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YSSP Report

Young Scientist Summer Program

How to prepare for and adapt to a climate tipping point

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ABSTRACT

Climate tipping points are abrupt regime shifts in the Earth's climate systems that are mainly driven by the anthropogenic increase in global mean temperature. Given the concrete possibility that increasingly frequent and severe climate-related catastrophes will negatively impact the world economy, how should we model our policies to optimally prepare for and adapt to such events?

We formulate a continuous-time version of the DICE model by W. Nordhaus, and include a climate tipping point as a random instant whose hazard rate increases with the global mean temperature. The resulting model is a two-stage optimal control problem with a random switching time, that we then approach with the Maximum Principle for heterogeneous systems developed by V. M. Veliov (see Wrzaczek et al. [2020], Veliov [2008]).

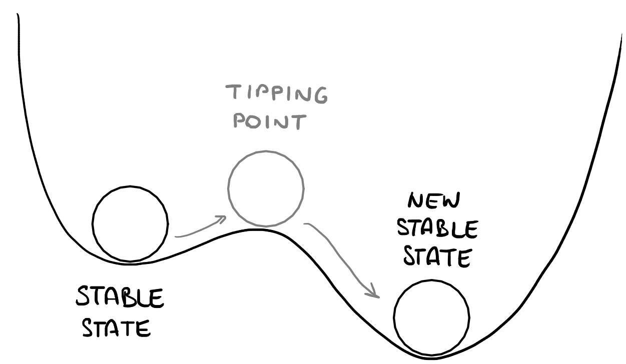
Among our results, we find analytical conditions for the optimal savings and emission abatement policy, and a decomposition of the growth rate of the social cost of carbon. Numerical simulations are provided for different tipping point scenarios.

Contents

1 Introduction

1.1 Tipping points

Consider a dynamical system in a stable state. By definition, a *tipping point* occurs when, by the influence of an external agent possibly triggering feedback mechanisms, the system shifts and crosses a threshold beyond which it is drawn away from its original state and into a different one.



Although the concept of tipping point can apply to any type of system shifting from one state to another as described, the scientific community is increasingly interested in *climate* tipping points, i.e., tipping points that occur in the Earth's climate system. This interest towards climate tipping points is fueled by the tangible possibility that increasingly frequent and severe climate-related catastrophes will negatively impact global economy in the best case scenario, or even lead to human extinction in the worst. To assess resilience – i.e., the ability to recover from a shock in the long run – in any socio-economic and/or climate model, it is therefore key to take into consideration the occurrence of one, or more, tipping points.

The main driver behind climate tipping points is the anthropogenic increase in global mean temperature triggered by GHGs emissions and, according to recent IPCC Special Reports (2018, 2019), their thresholds could lie between 1 and 2°C of global warming above pre-industrial levels, instead of 5°C as it was previously thought (Lenton et al. [2019]). It is therefore plausible that multiple tipping elements will occur within this century under anthropogenic climate change (Lenton et al. [2008]).

To give some concrete examples, the most critical climate tipping points we are likely to face are, according to Lenton et al. [2019], the melting of the Arctic sea ice and of the Greenland and Antarctic ice sheets, the slowdown of the Atlantic circulation, and several transformations of carbon sinks into carbon sources (e.g., destruction of the Amazon rainforest and of the Boreal forest, die-off of coral reefs, thawing of the Permafrost).

Addressing the matter only in the attempt to prevent the tipping point would be too demanding for the economy, and there is a strong possibility that actual tentative implementations of preventive policies may come as too little, too late. This is why we also need to consider building resilience to tipping points into the global system: how can we (optimally) change our policies and behaviors in preparation for and in adaptation to the occurrence of a climate tipping point?

We implement the problem in the Optimal Control environment: we suppose that one single policymaker controls the world's emission abatement and economic investments – which in turn influence the evolution of the Earth's climate system and of the population's goods and services – in order to maximize a social welfare function. The occurrence of a climate-related tipping point is modeled as a random instant whose hazard rate increases with global mean temperature, after which the system's dynamics are irreversibly changed. We then apply the Maximum Principle for heterogeneous systems to derive necessary conditions for the optimal savings and abatement policies.

The main advantage of treating this problem with Optimal Control Theory is being able to learn about the interplay of the different variables, actions, and factors, thus gaining more in-depth knowledge behind what constitutes an optimal solution. This precise disentanglement of the various effects, which cannot be achieved through other optimization techniques (e.g., classic nonlinear optimization), is important in order to correctly implement decisions and adaptations of decisions.

In our fragmented world, different countries have different priorities when it comes to climate and economy, especially considering the issue of climate justice: poor countries suffer the most from the climate change that has been mainly caused by past emissions of the rich countries. Because of this disparity, modeling one single policymaker may be unrealistic, but it results in the best possible *aggregated* behavior. Indeed, in decentralized models, this 'single planner' solution acts as a benchmark to tend to by setting the correct carbon taxes.

An objective limitation of our current work is the inability to introduce more than one tipping point into the system: in fact, due to the strong interconnections between different tipping points in the form of feedback mechanisms and domino effects, crossing even only one of them is likely to trigger a tipping point *cascade*. This phenomenon will be investigated in future research, as an application of a new theoretical framework to study multi-stage OC problems with a sequence of random switching times.

