start a career in the arts but end up in different professions (often as teachers, e.g., for music).

For this purpose, we propose the following variation of the Ramsey model of maximizing intertemporal (using the constant discount rate $r > 0$) benefit (b) from consumption (c), amended for a stock effect $v(k)$,[9]

$$\max \int_0^\infty e^{-rt}(b(c) + v(k))dt.$$

Consumption is, as in the Ramsey model, given by the difference between output $f(k)$ and investment (u),

$$c = f(k) - u.$$

A crucial deviation from the usual Ramsey setup is the assumed convexity of the production function, $f' > 0$ and $f'' \geq 0$, which will be economically justified below;[10] capital accumulation is defined as usual but with the twist that investment expenditures are subject to diminishing returns, $\left(\alpha'' > 0\right)$,

$$\dot{k} = u - \alpha(u) - \delta k,\ k(0) = k_0,$$

[5] See Settimi (2020).
[6] See Piastowski (2020).
[7] See Duncan (2020).
[8] In the EU, labor laws mostly dictate the scientists' wage distributions, which are more equal than the *Law of Lotka* for publication distributions.
[9] As, e.g., Hof and Wirl (2008) show that stock spillovers are crucial for thresholds in concave setups of the Ramsey model based on Barro and Sala-i-Martin (1995).
[10] Skiba (1978) and many follow-ups, e.g., Brock and Dechert (1985) consider convex-concave production functions.

because large investments are less effective in expanding the capital stock; $\delta > 0$ denotes the deprecation rate. Preceding the interpretations below and with reference to our own profession, purchasing many software licenses (say Mathematica, MATLAB, SPSS) and books (e.g., for this paper, Barro and Sala-i-Martin (1995), works of other economists as well as of sociologists including Merton) at once cannot all be effectively used immediately. That is, the speed of learning is limited due to constraints, in particular of time, so that a piecemeal strategy will be more effective in turning investment into productive human capital.

In order to simplify as much as possible and to allow for explicit, at least numerical, calculations we assume linear and quadratic specifications leading to the following model (dropping the arguments of time t):

$$\max_{u(t)\geq 0} \int_0^\infty e^{-rt}((mk^2+\beta k-u)+hk)dt, \tag{1}$$

$$s.t. \qquad \dot{k}=u-au^2-\delta k,\ k(0)=k_0,\ k\geq 0, \tag{2}$$

$$c(k,u):=mk^2+\beta k-u\geq 0. \tag{3}$$

We assume $0<\beta<\delta$ because this inequality is necessary[11] for the existence of a (positive) Sisyphus point. This assumption expresses that small capital stocks are not sustainable and, hence, rules out perennial growth, which is impossible for individuals in contrast to economies. Many talents compete in art, sports and science for a limited amount of positions that allow to earn the holder a living. Thus, there needs to be a selection mechanism and we propose one based on insufficiently high capital stocks as a selection threshold. This reflects the situation where talents without financial support cannot continue, apart from extremely high human capital. Our model appears similar to Hartl and Kort (2004) but has the following crucial differences: (i) the adjustment costs associated with large investments appear in the state equation instead of in the objective, (ii) investment must be paid from current revenues (no debt), and (iii) consumption must be nonnegative. Last but not least, (iv) the Hartl & Kort model does not allow for the kind of dynamics that we are interested in and that are relevant for the above mentioned fields.

A crucial implication of (1)–(3) is the existence of Sisyphus points, denoted k_s, i.e., a level of (human) capital at which consumption turns zero and therefore all initial conditions to the left of it must end up in the origin, $k\to 0$. More precisely, departing from the constraint (3), we can define the maximal level of feasible investment,

$$u\leq u^{\max}:=f(k)=\beta k+mk^2. \tag{4}$$

Furthermore, we define the investment level $u^{\min}(k)$, for a given stock k, as the lowest nonnegative investment level for which the stock does not decline, that is

$$u^{\min}(k):=\min\{u\geq 0:\dot{k}\geq 0\}. \tag{5}$$

This implies that if $k_s=\inf\{k>0:u^{\min}(k)<u^{\max}(k)\}$ satisfies $k_s>0$, then $\dot{k}<0$ for all $0<k<k_s$. The point k_s is then determined by the equation

$$\psi(k):=u^{\max}(k)-u^{\min}(k)=0. \tag{6}$$

We define the Sisyphus point as the largest stock of capital where only consumption reduced to zero avoids a decline. Arithmetically,

Definition The *Sisyphus point* (k_s) is the positive real root $k\in(0,k^{\max})$ of the third order polynomial defined in (6), so that $u^{\max}(k_s)=u^{\min}(k_s)<1/(2a)$, if such a root exists.

Therefore, $\dot{k}<0$ inevitably for $k\in(0,k_s)$ since $u^{\max}(k)<u^{\min}(k)$, see Fig. 1 (and thus also for $k\in(k^{\max},\infty)$).[12]

Figure 1 plots the crucial terms, $u^{\min}$ and $u^{\max}$ with their intersection determining k_s (also magnified). The dashed line shows the larger root of the equation $\dot{k}=0$ with $u>1/(2a)$. This locus, however, is irrelevant because gross capital formation is declining for too large investments and thus being dominated by them,

$$u\leq \frac{1}{2a}=\arg\max_u u-au^2. \tag{7}$$

Therefore, a solution with $u\geq 1/(2a)$ cannot be a candidate for maximizing (1) even if it satisfied the first-order optimality conditions. As a consequence,

$$k^{\max}=\frac{1}{4a\delta} \tag{8}$$

defines the upper bound of the capital stock (see Fig. 1 (a)).

[11] As shown below in (9).

[12] Further arithmetical conditions, in particular concerning the existence and the optimality of a Sisyphus point, are given in the Appendix.

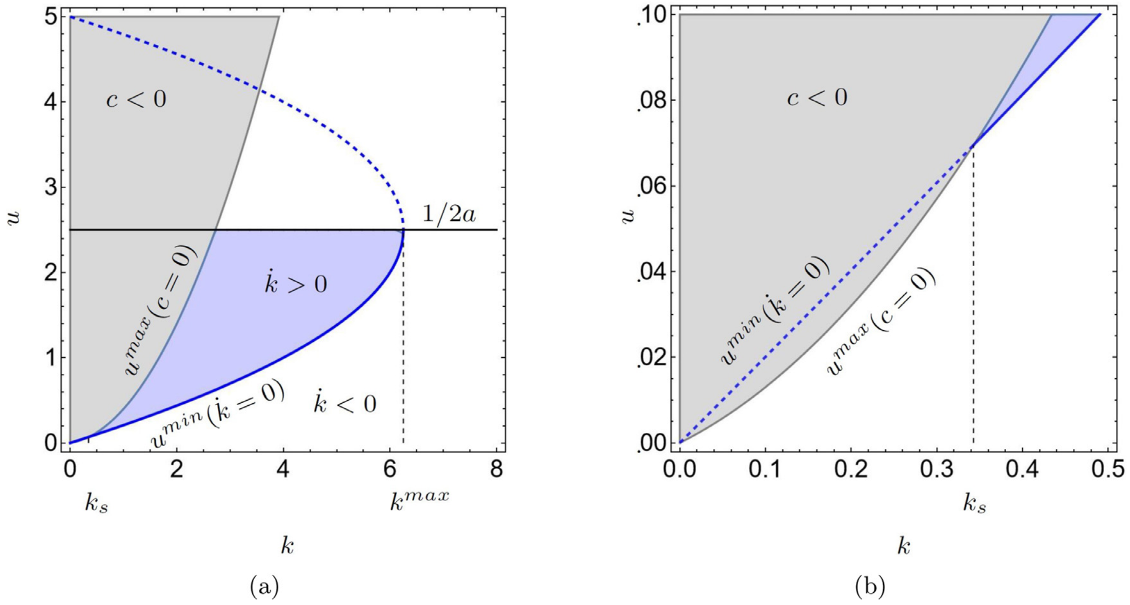


Fig. 1. Plot of the curves $u^{\min}$ ($\dot{k} = 0$) and $u^{\max}$ ($c = 0$) in (a) and the magnification to the right of the origin in (b). The blue area shows where human capital can grow whereas the gray area highlights infeasibility due to negative consumption. Maximal economical investment is $u = 1/2a$. Therefore the blue area above $1/2a$ is uneconomical . (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

A necessary condition for $k_s > 0$ is that

$$\frac{du^{\max}}{dk} < \frac{du^{\min}}{dk} \text{ at } k = 0,$$

which implies

$$\beta < \delta. \tag{9}$$

This explains the assumption made following the description of the model (1)–(3).[13] Interesting are the cases in which a steady state, k_∞, exists such that

$$0 < k_s < k_\infty < k^{\max}.$$

Application of the implicit function theorem to

$$mk^2 + \beta k - \frac{1 - \sqrt{1 - 4a\delta k}}{2a} = 0,$$

$$mk^2 + \beta k - a\left(mk^2 + \beta k\right)^2 - \delta k = 0,$$

implies that a larger value of the parameter m leads to a decline of the Sisyphus point and to an increase in $k^{\max}$, and thus to an expansion of the area $k_s < k < k^{\max}$ in both directions. This leads to the conjecture that k_∞ increases w.r.t. m too but this requires further analysis.

$v(k) > 0$ accounts for a talent's direct benefit from her human capital stock, e.g., from reputational status within her field. As mentioned above, intuitively, one can stay at the Sisyphus point forever only for a positive spillover from capital, $v(k) > 0$.[14] The economic reason is that staying at k_s for $v = 0$ yields zero for the integrand of the objective functional and staying there is thus dominated by any strategy of positive consumption by running down the capital stock.