1.2 Research question

Suppose that the policymaker is aware of the risk of a tipping point, that they know how this risk grows with global temperature, and what effect it will have (it could be either on the Earth's climate system, or on economic production, or both).

Given this context, the main objectives of this study are the following. First: we aim to derive the optimal emission abatement policy that should be implemented under the uncertainty of a pending climate catastrophe. Such a policy is expressed as the portion of reduced emissions as a function of time, and it can be adjusted upon occurrence of the tipping point. Second: we aim to analyze the evolution of Social Cost of Carbon over time and the different factors that contribute to its composition.

We will show how the expectation of different effects produces different behaviors in anticipation of the event, by considering four different types of tipping point: A) the melting of the Arctic sea ice, which compromises the self-cooling ability of the atmosphere, B) the destruction of capital due to an environmental catastrophe, which instantaneously decreases the available capital without bringing any long-term changes, C) the transformation of a carbon sink into a carbon source, which alters the rates of carbon uptake and release between the atmosphere and the biosphere/shallow ocean, and D) an increased impact of temperature on production, where the rate of economic production becomes less efficient.

1.3 Literature review

A few articles feature tipping points in theoretical dynamic approaches, and they can be divided into two main streams of literature: continuous-time consumption/pollution models and discrete-time integrated-assessment models based on DICE.

Among the stream of continuous-time consumption/pollution models, Cropper [1976], Clarke and Reed [1994], Gjerde et al. [1999] feature an unforeseen and sudden drop in society's level of consumption due to the environment's vulnerability to pollution-related catastrophic collapse. The drop in consumption may be complete or partial, reversible or irreversible. Tsur and Zemel [1996, 1998, 2016] feature an unpredictable and detrimental event, associated with the greenhouse effect, that inflicts damage in the form of an instantaneous penalty upon occurrence. In particular, Tsur and Zemel [1998] also features irreversible events, i.e., those which result in a truncation of the time horizon, and multiple-occurrence reversible events. In Cropper [1976], Tsur and Zemel [1996] the uncertainty concerning the time of occurrence refers to our ignorance about the exact pollution level required to trigger the event, whereas in Clarke and Reed [1994], Gjerde et al. [1999], Tsur and Zemel [1998, 2016] it stems from the intrinsic stochastic nature of the environmental processes that control occurrence. In the latter case, the hazard rate may increase with the GHG stock (Clarke and Reed [1994], Tsur and Zemel [1998, 2016]) or with global mean temperature change (Gjerde et al. [1999]). Differently from all of the above, van der Ploeg and de Zeeuw [2018] features a productivity shock, whose hazard rate increases with the pollution stock. It may consist in a sudden drop in technological stock or in the production's response to temperature. Also recoverable catastrophes are considered, like a sudden rise in the carbon stock or a partial destruction of capital.

Among the stream of discrete-time integrated-assessment models based on DICE, Keller et al. [2004] features a DICE model with the addition of central uncertainties and stochastic damages, caused by an uncertain environmental threshold imposed by an ocean circulation change, giving rise to a probabilistic optimization problem. Cai et al. [2013] features an extension of the DICE model by Nordhaus featuring a stochastic economy and with the addition of uncertain economic impact of possible (multiple) climate tipping events. The latter enters the climate damage function as a discrete Markov chain, starting from zero in the pre-tipping stage, with nondecreasing fractional values over time. Lemoine and Traeger [2014] includes a climate tipping point in Nordhaus's DICE model, which is then solved in a dynamic programming setting where the policymaker learns about a threshold's location by observing the system's response in each period. Lontzek et al. [2015] features a stochastic IAM based on DICE where the likelihood of tipping points increases with global warming, but it is uncertain.

Our work aims to bridge the existing gap between these two streams by adapting a continuous-time formulation of DICE to include a tipping point.

2 Methods and Discussion

We implement a finite-horizon, two-stage Optimal Control problem with a random switching time: we suppose that one single policymaker controls the world's emission abatement and economic investments in order to maximize a social welfare function. These control variables enter the dynamical system that governs the evolution of the state variables describing the Earth's climate – such as temperature and carbon stock – and the population's goods and services.

The occurrence of a climate-related tipping point is modeled as a random instant whose hazard rate increases with global mean temperature. Before the switch, the system is a continuous-time version of the DICE model by W. Nordhaus; after the switch, the system enters a new regime, that is characterized – depending on the nature of the tipping point – either by different dynamics or running payoff, or by a jump discontinuity in the state variables.

We then apply the Maximum Principle for heterogeneous systems developed by V. M. Veliov (see Veliov [2008], Wrzaczek et al. [2020]) and derive necessary conditions for the optimal savings and abatement policies.

A feature of the Maximum Principle is that it involves the computation of the co-state trajectories: each one of these functions represents the evolution of the *shadow-value* of the corresponding state variable over time, i.e., the marginal increase in the optimal payoff for an instantaneous increase in the state variable. This plays a crucial role in the analysis of the Social Cost of Carbon: due to its relation with the shadow-value functions, we are able to isolate the contributions of the elementary factors that add up to compose the Social Cost of Carbon.

2.1 Continuous-time DICE model

Before the occurrence of the tipping point, the model is a continuous-time version of the DICE that is described in Nordhaus [2014]. We use the continuous-time version of the functional forms and parameters in Freiberger et al. [2022], that were extracted from the official GAMS code DICE-2016R.

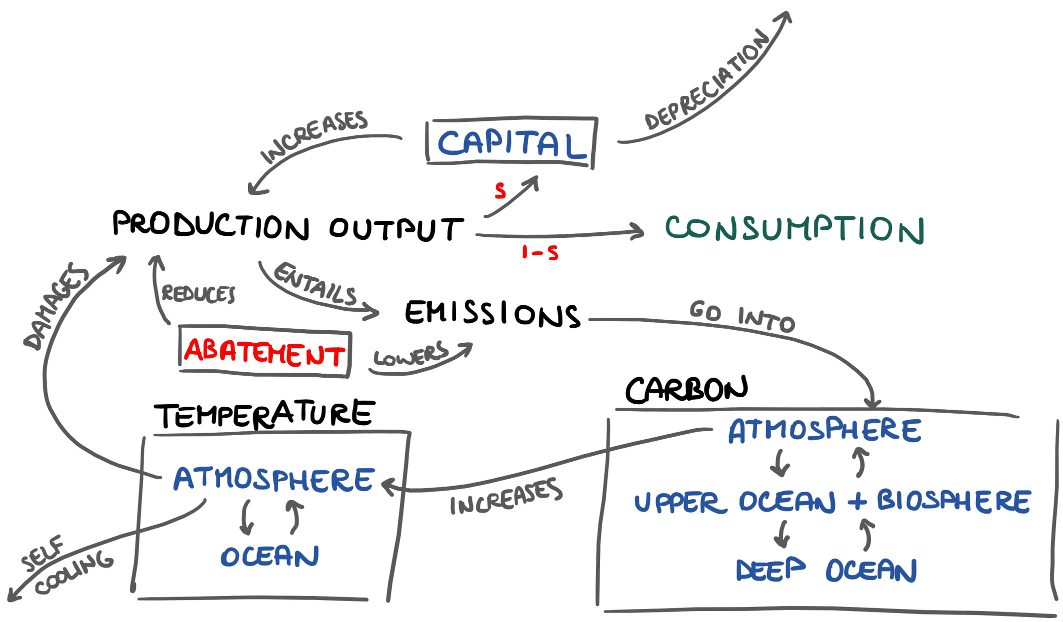
Control and State Variables

At every time, the state of the system is described by the following state variables: the *capital* stock K; the *carbon* stock vector $M = (M^{AT}, M^{UP}, M^{LO})$ has three components which measure the amount of carbon that is accumulated in the atmosphere, in the biosphere and shallow ocean, and in the deep ocean, respectively; the *temperature* vector $T = (T^{AT}, T^{OC})$ has two components which measure the global mean temperature change in the atmosphere and in the deep ocean, respectively.

There is a single policymaker who, at every time, sets the values the following control variables: the *savings* rate $s \in [0, 1]$ represents the fraction of the net economic output that is re-invested in capital; the *abatement* rate $\mu \in [0, 1]$ represents the fraction of industrial carbon emissions that is cut.

Controls:		**States:**	
$s \in [0,1]$	savings	K	capital
$\mu \in [0,1]$	abatement	M^{AT}	carbon (atmosphere)
		M^{UP}	carbon (shallow ocean + biosphere)
		M^{LO}	carbon (deep ocean)
		T^{AT}	temperature change (atmosphere)
		T^{OC}	temperature change (ocean)

The controlled dynamical system that describes the state variables' evolution, which is quite complex and involved, is presented in detail in the following pages. However, here is a graphic (and qualitative) representation of the mechanisms behind the mathematical expression of the dynamics.



Objective Function

The policymaker adjusts their policy (the savings rate s and the abatement rate μ) in order to optimize a social welfare function. This function is the flow over time of the "generalized" consumption, i.e., not only of traditional market goods and services, like food and shelter, but also of nonmarket items such as leisure, health status, and environmental services. This is obtained by maximizing the discounted integral of the population-weighted utility of per-capita consumption:

$$\underset{s,\mu\in[0,1]}{\text{maximize}} \int_0^T e^{-\rho t}\, L(t)\, u\big(c(t)\big)\, dt$$

where $\rho > 0$ is the rate of time preference (a discount is applied on the economic well-being of future generations), $L(t)$ is the exogenous population/labor, $u(c)$ is the utility of consumption, and $c(t)$ is the per-capita consumption $C(t)/L(t)$. The total consumption $C(t) = \big(1 - s(t)\big)Q(t)$ amounts to the fraction of the net production output $Q(t)$ that is not reinvested in capital, and as such it depends on the control and state variables (as explained in the next paragraph). The utility of consumption is isoelastic:

$$u(c) = k\,\frac{c^{1-\upsilon}-1}{1-\upsilon} \quad \text{with } \upsilon \in (0,1) \quad \text{or} \quad u(c) = k\log(c)$$