By definition, the Sisyphus point k_s is located at the intersection of the two curves $u^{\max}(k)$ and $u^{\min}(k)$ and thus at the intersection of the sets $\dot{k} \geq 0$ and $c \geq 0$. Any trajectory crossing the level k_s implies $\dot{u} \leq 0$ to its left while both $\dot{u} \geq 0$ and $\dot{u} < 0$ are possible to its right.[15] Since $k = 0$ is always a feasible long run solution, some optimal trajectories can pass through the Sisyphus point on their way to $k = 0$. Another property of the Sisyphus point is that it can be optimal to stay there forever. This is the standard outcome for thresholds in concave optimization problems (compare Wirl and Feichtinger, 2005)

[13] For sufficiency conditions on the existence of a Sisyphus point see Prop. A1.

[14] The precise criteria are given in the Appendix.

[15] This implies declining investments so that only the origin is attainable, not only for the optimal but for all feasible controls (i.e., of non-negative consumption).

but almost entirely ignored in dynamic optimization problems with convex-concave objectives (Hartl and Kort, 2004 draw attention to the possibility of a continuous policy function although the Hamiltonian is convex with respect to the state). Although nothing is consumed at the Sisyphus point ($c = 0$) since everything is invested ($u^{\min} = u^{\max}$ is required in order to avoid the decline, $k \to 0$), the payoff, i.e., the integrand in (1), can be positive if it includes a direct benefit from the state. Indeed, if $\nu > 0$, the Sisyphus point can be optimal, as shown in the examples below.

If an agent has no access to credit in order to expand her human capital, all paths, $0 < k(0) < k_s$, must end up in the origin. Contrary to usual thresholds, this attraction of the origin applies not only to optimal but to all feasible paths. This suggests an analogy to what is commonly called in physics a 'black hole', because there is no way to avoid this limiting outcome ($k \to 0$) once the 'horizon' $k_s > 0$ is crossed to the left, given the constraints that the agent faces. On the other hand, trajectories that expand human capital can and do exist on the right-hand side in the neighborhood of the Sisyphus point (but need not be optimal).

3.1. Optimality conditions

We define the (current value) Hamiltonian of the optimal control problem (1)–(3),

$$H = \lambda_0(mk^2 + (b+h)k - u) + \lambda(u - au^2 - \delta k) \tag{10}$$

and set, as usual, $\lambda_0 = 1$.[16] Maximizing the Hamiltonian (H) with respect to the control and accounting for the constraints, $c = mk^2 + \beta k - u \geq 0$ and $u \geq 0$, yields

$$u^* = \begin{Bmatrix} mk^2 + \beta k & & \frac{\lambda-1}{2a\lambda} > mk^2 + \beta k, \\ \frac{\lambda-1}{2a\lambda} & \Leftrightarrow & 0 \leq \frac{\lambda-1}{2a\lambda} \leq mk^2 + \beta k, \\ 0 & & \frac{\lambda-1}{2a\lambda} < 0. \end{Bmatrix} \tag{11}$$

The other first-order conditions[17] determine the evolution of the costate (λ) according to (12), which together with the state equation after substituting the optimal control (13) yields the canonical equation system. This system is given below for the interior solution of (11):

$$\dot{\lambda} = \lambda(r+\delta) - 2mk - \beta - h, \tag{12}$$

$$\dot{k} = \frac{\lambda - 1}{2a\lambda} - a\left(\frac{\lambda - 1}{2a\lambda}\right)^2 - \delta k. \tag{13}$$

Lemma 3.1. *The canonical equations allow for at most two steady states in the admissible domain $k > 0$.*

Proof. We carry out the proof for the interior control, i.e. $c(k,u) > 0$.[18] Equating the time derivatives to zero, we get the following algebraic system to determine the steady state(s) of the above dynamic system:

$$k = \frac{\lambda(r+\delta) - \beta - h}{2m},$$
$$k = \frac{\lambda^2 - 1}{4\delta a\lambda^2}.$$

The first equation defines a straight line in the (λ, k) plane with positive slope and a root at $\lambda = (\beta + h)/(r + \delta)$. The second function $k(\lambda)$ has a singularity at $\lambda = 0$ and two roots at $\lambda = \pm 1$. After equating the above two terms we get the following cubic equation in λ characterizing any steady state:

$$g(\lambda) := 4\delta a(r+\delta)\lambda^3 - \lambda^2((2m + 4\delta a(\beta + h)) + 2m. \tag{14}$$

Only the positive out of the three roots are of interest. Since all parameters are positive, $g \to -\infty$ for $\lambda \to -\infty$, $g \to +\infty$ for $\lambda \to +\infty$, $g(\lambda)$ has two extrema. There exists a local and positive maximum at $g(0) = 2m > 0$ and a local minimum at $\lambda > 0$. Therefore $g(\lambda)$ must have one and only one negative root and the remaining root(s) must be either positive or a pair of conjugate complex numbers.■ □

[16] However, we have to return to this implicit assumption of *normality* when deriving the optimal paths, because the case $\lambda_0 = 0$ cannot be ruled out. The corresponding abnormal solutions are derived in the Appendix.

[17] $\dot{\lambda} = r\lambda - H_k$ and $\dot{k}$ in (2) with u^* as given in (11).

[18] For the case $c(k,u) = 0$ this results from the proof of Prop. (A1) in the Appendix.

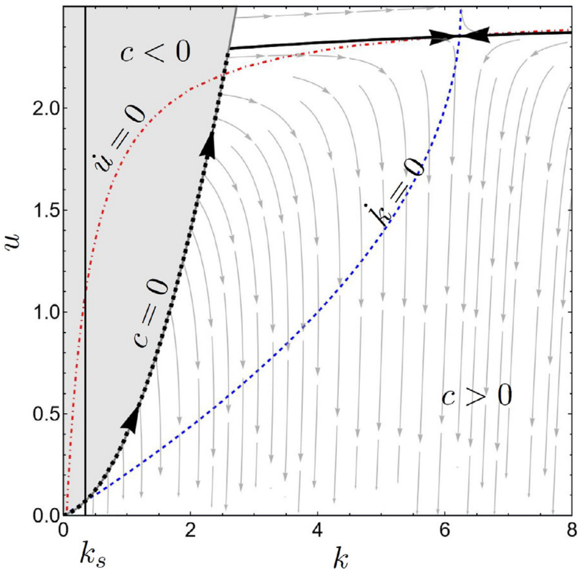


Fig. 2. The vector field, the isoclines and the feasible zone $c > 0$ for a rather low Sisyphus point and a large saddle point path, more precisely for (15) and $(m, \delta) = (0.3, 0.2)$. The path from the Sisyphus point to the saddle point path (solid black) includes the corner solution (c = 0) unless k_0 is sufficiently large.

4. Results

4.1. Bifurcation analysis

We choose the following parameters,

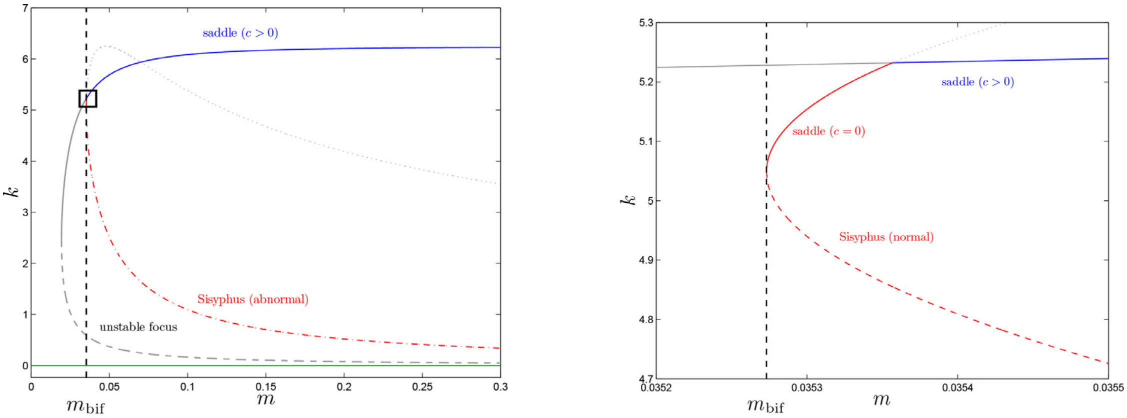
$$r = 0.03,\ a = 0.2,\ \beta = 0.1,\ h = 0.1, \tag{15}$$

i.e., the subjective discount rate is 3% per annum, the adjustment cost parameter limits investment to $u < 2.5$, and the linear earning and the direct benefit parameters are both set to 0.1. Numerical means are necessary, because it is impossible to determine by analytical means first the steady states and then which of the paths is optimal for a given initial condition. We derive the different cases by varying the parameter that governs the Matthew effect (m) and the rate of depreciation (δ).

Figure 2 depicts a typical phase portrait of the canonical equations but shows the control, investment u, instead of the costate (λ) assuming $m = 0.3$ and $\delta = 0.2$ in addition to the parameters in (15). The (unconstrained) system has a negative and stable steady state, which is irrelevant due to $k \geq 0$ and is replaced by the corner solution $k \to 0$ as a possible long run outcome. The other two steady states are positive of which the lower one is a repelling focus (complex eigenvalues with positive real parts) and the third and largest steady state is a saddle point. Figure 2 also shows the isoclines, $\dot{u} = 0$ and $\dot{k} = 0$, the stable manifold for the unconstrained problem in bold (its extension to the left requires $c < 0$), the feasible set, $c \geq 0$, and how the impossibility of getting credit (i.e., of non-negative consumption) affects the policy heading towards the high steady state: in order to make up for the low but steeply increasing investment along the boundary ($c = 0$), it is then flat in the interior along the saddle point path and close to the maximum due to the large Matthew effect. What this figure cannot tell us is, which of the strategies, following the saddle point path to the high steady state or converging to the origin (or to somewhere else), is optimal and for which initial conditions this is the case. All we can infer so far is that initial conditions $k_0 < k_s$ must end up at $k = 0$.