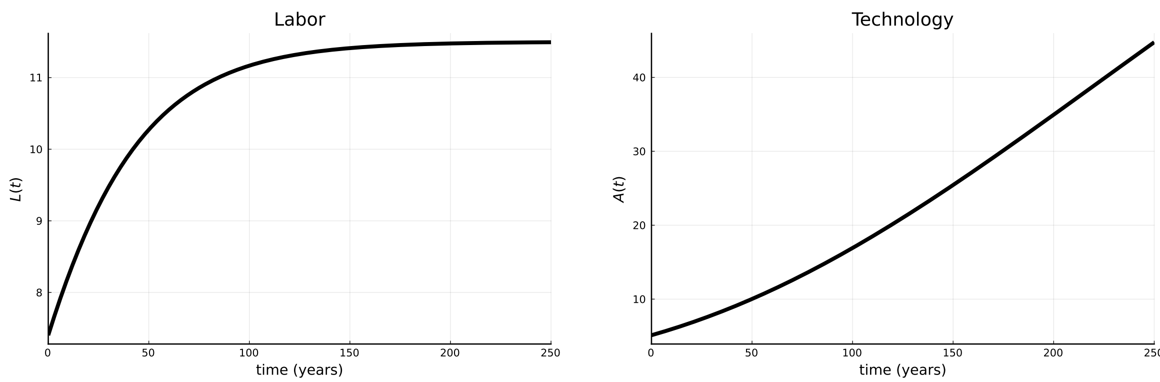
Economic Variables

The exogenous function $L(t)$ counts the world population in billions. Its growth rate declines over time so that the total population approaches a limit of 11.5 billion.

The global production function $Y(t,K)$ is assumed to be a constant-returns-to-scale Cobb-Douglas production function in capital K, exogenous labor $L(t)$, and Hicks-neutral exogenous technological change $A(t)$ (a coefficient whose growth represents technological advancement):

$$Y(t,K) = A(t)K^{\gamma}L(t)^{1-\gamma}$$

where $\gamma \in (0,1)$ is the capital elasticity.



The rising atmospheric temperature entails damages to production. Such damages include estimated damages to major sectors (such as agriculture), the cost of sea-level rise, adverse impacts on health, and nonmarket damages. They are assumed to be proportional to world output and are polynomial functions of global mean temperature change. Taking the temperature damage into consideration, production is thus resized by a factor $(1+\Omega(T^{AT}))^{-1}$ where

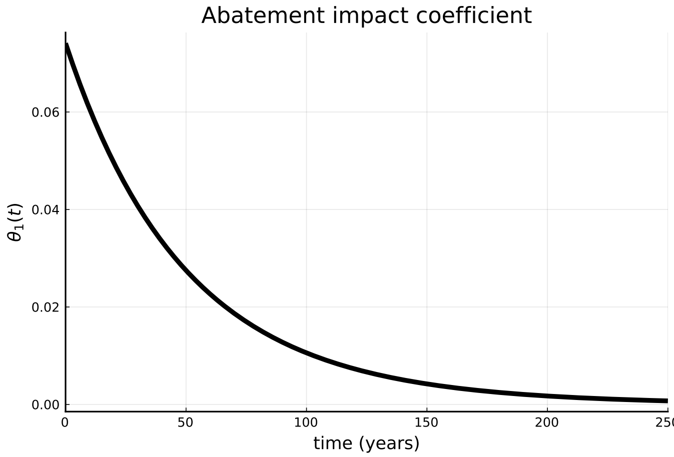
$$\Omega(T^{AT}) = \psi_1(T^{AT})^{\psi_2},$$

with ψ_1, ψ_2 parameters.

Abatement of carbon emissions also constitutes costs in terms of production. Abatement costs are assumed to be proportional to global output and to a strictly convex polynomial function of μ. A backstop technology, i.e., an all-purpose environmentally benign zero-carbon energy technology that can replace all fossil fuels, is introduced into the model by setting the exogenously time-dependent coefficient in the abatement-cost to be equal to the backstop price for each year. Such a price is assumed to be initially high and to decline over time with carbon-saving technological change. Considering abatement costs, production is again resized by a factor $(1-\Lambda(t,\mu))$ with

$$\Lambda(t,\mu) = \theta_1(t)\mu^{\theta_2},$$

where the coefficient $\theta_1(t)$ is the price of the backstop technology, in units of capital, and $\theta_2 > 1$ is a parameter.



After applying temperature-related damages and emission abatement costs to the total production, the net economic output amounts to:

$$Q(t, \mu, K, T^{AT}) = \frac{1 - \Lambda(t, \mu)}{1 + \Omega(T^{AT})} Y(t, K)$$

Assuming that the capital depreciates at a rate δ_K, and that a fraction s of the net output Q is reinvested, the capital K evolves according to the following dynamics:

$$\dot{K} = sQ(t, \mu, K, T^{AT}) - \delta_K K$$

The rest of the net output is consumed:

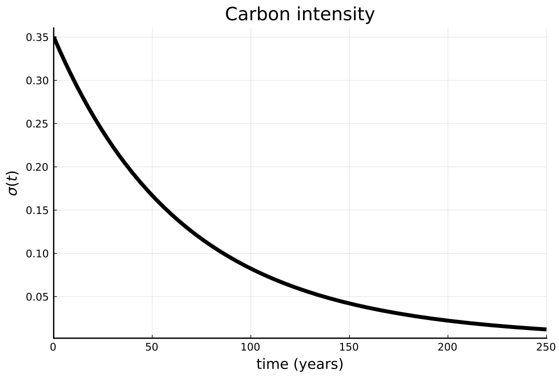
$$C(t) = \big(1 - s(t)\big) Q\big(t, \mu(t), K(t), T^{AT}(t)\big)$$