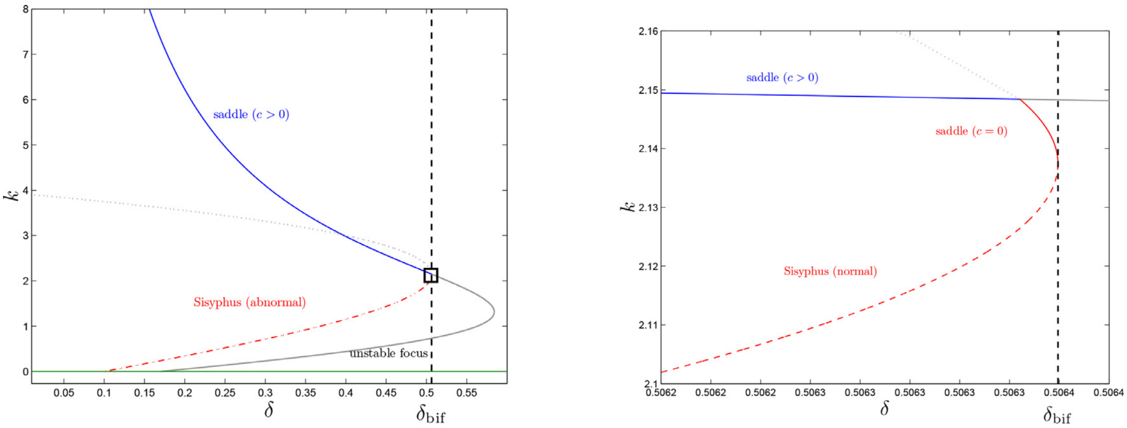
Figure 3 shows how the positive steady states of (12) and (13) depend on the parameter m, measuring what we call the Matthew effect. Positive roots of (14) require at least $m > 0.02\ldots$ so that an enterprise with only weakly convex returns but convex investment costs is doomed to fail. Of the two positive steady states the upper one is a saddle point with an asymptote at $k_\infty \to 6.25$ for $m \to \infty$. The lower one vanishes for large Matthew effects, i.e., $k_s \to 0$ for $m \to \infty$. The line between the two (positive) steady states (in blue) shows at which capital stocks the constraint $c \geq 0$ is binding, i.e., the Sisyphus point, k_s, depending on m. Therefore, the lower steady state is always located in the infeasible domain $c < 0$. The Sisyphus line intersects the upper steady state line in the area of the corner equilibria.[19] The gray curves refer to steady states of the canonical system that do not correspond to equilibria of the optimal solution (corresponding to the dominated part shown in Fig. 1 by the dashed curve). The colored curves refer to optimal long run outcomes. For $m < m_{bif}$ (= the

[19] The term *corner* refers to the active constraint $c = 0$.



(a) $m - k$ bifurcation diagram (b) Near intersection of red and blue curve

Fig. 3. This bifurcation shows the different longrun outcomes for different levels of the Matthew effect (m). The small rectangle located at the intersection of the red and the blue lines on the left-hand side are enlarged in the right-hand side. The bifurcation diagram is shown with respect to m for (15) and $\delta = 0.2$. The gray curves denote the steady states of the canonical system that cannot be optimal. Colored curves show the steady states of the optimal solution, where the color blue denotes saddles in the interior, whereas for the red and green curves the control constraint is binding. The green curve depicts the origin which is also a steady state of the canonical system and the red curve the equilibria with an active constraint but not satisfying $u^{\min} = u^{\max}$. The dashed part of the red curve (better visible in the right panel) corresponds to the normal case and the dashed-dotted part of the red curve corresponds to the abnormal case, see Prop. A2 in the Appendix. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)



(a) $\delta - k$ bifurcation diagram (b) Near intersection of red and blue curve

Fig. 4. The bifurcation diagram w.r.t. δ for (15) and $m = 0.3$. The colors indicate the optimal paths: convergence to the saddle point (blue), staying at the Sisyphus point (red, only visible in the enlargement on the right-hand side) and converging to the origin (green, for all points below $k_s(\delta)$, red dashed). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

bifurcation value) only the origin is feasible and thus represents the unique optimal long run outcome. The Sisyphus point appears at $m = m_{bif}$ and the Sisyphus curve separates the interior outcomes ($c \geq 0$ is not binding and identified in Fig. 3 by the blue curve) from the corner solutions, $k \to 0$. The magnification in Fig. 3b shows the novel outcome of an interior equilibrium at which $c = 0$ is binding, yet, staying at the corresponding Sisyphus point is optimal.

Many, if not most, applied papers follow Skiba (1978) and others (e.g., Brock and Dechert, 1985; Dechert and Nishimura, 1983 and more recent applications to shallow lakes, Mäler, 2000; Mäler et al., 2003), sketch phase diagrams and infer from these the existence of multiple long run outcomes which are separated by a threshold in the state space. This reasoning is insufficient, because one branch can be, and in many cases (also in our model) is, the globally optimal one. Therefore, instead of the local and geometric analysis, a global one is necessary; Wagener (2003) and later supplemented by Kiseleva and Wagener (2010) check explicitly for the optimality of different paths in shallow lake models. Also see Antoci et al. (2011) for a global analysis of a growth model with indeterminacy in the long run outcome. In fact, the determination of the optimal policy in our model faces additional complexities (as elaborated on in the Appendix) and thus requires advanced numerical techniques similar to the methods sketched in Grass et al. (2008) and in Grass (2012).

Figure 4 shows a bifurcation diagram for δ and a pattern similar to the one in Fig. 3: a high and (saddle point) stable steady state and a low steady state, which is not only unstable but is located in the infeasible domain. Lowering δ below

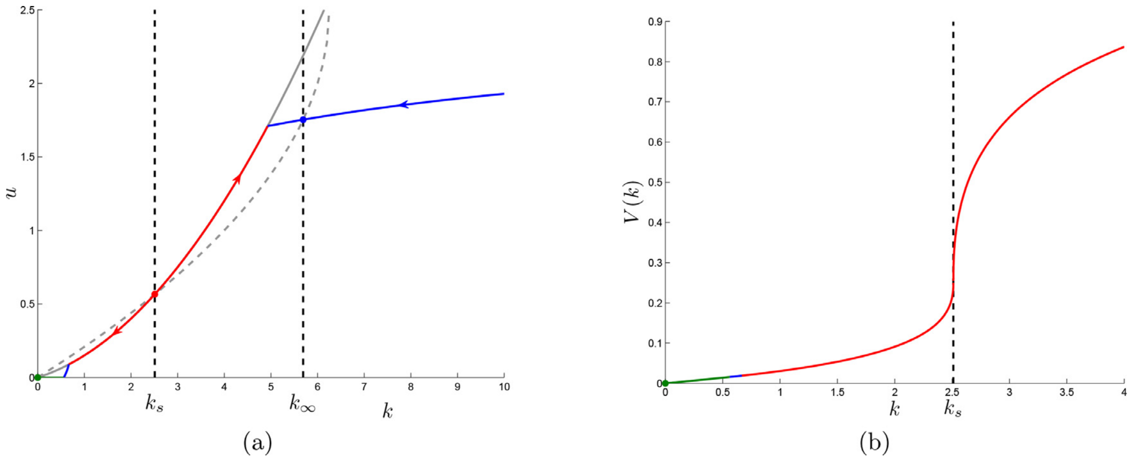


Fig. 5. The generic optimal policy for (15) and $(m, \delta) = (0.05, 0.2)$. The state space (panel a) is separated by a threshold at the Sisyphus point k_s: to the left, the optimal solution converges to zero and to the right it is optimal to move to an interior equilibrium k_∞. For $k(0) = k_s$ it is optimal to stay put. The circles denote the equilibria and the color code refers to different (in)active constraints: blue no constraint is active, red if $c = 0$ and green of $u = 0$. The dashed gray line depicts the $\dot{k}$-isocline and the solid gray line corresponds to the active constraint $c = 0$. Panel (b) shows the value function $V(k)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

β (here 0.1) increases the upper saddle.[20] No positive roots exist for too large depreciation rates, $\delta > 0.6$, rendering again the origin as the only possible long run outcome. Figure 4 includes also an identification of the optimal policies conditional on the bifurcation parameter (δ) and the initial condition ($k = k_0$). For $\delta < 0.5063\ldots$ and sufficiently large initial capital, $k_0 > 2.149\ldots$, the saddle point path heading towards the high steady state is the optimal policy (indicated by the blue line). The enlargement shows that for slightly larger depreciation rates and lower capital stocks, first the boundary policy ($c = 0$ and $k = k_s$, red) and then heading towards the origin ($k \to 0$) is optimal.

4.2. Optimal strategies

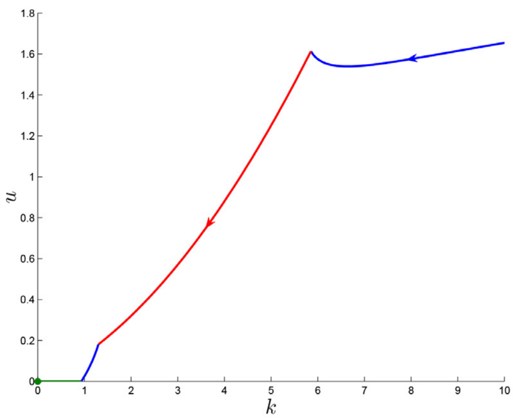
Figure 5 a shows the optimal strategies in the (k, u) space for $m = 0.05$ (small convexity, but the generic case) and all the other parameters as in (15) and Fig. 3. There are three possible long run outcomes depending on initial conditions. As conjectured, sufficiently large initial endowments with human capital lead to the high steady state as shown in the bifurcation diagram in Fig. 3. Low initial human capital requires one to leave the profession. However, if placed exactly at the Sisyphus point, then it is optimal to stay there forever! The two trajectories emerging from the Sisyphus point, $k_s \approx 2.5$, move either to $k = 0$ or to the upper saddle point equilibrium (denoted $k_\infty \approx 5.7$). However, both strategies start and evolve over a wide range of human capital along the constraint $c = 0$ (shown in red) but end up very differently either at the saddle point path (at least close to the steady state k_∞ shown in blue) or at another border solution ($u = 0$ close to $k = 0$, in green). Figure 5b shows the value function $V(k)$ for the same case highlighting the steep, actually infinite, slope $V' = \infty$ at k_s. This property is important from both an economic (see the discussion in the following subsection) and a mathematical point-of-view because of a violation of the assumption of normality. More precisely, the usually to 1 normalized coefficient λ_0 of the objective in the Hamiltonian (10) turns 0 and the costate λ diverges to infinity.[21] The Sisyphus point k_s serves as a threshold between the two attractors, $k \to 0$ and $k \to k_\infty > 0$, and ensures continuity of the control (investment) instead of the jump typically linked to such non-concave dynamic optimization problems. Furthermore, even if starting to the right-hand side of the Sisyphus point and thus continuing with one's profession, it is optimal to do so with minimal consumption (i.e., at the boundary, $c = 0$) in order to invest the maximum possible subject to the unattainability of credit.

Remark. An economic interpretation of abnormality of the problem in the Sisyphus point is the following. In the Sisyphus point the optimal control is on the edge. There is no other possibility than to choose $u^{\max}(k_s)$ and the slightest change in the state value yields a sharp (infinite) relative increase/decrease in the optimal profit.

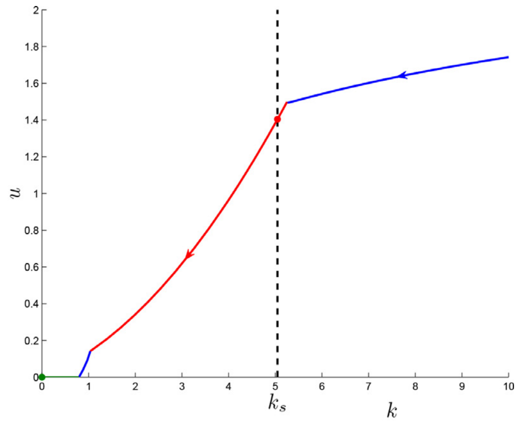
Figure 6 shows the structural changes. At $m = 0.03$, which is below the case in Fig. 5, the optimal policy is to move to $k = 0$ (to give up) for all initial conditions. An increase of the Matthew parameter to $m = 0.03527$ leads to the emergence of a Sisyphus point along the optimal path, but it is then only passed on the way to the origin. Further increases of m render the high steady state feasible, which coincides with the Sisyphus point for $m = 0.03531$. That is, a positive steady state is optimal at least for initial capital exceeding the Sisyphus point, $k_0 > k_s$, but still at the boundary, i.e., $c = 0$. Further

[20] The set of feasible saddles is not bounded for $\delta \geq 0$. Contrary to the case of Fig. 3, this means that for a certain low depreciation the final outcome can be arbitrarily large.

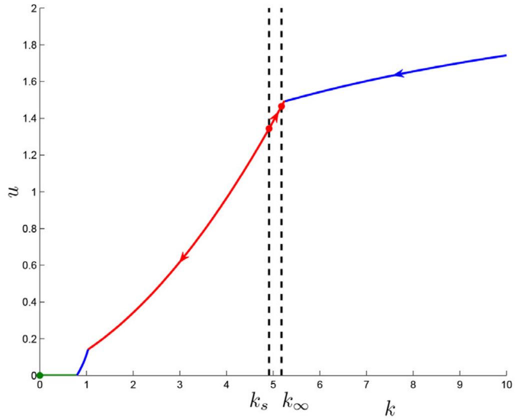
[21] The details, including the numerical treatment of the abnormal case, are given in the Appendix.



(a) $m = 0.03$, $k \longrightarrow 0$ is the only solution



(b) $m = 0.03527$, the Sisyphus point existsbe-ing a semi-stable equilibrium



(c) $m = 0.03531$, the Sisyphus point and an interior equilbrium appear

Fig. 6. Structural change in the (k, u) space for (15) and $\delta = 0.2$ and for m in the neighborhood of the lowest Sisyphus point (compare Fig. 3). Circles refer to equilibria, the colors to the different (in)active constraints: blue means no constraint is active, red indicates that the constraint $c = 0$ is active and green $u = 0$ is active . (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

increases of m move the higher steady state into the interior allowing for $c > 0$, as shown in the example in Fig. 5. Even larger Matthew effects, such as $m = 0.3$ corresponding to the phase portrait in Fig. 2, render the high steady state attracting over a much wider range of initial conditions but may still require consumption at the boundary (see Fig. 2) unless endowed with large human capital.

5. Economic implications

Although the model is introduced formally, its features are crucial in different fields in which individual talent can lead to a very unequal distribution of incomes. Familiar examples are: sports, arts and also science according to Merton (1968) if reputation and fame are accounted for. The evaluation of performance is most objective in sports but highly subjective in arts and thus also depends on luck, advertising and access to media and markets (e.g., compare Yegorov et al., 2016). The situation in science is presumably in between the other two.

A crucial observation in all these examples is that the individual reward is linear in own human capital (k) but convex in prominence or fame, i.e., relative to the competitors in a particular field. Assume for simplicity that the reward per unit of human capital is affine,

$$\tilde{m}\frac{k}{K} + \beta,$$

in which β describes the individual productivity per unit of individual human capital and the first term accounts for the increasing returns due to prominence or fame. More precisely, the relative position of individual human capital (talent, ability, visibility, etc.) matters with respect to a reference point denoted by K. For example, K could describe the average over all other actors active in a field and is thus exogenously given at the individual level. Treating K as a constant,[22] we define a reward coefficient,

$$\rho(k) := mk + \beta, \; m := \frac{\tilde{m}}{K} > 0, \; \delta > \beta > 0 \tag{16}$$

and obtain linear quadratic revenues (y) with respect to individual human capital,

$$y(k) = k\rho(k) = mk^2 + \beta k,$$

as stipulated in (1). In traditional industries, $m = 0$, yet $m > 0$ holds in branches in which recognition, talent or prominence lead to excessive returns. The additional payoff term, hk, accounts for individual satisfaction from a certain level of human capital (whether in absolute or in relative terms does not matter given our assumption about K).

For a given population of athletes, artists, scientists or small businesses, with the same initial human capital $k(0)$ but different $\tilde{m}$ (or respectively, m), their personal Sisyphus points and long run attainments will differ. As a consequence, some of them will have to exit the market (those with low m and low k_0), while others with the same (or higher) m and $k_0 > k_s$ will persist. Therefore, success is unevenly distributed leading to the Matthew effect observed in empirical studies.

The assumption of no debt at any point in time accounts for the uncertainties banks face about the skills of an applicant (e.g., a young painter asking for credit to travel to and learn from a famous master or at a foreign academy). Therefore, the applicant will not be offered credit, or if then only at prohibitively expensive terms. High bankruptcy rates that are characteristic for certain kinds of businesses provide another reason for credit restrictions.

5.1. Science

We start with science as our first example due to the original and stimulating work of Merton (1968) who coined and related the Matthew effect to cumulative advantage: eminent scientists get disproportionate credit for their contributions to their field, while relatively unknown ones get disproportionately little. Compensation in science consists of two parts (Stephan, 1996): one is paid regardless of an individual's success, the other (including prestige, journalistic citations, paid speaking invitations, and other such rewards) reflects the contribution to the scientific field.[23] Therefore, the recognition for scientific work is skewed in favor of established scientists and additional factors reinforce the process of cumulative advantage: differences in individual capabilities, unequal access to resources, inequality of peer recognition, and differences in scientific productivity.

In our notation, (k), the scientific human capital of a researcher, is the only production factor. $\rho(k)$ denotes the scientist's payoff from a particular piece of work (say a paper) accounting for the non-linear Matthew or recognition effect. The reward (y) can be used for consumption (c) and investment (u). The additional term (hk) accounts how a researcher values own achievements irrespective of her public evaluation (compare Bénabou and Tirole, 2006, for the consequences of such intrinsic motives).

5.2. Sports

The Matthew effect is noticeable in many kinds of sports, because the winners take a disproportionately large share of the pie (but not all as competition is a *conditio sine qua non* for winning), monetarily, but even more so in terms of fame. Indeed, everyone is familiar with the name of the winner, say of the Tour de France, but only few know the names of those finishing in second place.[24] Sports provides also a good example how to link the individual Matthew effect to the aggregate. Considering individual talents for different kinds of sports, e.g., in Austria, entering alpine skiing brings about fierce competition and thus a large K, while entering a related field like ski jumping will allow one to face a lower K; of course, payoffs are also larger in fields populated by many competitors (in the US football or baseball versus soccer). And sports is full of examples, where people invest not only time but also their own money to make it to the top: in skiing (the Kostelic sisters were coached by their father), racing (Niki Lauda, three times World Champion of Formula 1 racing, spent his own money in order to be able to enter racing events), and in tennis (from Steffi Graf to Dominik Thiem to the Williams sisters, all coached by their fathers). However, those who did not make it, quit (with countless but unknown examples).

The examples from sports are not limited to individuals but also include collectives. Recently, The Economist (2020) reports on the unequal situation between the Premier League and lower league professional football teams in England. For example, Bury FC, just north of Manchester, was kicked out of professional football after it failed to service its debt, while nearby Manchester City's Emirati owners generously paid the players' salaries, by far exceeding the club's revenues. Zoë Hitchen, a fan of Bury, said, “The system ... always lets people down at the bottom. It never lets down the people at the top.”[25]

[22] The extension for a competitive equilibrium of agents having different abilities and starting from different initial conditions is left for future research.

[23] And since scientists value both income and prestige, scientists are willing to give up some income in exchange for greater ability to publish according to Stern (2004).

[24] An exception maybe Raymond Poulidor, because he finished several times second behind Anquetil.