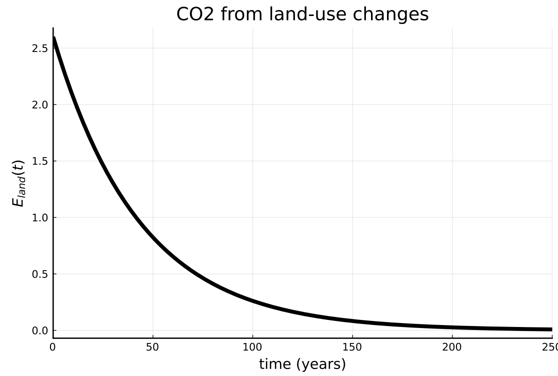
In absence of emission abatement, the industrial CO_2 emissions are given by the carbon intensity, i.e., the emission per unit of production output, times the total production output. The carbon intensity $\sigma(t)$ is assumed to be an exogenous function of time that is linked to technological advancement. Actual industrial emissions are then reduced by the abatement rate μ, yielding:

$$E_{ind}(t, \mu, K) = (1 - \mu)\sigma(t) Y(t, K)$$

Whereas industrial CO_2 emissions are endogenous, the CO_2 arising from land-use changes is assumed to be exogenous (other GHGs are also accounted for, in the exogenous part of the radiative forcing term in the temperature dynamics, as we explain later). The total flow of CO_2 emissions is thus

$$E(t, \mu, K) = E_{ind}(t, \mu, K) + E_{land}(t)$$





Geophysical Equations

The carbon cycle is represented by a three-reservoir model: we consider the stock of carbon in the atmosphere M^{AT}, a quickly mixing reservoir in the shallow ocean and the biosphere M^{UP}, and the carbon stored in the deep ocean M^{LO}. CO_2 emissions are released into the atmosphere (coefficient $\alpha = 3/11$ converts the mass of CO_2 to the mass of carbon), and there is a constant exchange of carbon between adjacent layers: the biosphere takes up atmospheric carbon at a rate δ_1 and releases it back into the atmosphere at a rate δ_2, whereas the deep ocean absorbs carbon from the shallow ocean at a rate δ_3 and releases it back at a rate δ_4.

$$\begin{cases} \dot{M}^{AT} = -\delta_1 M^{AT} + \delta_2 M^{UP} + \alpha E(t, \mu, K) \\ \dot{M}^{UP} = \delta_1 M^{AT} - (\delta_2 + \delta_3) M^{UP} + \delta_4 M^{LO} \\ \dot{M}^{LO} = \delta_3 M^{UP} - \delta_4 M^{LO} \end{cases}$$

By introducing the following vector notation,

$$M = \begin{pmatrix} M^{AT} \\ M^{UP} \\ M^{LO} \end{pmatrix}, \qquad \Phi = \begin{bmatrix} -\delta_1 & \delta_2 & 0 \\ \delta_1 & -(\delta_2 + \delta_3) & \delta_4 \\ 0 & \delta_3 & -\delta_4 \end{bmatrix}, \qquad e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

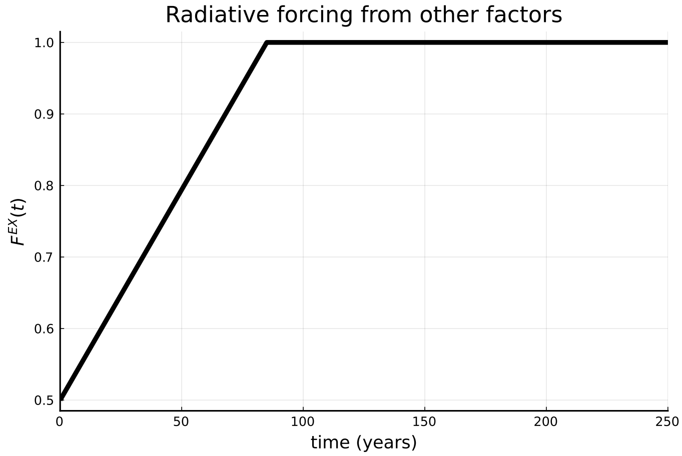
we can reformulate the carbon dynamics as

$$\dot{M} = \Phi M + \alpha E(t, \mu, K) e_1$$

Radiative forcing, i.e., the greenhouse effect, is the mechanism that links the accumulation of GHGs in the atmosphere to warming at the earth's surface. Although the main driver of such phenomenon is the atmospheric CO_2, exogenous forcing from other GHGs (e.g., aerosols and ozone) is also present. The increase in atmospheric temperature due to radiative forcing is given by:

$$F(t, M^{AT}) = \eta \log_2 \left(\frac{M^{AT}}{\tilde{M}} \right) + F^{EX}(t)$$

where $\tilde{M}$ is the pre-industrial level of atmospheric carbon, and η is a parameter. Furthermore, the atmosphere has a self-cooling ability and it also constantly exchanges heat with the ocean



through diffusion.