[25] However, the UEFA expelled Manchester City from Europe's football contests for the following two years percisely because of that.

5.3. Arts

Similar to sports, and maybe even larger, are the uneven returns in many disciplines of the arts. For example, in 2018 David Hockney earned above 90 million dollars for a single painting (Callum, 2019), while countless other painters earn just a few dollars for their work with a value in most cases for sure above $1/10^6$ of Hockney's painting. In May 2019 Jeff Koon's sculpture of a rabbit was bought for 91 million dollars (Kazakina, 2019). Moving back in history, only a handful of painters (e.g., Giotto, Dürer, da Vinci,[26] Raffael, Picasso, etc.) out of thousands, if not millions, account for an overwhelming share of the total value of all paintings; similarly for composers, of which only few remain known and played. This is implicitly confirmed by the comparatively short (but kept secret) grab lists even famous museums like the Louvre have in the case of an emergency due to fire.[27]

Yegorov et al. (2016) present a model that addresses the different opportunities artists have when they try to access a market. This depends on cultural specifics (like language for writers, taste for the kind of music for musicians and composers but also for paintings) that can affect an artist's career choices. This explains, inter alia, the skewed distribution of authors, since writing in English offers immediate access to a much larger market than, say, writing in Albanian.[28]

5.4. Other examples

As already mentioned, the returns to small businesses and even start-ups can be also highly skewed and this return to prominence seems to be increasing over time due to search engines like Google (the power law describes the distribution of visits to homepages in many fields) and the scalability of business models with modern information technologies. Another topical application is to self-employment in service sectors in the era of digitization (e.g., as an Amazon *MTurk*). As industrial employment shrinks due to robotization, many workers move to the service sector and may offer new as well as traditional types of services, like yoga, sports and fitness instruction, massage, even writing articles and theses. In such cases, skills can at best be observed imperfectly and demand grows via word-of-mouth and nowadays via social networks. More talent or better advertising and/or initial luck can lead to earnings above the normal return so that the market return consists of two components, (i) proportional to skills, βk, and (ii) the gain due to prominence and marketing. Let m measure those marketing skills compared to the average, then for a homogeneous distribution of skills (m) the returns will be disproportionately distributed, where the term mk^2 captures the induced revenues.

Engaging in shady or illegal businesses, e.g., drug dealing is another example that fits our crucial assumptions: it is self-financed (it is hard to get credit to finance one's career in this field) and the returns are highly skewed: those at the top have luxury apartments and sports cars while those selling the drugs in the street earn less than the minimum wage. That is, they stay at or close to what we call the Sisyphus point and this in spite of the implied risk in the hope to move up on the career ladder, see Levitt and Venkatesh (2000).[29]

The model and our simulations describe the returns to talents accounting for individual accumulation of human capital and the possibility of (very) high returns or none at all. Therefore, we use the above results to derive a few economic implications for arts and sports, and to some extent also for science.

1. There always exists an area of initial conditions, $k_0 \in [0, k_s]$, for which it is impossible to grow. This explains why most do not enter particular fields of arts, sports and science and why many of those who enter give up.
2. Our results suggest the need for scholarships in order to foster young talents, artists, sportsmen and scientists, before they can make a living from the market's returns. If the duration of the scholarship is short relative to the applicant's maturity, he or she gives up (actually, has to) and the talent is lost (over time). Hence, the share of lost talents depends on competition in a sector and on the availability of scholarships (public or private).[30]
3. Even if a talented individual accumulates human capital beyond the Sisyphus point, the following period is tough, because Fig. 5 and also Fig. 2 (for a different scenario) suggest that all returns have to be invested $(c = 0)$ at this stage and this for quite some time. The reason is that the return to one's human capital is very large in this domain (actually infinite at the Sisyphus point). This ascent is followed by a period of slow growth (the blue line) towards the steady state along which the agent can already enjoy the fruits of her work and talent. The story of Martin Eden (by Jack London) explains this phenomenon very well. Many of the impressionists did not become famous during their creative time – and of those many only posthumously. This explains also why in the past many artists and scientists came from wealthy and often noble families or depended on rich patrons or already famous individuals (e.g., Giotto on Cimabue) because of the need to self-finance education.

[26] The painting *Salvator Mundi* was sold for 450.3 Mio. dollars to Prince Badr bin Abdullah, setting a new record for the most expensive painting.

[27] See The Economist (2021).

[28] Nevertheless, Isaac Bashevis Singer received the Nobel prize for literature albeit writing in Yiddish. However, he lived in New York and was readily translated.

[29] Tragler et al. (2001) analyze the intertemporal trade-off between the social costs of illicit drug consumption and the expenditures for controlling the US-cocaine epidemics.

[30] This explains the dominance of the former Socialist economies in particular when the German Democratic Republic, GDR, beat the Federal Republic of Germany, FRG, in terms of medals at many Olympic games (with a population of less than a third).

4. The bifurcation diagram w.r.t. the depreciation-parameter δ in Fig. 4 implies: a very low depreciation allows for unbounded growth (as in the AK-model of Rebelo, 1991), while no other steady state than the boundary solution, $k \to 0$, exists for high depreciation rates. Depreciation depends not only on personal abilities but also on trends (in science) and en vogue fashion (in arts, recall the fate of good naturalistic painters during the 20th century) that can depreciate one's particular kind of human capital. While in some fields it is possible to use old knowledge, the need for particular skills changes very rapidly in others, for example, in computer science (people are forced to learn new versions of software every few years).[31] Therefore, working in a field characterized by less erosion of the usage of particular techniques renders a comparative advantage. However, this effect may be countered by excessive entry of young talents who prefer to enter a stable rather than a volatile field, just think of the many students flocking to literature and history of arts. The parameter β captures just the opposite of δ, in fact, only $|\beta - \delta|$ matters.
5. The bifurcation diagram in Fig. 3 w.r.t. the parameter m and the analyses of the corresponding optimal paths in Figs. 5 and 6 show: i) the Sisyphus point shrinks as m grows; ii) the location of the upper saddle grows, but only slowly, and so does its domain of attraction. However, the first effect is stronger. Indeed, given the optimality of the high steady state, the existence of the Sisyphus point ensures that $k \to 0$ is optimal for sufficiently low endowments, at least for $k_0 < k_s$. Therefore, agents characterized by a large value of m are able to survive even if starting at relatively low initial human capital (of course larger than k_s). A high value of m need not be due to human capital itself but could arise from exceptional ability to sell or market one's talent or to have access to networks and to large markets.
6. The parameter h accounts for non-monetary benefits. This explains why (some) artists continue to practice their profession even if that means fighting for survival due to the lack of sufficient proceeds from their work.[32]

6. Conclusions

We have formulated an intertemporal optimization problem about the career decisions of an individual talent (in arts, sports or science) accounting for the difficulty (more precisely, impossibility in our model) to get credit and the convex returns to human capital that capture the Matthew effect observed in different professions. The proposed model leads to multiple steady states and thus to thresholds due to a non-concave maximization problem. Only sufficiently high initial values of human capital allow for convergence to the high (saddle point) equilibrium. Therefore, whether one should pursue or stop such a profession depends on initial conditions. A novel feature is the appearance of a Sisyphus point, i.e., a point at which consumption is at the subsistence level ($c = 0$), because all proceeds must be invested in order to avoid the decline of human capital. It turns out that this point can indeed be optimal and can determine the location of the threshold. Furthermore, the no-debt constraint affects the outcome substantially: first, it eliminates otherwise feasible interior solutions; second, even if it is optimal staying in business, it determines investment. This finding is in line with Caucutt and Lochner (2020) who show, in an entirely different context, that (life-cycle) borrowing constraints severely limit investments into human capital.

The model differs from previous socio-economic studies of thresholds by the emergence of new and special properties at a point that we call a Sisyphus point. In particular, it can be optimal to stay at this point forever (but not necessarily) and it can serve as a threshold between attractors to leaving (convergence to the origin of the state space) or to attaining a profitable outcome in one's profession. Therefore, Sisyphus points provide a sharp differentiation about the career prospects and how they depend on initial human capital, e.g., in science after receiving a Ph.D. If the path to $k = 0$ is the optimal (or the only viable) outcome for many, then it describes the Matthew effect and explains why some never publish a paper and quit science. This phenomenon is also observable in the arts (with teaching as an exit option for painters and musicians), in sports (e.g., offering tennis or skiing lessons after exiting) and for small businesses. The reason for this separation of outcomes is as follows. The reward continuously grows at k^2. The linear term dominates the necessary investments for capital expansion (i.e., $u > u^{\min}$) at small levels of k while the quadratic term dominates for large values of k (see Fig. 1). An initial condition, $k_0 > k_s$, corresponds to the following situation, e.g., in research: a young researcher publishes a paper already prior to receiving a Ph.D., which helps in the post-graduate job market and thereby reinforces further growth (e.g., landing at a famous institution).[33] Those less lucky, can choose between staying at their respective Sisyphus point or quitting (human capital converges to zero). Indeed, many young scientists face this problem and even relatively established scientists can temporarily find themselves in a Sisyphus trap, and such outcomes are even more frequent in the arts.

The model presented in this paper can be applied to and extended into many directions. Our treatment of abnormality in an optimal control problem should be applicable to other problems lacking the property of normality. In terms of theory, a possible extension is for uncertainty (continuous as well as jumps) due to the presence of luck, in particular, in arts and also in science, but less so in sports; another one is to construct alternative formulations of both, the objective and the dynamic constraint. In terms of applications, the presented or related frameworks may provide insights into other fields and can lead to similar complex dynamic patterns including the possibility of Sisyphus points. In terms of economic policies, the

[31] One reason why LaTeX is so successful.
[32] Unlike rats irrational individuals may fail to abandon a sinking ship.
[33] However, things need not be that straightforward, because a recent article in The Economist (2019) argues that "if at first you do not succeed, try, try, try again".

existence of Sisyphus points can create socially unfavorable outcomes if too many talents cannot finance their investment into human capital due to credit constraints.