$$\begin{cases} \dot{T}^{AT} = \xi_1 \left[-\xi_2 T^{AT} + \xi_3 (T^{OC} - T^{AT}) + F(t, M^{AT}) \right] \\ \dot{T}^{OC} = \xi_4 (T^{AT} - T^{OC}) \end{cases} \tag{1}$$

By introducing the following vector notation,

$$T = \begin{pmatrix} T^{AT} \\ T^{OC} \end{pmatrix}, \qquad \zeta = \begin{bmatrix} -\xi_1(\xi_2 + \xi_3) & \xi_1 \xi_3 \\ \xi_4 & -\xi_4 \end{bmatrix}, \qquad e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

we can reformulate the temperature dynamics as

$$\dot{T} = \zeta T + \xi_1 F(t, M^{AT}) e_1$$

Unlike the original DICE model, we do not include the following features: the resource constraint on carbon fuels

$$\int_0^T E_{ind}(t, \mu(t), K(t)) \, dt \leq CCum,$$

nor the possibility of negative emissions after some time

$$\mu(t) \in [0, 1.2] \quad \text{for } t \geq t_0,$$

nor the constraint on atmospheric temperature

$$T^{AT} \leq \mathcal{T}.$$

There are two reasons behind this choice: because the theory we apply to solve our problem does not feature this kind of constraints or changing control sets, and because these specifications would not provide further insight for our purposes anyway.

Full model

Putting it all together, the DICE model in continuous time results in the following optimal control problem:

$$
\begin{aligned}
&\underset{s,\mu\in[0,1]}{\text{maximize}} \int_0^T e^{-\rho t}\, L(t)\, u\big(c(t)\big)\, dt \\
&\text{subject to} \quad \begin{cases} \dot K(t) = sQ(t,\mu,K,T^{AT}) - \delta_K K & K(0) = K_0 \\ \dot M(t) = \Phi M + \alpha E(t,\mu,K) e_1 & M(0) = M_0 \\ \dot T(t) = \zeta T + \xi_1 F(t, M^{AT}) e_1 & T(0) = T_0 \end{cases} \\
&\text{where} \quad c(t) = \big(1 - s(t)\big)\, Q\big(t, \mu(t), K(t), T^{AT}(t)\big) \,/\, L(t)
\end{aligned}
$$

This is the problem that one should solve in a scenario without any tipping point. However, it is also the problem that a policymaker solves if they are not *expecting* a tipping point, regardless of it then occurring or not. For this reason, in a scenario where a tipping point is indeed under way, we can call this the *myopic* problem: the agent plans their policy as if no event is pending, and is caught by surprise by its occurrence.

Below we show part of the output that was generated by the numerical simulation. The GAMS calibration was used for the parameter values.

The drop in savings rate and emission abatement toward the end of the time horizon is due to the absence of a scrap value function: indeed, if one is indifferent to the system's conditions at the final time, it makes sense to stop abating (to maximize production) and to stop saving (to maximize consumption). Usually, in this type of models, the plots are truncated to get rid of this trivial effect.

As for the controls, here we see that the savings rate s is quite stable slightly above 25%, whereas the abatement rate μ starts slightly below 20% and gradually increases towards net zero over the course of about 115 years.

The global atmospheric temperature change T^{AT} increases linearly from 0.85°C to about 3.5°C in the first 100 years, then its growth rate declines and the value settles at about 4.3°C.