Declaration of Competing Interest

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors. The authors have no competing interests to declare.

Acknowledgments

The authors acknowledge TU Wien Bibliothek for financial support through its Open Access Funding Programme.

Appendix A.

The Sisyphus point and its properties

We start deriving the necessary optimality conditions for the two cases $c(k,u) > 0$ and $c(k,u) = 0$. Thus, we consider the Lagrangian[34]

$$L(k,u,\lambda,\nu,\lambda_0) = H(k,u,\lambda,\lambda_0) + \nu(mk^2 + \beta k - u), \tag{17}$$

where $H(k,u,\lambda,\lambda_0)$ is the Hamiltonian (10).

Let $(k^*(\cdot), u^*(\cdot))$ be an optimal solution of problem (1–3). Then there exists a real value λ_0, a piecewise continuously differentiable costate $\lambda(\cdot)$ and a piecewise continuous multiplier function $\nu(\cdot)$, satisfying the maximum condition

$$u^*(t) = \max_{c(k^*(t),u)\geq 0} H(k^*(t),u,\lambda(t),\lambda_0). \tag{18a}$$

The costate $\lambda(\cdot)$ follows the adjoint equation

$$\begin{aligned}\dot{\lambda}(t) &= r\lambda(t) - L_k(k^*(t),u^*(t),\lambda(t),\nu(t),\lambda_0)\\ &= (r+\delta)\lambda(t) - \lambda_0(2mk^*(t)+\beta+h) - \nu(t)(2mk^*(t)+\beta)\end{aligned} \tag{18b}$$

together with the transversality condition,

$$\lim_{t\to\infty} \mathrm{e}^{-rt}\lambda(t) = 0. \tag{18c}$$

The multiplier function $\nu(\cdot)$ satisfies for all t the complementary slackness condition, (18d)

$$\nu(t)c(k^*(t),u^*(t)) = 0 \tag{18e}$$

and the non-negativity condition,

$$\nu(t) \geq 0. \tag{18f}$$

Finally the non-zero condition has to be satisfied

$$(\lambda_0, \lambda(t)) \neq (0,0), \quad t \geq 0. \tag{18g}$$

If the necessary optimality conditions (18) are satisfied for $\lambda_0 = 1$ the optimal solution $(k^*(\cdot), u^*(\cdot))$ is called *normal*, otherwise if $\lambda_0 = 0$ the optimal solution is called *abnormal*.

Remark. We note that for every $k_0 \geq 0$ problem (1–3) exhibits an optimal solution, see, e.g., (Seierstad and Sydsæter, 1987, Theorem 10).

In most of the textbooks on optimal control theory a proof can be found that for finite time horizon problems with free end-states every optimal solution is normal, see, e.g., Takayama (1985) and Grass et al. (2008). For infinite time horizon problems with free end-states Halkin (1974) showed that optimal solutions may be abnormal. More recently

[34] For the sake of brevity we do not explicitly consider the constraint $u \geq 0$. Its derivation is straight forward.

Aseev and Veliov (2015) proved the normality of optimal solutions for a general class of infinite time horizon problems. In Grass et al. (2021) it has been shown for a simple economic application that abnormal solutions are not in general degenerate but can have meaningful interpretation.

The actual problem (1–3) is another example of a meaningful model that exhibits abnormal solutions. Subsequently we derive sufficient and necessary conditions that ensure that an optimal Sisyphus point k_s is (ab)normal.

We note that due to the uniqueness of the optimal control value, given by the maximum condition (18a), the optimal control $u^*(\cdot)$ can be chosen continuously. This implies the continuity of the multiplier function $\nu(\cdot)$ and the costate $\lambda(\cdot)$ being continuously differentiable except on possible switching times τ. At a switching time $\tau < \infty$ the costate is continuously differentiable from the left and from the right.

From the maximum condition (18a) analytic functional forms can be derived for the optimal control $u^*(\cdot)$ and multiplier function $\nu(\cdot)$ yielding

$$u^*(t) = \begin{cases} u^{\text{int}}(\lambda(t), \lambda_0) & c(k^*(t), u^*(t)) > 0 \\ u^{\max}(k^*(t)) & c(k^*(t), u^*(t)) = 0 \end{cases} \tag{19}$$

with

$$u^{\text{int}}(\lambda, \lambda_0) := \frac{\lambda - \lambda_0}{2a\lambda}, \tag{20}$$

$$u^{\max}(k) := mk^2 + \beta k. \tag{21}$$

For the multiplier function $\nu(\cdot)$ we find

$$\nu(t) = \begin{cases} 0 & \text{if } c(k^*(t), u^*(t)) > 0 \\ \nu^{\max}(k^*(t), \lambda(t), \lambda_0) & \text{if } c(k^*(t), u^*(t)) = 0 \end{cases} \tag{22}$$

with

$$\nu^{\max}(k, \lambda, \lambda_0) := -\lambda_0 + \lambda(1 - 2a(mk^2 + \beta k)). \tag{23}$$

Therefore the canonical system appears in two different specifications[35]

$$\dot{k}(t) = u^{\text{int}}(\lambda(t)) - au^{\text{int}}(\lambda(t))^2 - \delta k(t), \tag{24a}$$

$$\dot{\lambda}(t) = (r + \delta)\lambda(t) - \lambda_0(2mk(t) + \beta + h) \tag{24b}$$

and

$$\dot{k}(t) = u^{\max}(k(t)) - au^{\max}(k(t))^2 - \delta k(t), \tag{25a}$$

$$\dot{\lambda}(t) = \zeta(k(t), \lambda_0)\lambda(t) - h \tag{25b}$$

with

$$\zeta(k, \lambda_0) := r + \delta + (2ak(mk + \beta) - \lambda_0)(2km + \beta). \tag{25c}$$

Next we formulate conditions for the existence of a Sisyphus point k_s and an equivalent characterization to the definition of these points.

Proposition A1. *Let* $a, m, \delta > 0$ *and* Δ *be defined by*

$$\Delta := \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 \tag{26}$$

with

$$p := \frac{a\beta^2 - m}{am^2} - \frac{4}{3}\frac{\beta^2}{m^2}$$

[35] If $u^*(t, k_0) = 0$ Eq. (24b) is not altered. That is the reason why we do not analyze this constraint explicitly.

$$q := \frac{16}{27}\frac{\beta^3}{m^3} - \frac{2}{3}\frac{\beta(a\beta^2 - m)}{m^3 a} + \frac{\delta - \beta}{am^2}.$$

and let $f(k, u)$ be the right hand side of state equation (2), i.e.,

$$f(u, k) := u - au^2 - \delta k. \tag{27}$$

Then

- *a Sisyphus point k_s exists iff the following conditions are satisfied*

$$\Delta \le 0,\ \ \beta - \delta < 0,\ \text{ and } m > a\beta^2. \tag{28}$$

- *$k_s > 0$ is a Sisyphus point iff k_s satisfies*

$$f(k_s, u^{\max}(k_s)) = 0 \tag{29a}$$

and for all k with $0 < k < k_s$ and u satisfying $c(k, u) \ge 0$

$$f(k, u) < 0. \tag{29b}$$

Remark. Conditions (29a) and (29b) can be used as an alternative definition of a Sisyphus point. This definition is in accordance with the introduction of a Stalling Equilibrium in Feichtinger et al. (2021) and Grass et al. (2021). It states that zero is the only reachable state for $k_0 < k_s$.

For the sake of clarity of the used denotation for k_s as an optimal Stalling equilibrium we define:

Definition. Let k_s be a Sisyphus point. If the solution,

$$(k^*(\cdot), u^*(\cdot) \equiv (k_s, u^{\max}(k_s)),$$

is an (ab)normal optimal solution of problem (1–3) for $k(0) = k_s$, then k_s is called a(n) *(ab)normal optimal Sisyphus point*. Otherwise k_s is called an *inferior Sisyphus point*.

Proof. To prove the existence of k_s we note that $f(k_s, u^{\max}(k_s)) = 0$ yields the quartic equation

$$\beta k + mk^2 - a(\beta k + mk^2)^2 - \delta k = 0$$

which reduces to a cubic equation dividing by $k > 0$

$$p_s(k) := \beta + mk - ak(\beta + mk)^2 - \delta = 0 \tag{30a}$$

with derivative

$$p_s'(k) = 3am^2k^2 + 4a\beta mk - m + a\beta^2. \tag{30b}$$

Applying Cardano's formula to Eq. (30a) one finds the discriminant Δ given by Eq. (26a).
The cubic polynomial Eq. (30a) evaluated at 0 yields

$$p_s(0) = \beta - \delta. \tag{30c}$$

The solutions of the quadratic Eq. (30b) yielding the local minimum and maximum of $p_s(\cdot)$ are

$$k_{l,h} = \frac{-2a\beta \pm \sqrt{(a\beta)^2 + 3am}}{3am},\quad 0 < k_l < k_h. \tag{30d}$$

[36] Let the conditions (28) be satisfied. For every cubic equation a root $k_0 \in \mathbb{R}$ exists. The inequality $\beta - \delta < 0$ together with

$$\lim_{k\to\mp\infty} p_s(k) = \pm\infty$$

[36] For an english translation of Cardano's *Ars Magna* (1545) see Cardano (1993).

yields

$$k_0 < k_l < 0 \quad \text{and} \quad p_s(k)\begin{cases} > 0 & \text{if } k < k_0 \\ < 0 & \text{if } k_0 < k < 0. \end{cases}$$

For $\Delta < 0$ three real roots exist, and for $\Delta = 0$ two real roots exist. Let us start with the case of three real roots, $k_0,\ k_s$ and k_1. These values satisfy the inequality

$$k_0 < k_l < k_s < k_h < k_1.$$

Inequality $m > a\beta^2$ guarantees $k_h > 0$ and therefore the latter inequality specifically becomes

$k_0 < k_l < 0 < k_s < k_h < k_1.\Delta = 0$ implies $k_s = k_h = k_1$ and hence

$$k_0 < k_l < 0 < k_s = k_h = k_1.$$

Thus, if conditions (28) are satisfied a root $k_s > 0$ of $p_s(\cdot)$ exists and k_s satisfies the definition of a Sisyphus point. This proves that these conditions are sufficient for the existence of a Sisyphus point.

Let k_s be a Sisyphus point and let $\Delta > 0$, then a single root k_0 exists. We either have $k_0 < k_l < 0$ or $k_h < k_0$. In the latter case $p_s(k) > 0$ for $k < k_0$ yielding $f(k, u^{\max}(k)) > 0$. In both cases k_0 is not a Sisyphus point.

Let k_s be a Sisyphus point and let $\beta - \delta \geq 0$. We distinguish the cases $p_s(k_l) \leq 0$ and $p_s(k_l) > 0$. The first case yields the existence of two, $k_0 = k_1$ and k_2, or three real roots, k_0, k_1 and k_2, satisfying $k_0 = k_1 < 0 < k_2$ or $k_0 < k_1 < 0 < k_2$. None of these roots satisfy the conditions of a Sisyphus point. In the latter case a single root k_0 exists and we are in the situation of $\Delta > 0$.

Let k_s be a Sisyphus point and let $m \leq a\beta$. Then $k_h \leq 0$ and we are in one of the previous situations. Summing up, if one of the conditions (28) is violated no Sisyphus point exists, proving that these conditions are necessary.

Next we prove the second statement (29). Condition (29a) immediately follows from the definition of a Sisyphus point. For condition (29b) we note that k_s satisfies

$$u^{\max}(k_s) = u^{\min}(k_s),$$

where $u^{\min}(k)$ is the smaller root of the quadratic equation $f(k,u)]textwithrespecttou$, i.e.

$$u_{1,2}(k) = \frac{1 \mp \sqrt{1 - 4a\delta k}}{2a}$$

and

$$u^{\min}(k) := u_1(k).$$

For $0 < k < \dfrac{1}{4\delta}$ this yields

$$f(k,u) = \begin{cases} < 0 & \text{if } 0 < u < u_1(k) \\ > 0 & \text{if } u_1(k) < u < u_2(k) \\ < 0 & \text{if } u > u_2(k). \end{cases}$$

Considering the cubic polynomial $p_s(k)$ and the conditions for the existence of $k_s > 0$ we find that $u^{\max}(k) < u^{\min}(k)$ holds for all $0 < k < k_s$. Therefore $f(k,u) < 0$ for $0 < k < k_s$ and $0 \leq u \leq u^{\max}(k)$. This finishes the proof.■ □

We are now dealing with the analysis of the equilibrium (k_s, λ_s) in the state-costate space, where k_s is a Sisyphus point and

$$\lambda_s := \frac{h}{r + \delta - (\beta + 2k_s m)(1 - 2ak_s(\beta + mk_s))}, \tag{31}$$

the root of Eq. (25b).

In the following proposition we give a full classification of the equilibrium (k_s, λ_s).

Proposition A2. *Let $(k_s, \lambda_s) \in \mathbb{R}^2$ be an equilibrium of the canonical system Eqs. (25a)–(25b), $\lambda_0 = 1$ and $h > 0$. Then, the eigenvalues $\xi^s_{1,2}$ of the Jacobian at (k_s, λ_s) are*

$$\xi^s_1 := (\beta + 2k_s m)(1 - 2ak_s(\beta + mk_s)) - \delta > 0 \tag{32}$$

and

$$\xi^s_2 := r + \delta - (\beta + 2k_s m)(1 - 2ak_s(\beta + mk_s)) \lesseqqgtr 0 \tag{33}$$

and the eigenvector related to ξ^s_2 is

$$v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The equilibrium (k_s, λ_s) *is a saddle iff the costate value* λ_s *satisfies*

$$\lambda_s < 0, \tag{34}$$

and the according Lagrangian multiplier becomes

$$\nu^{\max}(k_s, \lambda_s, 1) < 0. \tag{35}$$

For $\xi_2^s = 0$ *the costate value is unbounded,* $\lambda_s = \infty$*.*

Proof. The Jacobian J_s at (k_s, λ_s) of the canonical system Eqs. (25a)–(25b) is of the form

$$J_s = \begin{pmatrix} r - \zeta(k_s) & 0 \\ \Omega(k_s, \lambda_s) & \zeta(k_s) \end{pmatrix}$$

with $\zeta(k)$ defined in Eq. (25c).

Due to the diagonal structure of the Jacobian J_s the eigenvalues $\xi_{1,2}^s$ and corresponding eigenvectors $\nu_{1,2}$ are

$$\begin{aligned} &\xi_1^s = r - \zeta(k_s), \quad \xi_2^s = \zeta(k_s), \\ &\nu_1 = \begin{cases} \begin{pmatrix} \dfrac{r}{\Omega(k_s, \lambda_s)} \\ 1 \end{pmatrix} & \text{for} \quad \Omega(k_s, \lambda_s) \neq 0, \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{for} \quad \Omega(k_s, \lambda_s) = 0, \end{cases} \\ &\nu_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned} \tag{36}$$

First we show that $r - \zeta(k_s) > 0$. Assume to the contrary that $r - \zeta(k_s) \le 0$, then $\zeta(k_s) \ge r > 0$ and (k_s, λ_s) is a saddle. Since the k-component of the according eigenvector, see Eq. (36), is non-zero there exists a stable path for some $k_0 < k_s$ and $\tilde{\lambda}$, i.e.

$$\lim_{t\to\infty}(k(t), \lambda(t)) = (k_s, \lambda_s) \text{ with } k(0) = k_0, \text{ and } \lambda(0) = \tilde{\lambda}.$$

This contradicts property (29b), stating that $f(k_0, u) < 0$ for all $k_0 < k_s$ and admissible control values u. Thus, we find $r - \zeta(k_s) > 0$ implying, (k_s, λ_s) is a saddle iff $\zeta(k_s) < 0$. Using definition (31) and $h > 0$ we find

$$\lambda_s = \frac{h}{r + \delta - (\beta + 2k_s m)(1 - 2ak_s(\beta + mk_s))} = \frac{h}{\zeta(k_s)},$$

proving statement (34).

Using the definition of the Sisyphus point k_s we find

$$u^{\max}(k_s) = u^{\min}(k_s) < \frac{1}{2a}$$

and hence

$$1 - 2a(mk_s^2 + \beta k_s) > 0.$$

This together with the expression of the equilibrium costate λ_s and the expression for the Lagrangian multiplier

$$\nu^{\max}(k_s, \lambda_s, 1) = -1 + \lambda_s(1 - 2a(mk_s^2 + \beta k_s)).$$

yields $\nu^{\max}(k_s, \lambda_s, 1) < 0$ if $\lambda_s < 0$.

For $\zeta(k_s) = 0$ the denominator of λ_s is zero and hence λ_s becomes infinite. This finishes the proof. ■ □

Remark. As a consequence of Prop. (A2) we find that for $h > 0$ and $\zeta(k_s) \le 0$ the equilibrium (k_s, λ_s) is not an admissible equilibrium of the canonical system.

Finally we derive necessary and sufficient conditions for the (ab)normal optimality of a Sisyphus point k_s.

Proposition A3. *Let* k_s *be a Sisyphus point then*

- *For* $h = 0$ *and* $\delta > 0$ *the Sisyphus point* k_s *is never optimal.*
- *Let conditions (28) be satisfied,* $k_0 < k_s$, $\lambda(\cdot, k_0)$ *be the costate corresponding to the optimal solution* $(k^*(\cdot, k_0), u^*(\cdot, k_0))$*. Then* k_s *is an optimal Sisyphus point iff one of the following conditions is satisfied*
  - $c(k^*(0, k_0), u^*(0, k_0)) > 0$ *for* $k_0 < k_s$ *and*

$$\lim_{k\to k_s^-} c(k^*(0, k_0), u^*(0, k_0)) = 0 \tag{37}$$

  k_s is a normal Sisyphus point and (k_s, λ_s) an unstable node.

- *$c(k^*(0,k_0),u^*(0,k_0))=0$ for some $\varepsilon>0$ and $k_s-\varepsilon<k_0<k_s$ and*

$$\lim_{k_0\to k_s^-}\lambda(0,k_0)=\lambda_s<\infty \tag{38}$$

k_s is a normal Sisyphus point and (k_s,λ_s) an unstable node.

- *$c(k^*(0,k_0),u^*(0,k_0))=0$ for some $\varepsilon>0$ and $k_s-\varepsilon<k_0<k_s$ and*

$$\lim_{k_0\to k_s^-}\lambda(0,k_0)=\infty \tag{39}$$

k_s is an abnormal Sisyphus point and (k_s,λ_s) a saddle point or $\zeta(k_s)=0$.

Remark. In the following proof we make use of the following properties of the optimal solution $(k^*(\cdot,k_0),u(\cdot,k_0))$ for $k^*(0,k_0)=k_0\le k_s$:

- For $k_0\le k_s$ the optimal solution is unique.
- For every $T>0$ the optimal solution $(k^*(\cdot,k_0),u(\cdot,k_0))$ converges uniformly to $(k^*(\cdot,k_s),u(\cdot,k_s))$ on $[0,T]$ and for $k_0\to k_s$. Specifically, we find

$$\lim_{k_0\to k_s}u^*(0,k_0)=u^*(0,k_s)$$

and

$$\lim_{k_0\to k_s}\lambda(0,k_0)=\lambda(0,k_s).$$

- The costate $\lambda(\cdot,k_0)$ is strictly decreasing for $k_0<k_s$.
- The solution of the adjoint Eq. (25c) yields[37]

$$\lambda(0,k_0)=\lim_{t\to\infty}h\int_0^t e^{-\int_0^s\zeta(k^*(l,k_0))\,dl}\,ds. \tag{40}$$

Proof. Let $h=0$ and $(k_s,u^{\max}(k_s))$ be the equilibrium solution. Then the objective value is

$$\int_0^\infty e^{-rt}(mk_s^2+\beta k_s-u^{\max}(k_s))\,dt=0,$$

which is always dominated by $(k(\cdot),0)$ with $k(0)=k_s$ yielding a positive objective value.