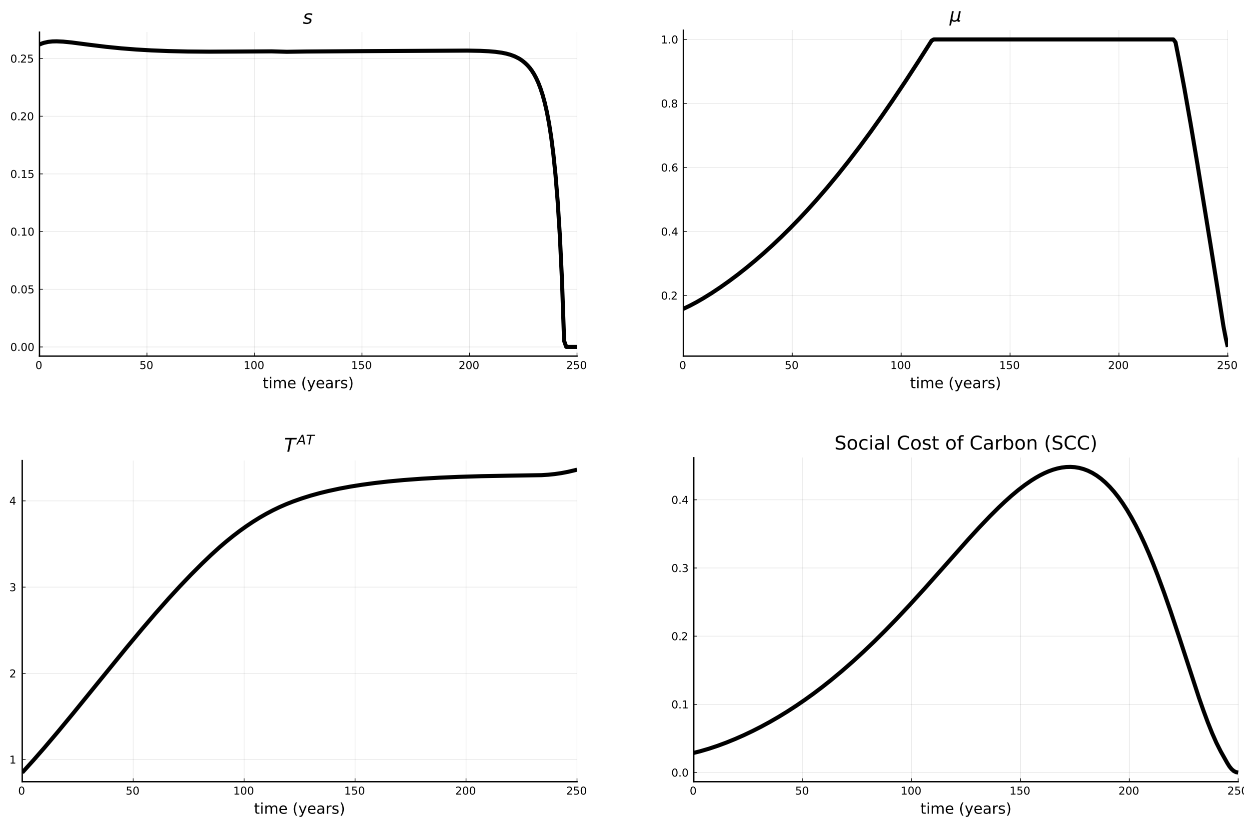
The Social Cost of Carbon grows from about 0.03 to about 0.45 in 173 years, then it sharply declines at the end as a result of the end-time effect mentioned earlier; however, since the SCC is the fraction of adjoints (see Appendix A), it starts decreasing about 50 years earlier than savings and abatement, which also include direct effects.

2.2 Integrating a Climate Tipping Point

Modelling a Stochastic Tipping Point

We add to the continuous-time DICE model described in section 2.1 the possibility of a stochastic climate tipping point, i.e., an unpredictable climate-related event that changes the system abruptly and irreversibly.

Due to current limitations in the mathematical tools that we use to tackle multi-stage optimal control problems with random switching times, we assume the tipping point to be unique, in that one and only one tipping point will eventually occur, either during the planning horizon or afterwards. This is insightful to study the separate effects of tipping



points of different natures one by one; however, once an adequate theoretical framework is developed, it will be fundamental to consider in future research the possibility of a sequence of multiple interconnected tipping points, as feedback mechanisms and domino effects may realistically trigger a tipping point cascade.

Let us denote by τ the instant at which the tipping point occurs, also called *switching time* because it constitutes a shift from one regime to another. We will call *Stage 1* the time interval $[0, \tau \wedge T)$[1], and *Stage 2* the time interval $[\tau, T]$ if $\tau < T$. We model τ as a random variable, taking values in the positive half-line $[0, \infty)$, whose hazard rate is assumed to increase with the global mean temperature change:

$$\lim_{dt \to 0^+} \frac{\mathbb{P}\big(\tau \leq t + dt \,|\, \tau > t\big)}{dt} = h\big(T(t)\big)$$

with $h_{T^{AT}}, h_{T^{OC}} > 0$.[2]

The effects of the tipping point may be a change in the state dynamics, or a jump discontinuity in the state trajectory at the switching time. We only consider regime shifts that concern the carbon cycle dynamics (in Stage 2 the carbon cycle matrix $\Phi_2 \neq \Phi$), the temperature dynamics (in Stage 2 the temperature dynamics matrix $\zeta_2 \neq \zeta$), the impact of temperature on production (in Stage 2 the function $\Omega_2(T^{AT}) \neq \Omega(T^{AT})$, yielding $Q_2 \neq Q$), and instantaneous change in capital stock ($K(\tau^+) \neq K(\tau^-)$). This last case is not technically

[1] We denote $\tau \wedge T = \min(\tau, T)$

[2] We denote f_x the partial derivative of a function f with respect to the variable x.

a tipping point, in that it does not change the nature of the system irreversibly, but since the concept of climate catastrophe entailing substantial economic damage *una tantum* is widely featured in the literature, we chose to include it in our analysis.

Although we are aware that we are considering only a limited range of regime shifts, this is only to lighten the notation. The same theory is applicable to any change in the system's dynamics, any change in the instant payoff, any jump discontinuity in the state trajectory of the kind $X(\tau^+) = \varphi\big(\tau, X(\tau^-)\big)$.

Information and Strategy

In Stage 1, the policymaker does not know when τ will occur, they know that it could occur at any time, and they are aware of its effects on the system. So, when planning a Stage 1 strategy in advance, one needs to plan for the whole time horizon (even if the switch then ends up happening during the time horizon): for all $t \in [0, T]$ we denote by

$$s_1(t), \qquad \mu_1(t)$$

the Stage 1 savings and abatement rate respectively, at time t.