To prove the second statement we note that due to the properties formulated in the previous remark we find for the case (37)

$$\lim_{k_0\to k_s^-}u^*(0,k_0)=\lim_{k_0\to k_s^-}u^{\text{int}}(\lambda(0,k_0))=u^{\text{int}}(\lambda(0,k_s)),$$

satisfying

$$\lim_{k_0\to k_s^-}c(k^*(0,k_0),u^{\text{int}}(\lambda(0,k_0))=0$$

and therefore

$$u^{\text{int}}(\lambda(0,k_s))=u^{\max}(k_s).$$

In the other cases (38) and (39) $u^*(0,k_0)=u^{\max}(k_0)$ already holds for some $\varepsilon>0$ and all $k_s-\varepsilon<k_0<k_s$. This implies that $(\tilde{k}(\cdot),\tilde{u}(\cdot))\equiv(k_s,u^{\max}(k_s))$ is an optimal solution and hence k_s an optimal Sisyphus point.

To prove that in the case (37) (k_s,λ_s) is an unstable node, we use

$$\hat{\lambda}:=\lim_{k_0\to k_s^-}\lambda(0,k_0)<\infty,$$

satisfying

$$u^{\text{int}}(\hat{\lambda})=\frac{\hat{\lambda}-1}{2a\hat{\lambda}}=mk_s^2+\beta k_s,$$

yielding

$$\nu^{\max}(k_s,\hat{\lambda},1)=0.$$

Since $\lambda(\cdot,k_0)$ is strictly decreasing we find for$\lambda(\cdot,k_s)$

$$\lim_{k_0\to k_s^-}\dot{\lambda}(0,k_0)=\dot{\lambda}(0,k_s)\le 0.$$

[37] This is only true if $c((k^*(\cdot,k_0),u(\cdot,k_0))=0$ for all $t\ge 0$. Since we are interested in the limit behavior $k_0\to k_s$ this formula holds at least in the limit.

Since $\dot{\lambda}(0, k_s) < 0$ contradicts the result that the equilibrium solution $(k_s, u^{\max}(k_s))$ is the unique optimal solution at k_s this yields $\dot{\lambda}(0, k_s) = 0$ and hence $(k_s, \hat{\lambda})$ is an admissible equilibrium, necessarily satisfying $\hat{\lambda} = \lambda_s < \infty$. Thus, the normal necessary optimality conditions are satisfied. Using Prop. A2 this implies that (k_s, λ_s) is an unstable node.

For the cases (38) and (39) we note that due to the uniform convergence and representation (40) we find

$$\lim_{k_0 \to k_s^-} \lambda(0, k_0) = \lim_{k_0 \to k_s^-} \lim_{t \to \infty} h \int_0^t e^{-\int_0^s \zeta(k^*(l, k_0))\, dl}\, ds$$

$$= \lim_{t \to \infty} \lim_{k_0 \to k_s^-} h \int_0^t e^{-\int_0^s \zeta(k^*(l, k_0))\, dl}\, ds$$

$$= \lim_{t \to \infty} h \int_0^t e^{-\int_0^s \zeta(k_s)\, dl}\, ds.$$

Therefore $\hat{\lambda} := \lim_{k_0 \to k_s^-} \lambda(0, k_s) < \infty$ implies $\zeta(k_s) > 0$ and $(k_s, \hat{\lambda}) = (k_s, \lambda_s)$ being an unstable node. To prove that this equilibrium is admissible we note that for all $k_0 < k_s$

$$\nu^{\max}(k_0, \lambda(0, k_0), 1) \geq 0.$$

This yields

$$\lim_{k_0 \to k_s^-} \nu^{\max}(k_0, \lambda(0, k_0), 1) = \nu^{\max}(k_s, \hat{\lambda}, 1) \geq 0.$$

In the third case $\lim_{k_0 \to k_s^-} \lambda(0, k_s) = \infty$ implies $\zeta(k_s) \leq 0$. Using Prop. A2 we find that (k_s, λ_s) is a saddle for $\zeta(k_s) < 0$ or does not exist for $\zeta(k_s) = 0$.

In this latter case the necessary optimality conditions are not satisfied for $\lambda_0 = 1$. Setting $\lambda_0 = 0$ the adjoint equation becomes

$$\dot{\lambda}(t) = (r + \delta + (2ak(t)(mk(t) + \beta)(2mk(t) + \beta)))\lambda(t) - h$$

and

$$\nu^{\max}(k, \lambda, 0) = \lambda(1 - 2a(mk^2 + \beta k).$$

Therefore with

$$\lambda_s^0 := \frac{h}{r + \delta + (2ak_s(mk_s + \beta))(2mk_s + \beta)} > 0,$$

(k_s, λ_s^0) is an unstable node of the abnormal canonical system satisfying

$$\nu^{\max}(k_s, \lambda_s^0, 0) > 0$$

and hence satisfies the abnormal necessary optimality conditions.

These cases cover all possibilities considering the solution structure of any possible optimal solution for $k_0 < k_s$.■ □

Remark. Noting that $\lambda(0, k_0) = \frac{d}{dk_0} V^*(k_0)$ the conditions in Prop. A3 read as

$$\lim_{k_0 \to k_s^-} \frac{d}{dk_0} V^*(k_0) < \infty$$

in the normal case and

$$\lim_{k_0 \to k_s^-} \frac{d}{dk_0} V^*(k_0) = \infty,$$

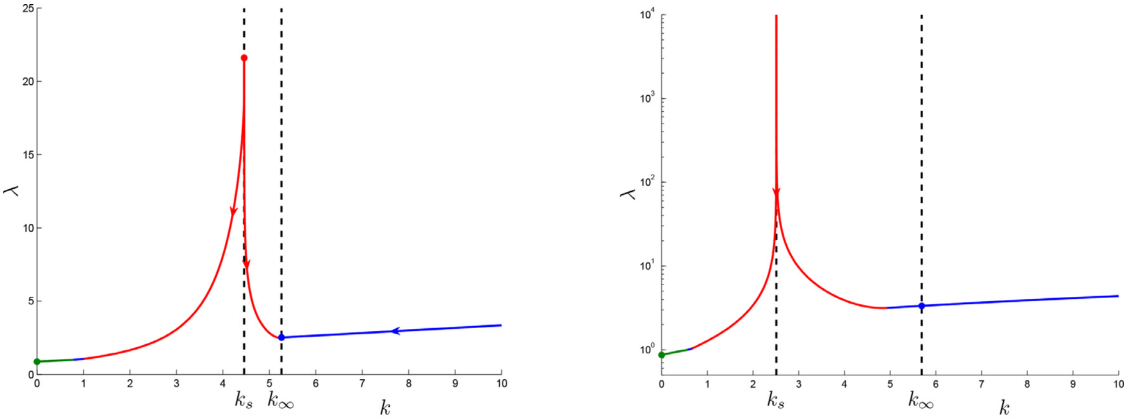
where

$$V^*(k_0) := \max_{u(\cdot)} V(k_0, u(\cdot)),$$

$$0 \leq u(t) \leq mk(t)^2 + \beta k(t), \quad t \geq 0.$$

in the abnormal case.

In our numerical examples presented in Section 5.1 there is a small region $m \in (0.0352\ldots, 0.0353\ldots)$, where (k_s, λ_s) is an unstable node and property (38) applies, see Fig. A.1a. For parameter values $m > 0.0353\ldots$ the equilibrium (k_s, λ_s) is a saddle and property (39) applies, see Fig. A.1b and for an overall presentation see the bifurcation diagram in (Fig. A.2).



(a) $m = 0.036$ (b) $m = 0.05$

Fig. A.1. In panel (a) the equilibrium (k_s, λ_s) is an unstable node and the problem for $k(0) = k_s$ is normal. In panel (b) the equilibrium (k_s, λ_s) is a saddle and the problem for $k(0) = k_s$ is abnormal, see Prop. A3. The parameter values are taken from (15).

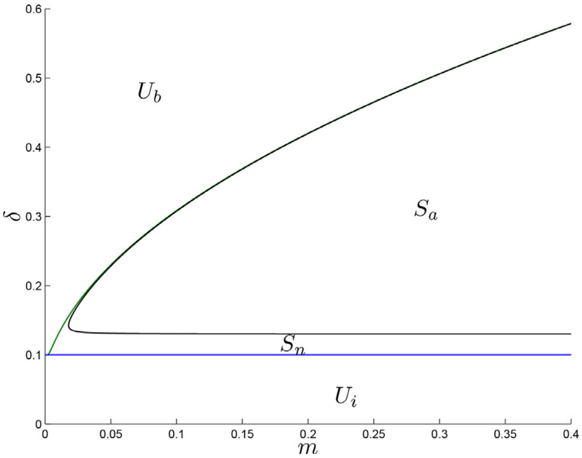


Fig. A.2. This figure shows the different solution regions for the base case and (m, δ)-space. In region S_a the equilibrium solution at the Sisyphus point k_s is abnormal, whereas in S_n the solution is normal. In these regions the state space is separated into $[0, k_s)$ and (k_s, ∞). Solutions starting below the Sisyphus point converge to zero and solutions starting above the Sisyphus point converge to a state $\tilde{k} > k_s$. Solutions starting at the Sisyphus point stay put. In region U_i the optimal solution converges to a state $\tilde{k} > 0$ and solutions lying in region U_b it is optimal to converge to zero. The green line is the limit point curve of the Sisyphus point, at the black cure $\zeta(k_s) = 0$ and at the blue curve the Sisyphus point becomes zero. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

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