In Stage 2, the policymaker knows that τ has occurred and when, and can therefore adjust their strategy accordingly. So, when planning a Stage 2 strategy in advance, one needs to plan for all possible occurrences of the switching time: for all $\tau \in [0, T]$ and for all $t \in [\tau, T]$ we denote by

$$s_2(t \,|\, \tau), \qquad \mu_2(t \,|\, \tau)$$

the Stage 2 savings and abatement rate respectively, at time t, given that the regime shift occurred at time τ. The same holds for the corresponding state trajectories: we will have $K_1(t), M_1(t), T_1(t)$ in Stage 1 and $K_2(t \,|\, \tau), M_2(t \,|\, \tau), T_2(t \,|\, \tau)$ in Stage 2.

In light of these considerations, and due to the stochastic nature of the switching time τ, the policymaker's objective is to maximize the *expectation* of the integral payoff over all possible realizations of τ:

$$\begin{aligned}
&\underset{s_1,\mu_1,s_2,\mu_2\in[0,1]}{\text{maximize}} \;\; \mathbb{E}_\tau\left[\int_0^{\tau\wedge T} e^{-\rho t}\, L(t)\, u\big(c_1(t)\big)\, dt + \int_{\tau\wedge T}^{T} e^{-\rho t}\, L(t)\, u\big(c_2(t \,|\, \tau)\big)\, dt\right] \\
&\text{subject to} \;\; \begin{cases}
\dot{K}_1(t) = s_1 Q(t, \mu_1, K_1, T_1^{AT}) - \delta_K K_1 & K_1(0) = K_0 \\
\dot{M}_1(t) = \Phi M_1 + \alpha E(t, \mu_1, K_1) e_1 & M_1(0) = M_0 \\
\dot{T}_1(t) = \zeta T_1 + \xi_1 F(t, M_1^{AT}) e_1 & T_1(0) = T_0 \\
\frac{d}{dt}\mathbb{P}(\tau > t) = -h\big(T_1(t)\big)\mathbb{P}(\tau > t) & \mathbb{P}(\tau > 0) = 1 \\
\dot{K}_2(t \,|\, \tau) = s_2 Q_2(t, \mu_2, K_2, T_2^{AT}) - \delta_K K_2 & K_2(\tau \,|\, \tau) = \varepsilon\big(T_1^{AT}(\tau)\big) K_1(\tau) \\
\dot{M}_2(t \,|\, \tau) = \Phi_2 M_2 + \alpha E(t, \mu_2, K_2) e_1 & M_2(\tau \,|\, \tau) = M_1(\tau) \\
\dot{T}_2(t \,|\, \tau) = \zeta_2 T_2 + \xi_1 F(t, M_2^{AT}) e_1 & T_2(\tau \,|\, \tau) = T_1(\tau)
\end{cases} \\
&\text{where} \quad c_1(t) = \big(1 - s_1(t)\big)\, Q_1\big(t, \mu_1(t), K_1(t), T_1^{AT}(t)\big) \,/\, L(t) \\
&\text{and} \quad c_2(t \,|\, \tau) = \big(1 - s_2(t \,|\, \tau)\big)\, Q_2\big(t, \mu_2(t \,|\, \tau), K_2(t \,|\, \tau), T_2^{AT}(t \,|\, \tau)\big) \,/\, L(t)
\end{aligned}$$

2.3 Optimality conditions

We address the problem with a new approach, that was recently developed by Wrzaczek et al. [2020]. This is the first time that this methodology is employed in a climate/economy model with random switching time: the existing literature has only been featuring the backward approach so far.

The method consists in introducing two auxiliary state variables, Z_1, Z_2 in Stage 1 and 2 respectively, that are linked to the distribution of the random variable τ, then reformulating the objective value using a vintage structure, and finally applying the Maximum Principle for heterogeneous systems by Veliov [2008].

The auxiliary state variables are

$$Z_1(t) = \mathbb{P}(\tau > t), \qquad Z_2(t \,|\, \tau) = h\big(T_1(\tau)\big) Z_1(\tau)$$

and as such they are subject to the following dynamics:

$$\begin{cases} \dot{Z}_1(t) = -h\big(T_1(t)\big) Z_1(t) \\ Z_1(0) = 1 \end{cases} \qquad \begin{cases} \dot{Z}_2(t \,|\, \tau) = 0 \\ Z_2(\tau \,|\, \tau) = h\big(T_1(\tau)\big) Z_1(\tau) \end{cases}$$

Observe that Z_1 is the tail of τ's distribution, whereas $Z_2(t \,|\, s) = -\dot{Z}_1(s) = -\frac{d}{ds}\mathbb{P}(\tau > s) = f_\tau(s)$, which is τ's probability distribution.

By explicitly computing the expectation using Z_1 and Z_2, and after some basic integral manipulations (e.g., integrating by parts and applying Fubini's theorem), we obtain the following vintage structure formulation for the problem:

$$\underset{s_1,\mu_1,s_2,\mu_2\in[0,1]}{\text{maximize}} \int_0^T e^{-\rho t} \left[Z_1(t)\, L(t)\, u\big(c_1(t)\big) + \int_0^t Z_2(t \,|\, \tau)\, L(t)\, u\big(c_2(t \,|\, \tau)\big)\, d\tau \right] dt$$

$$\text{subject to} \quad \begin{cases} \dot{K}_1(t) = s_1 Q(t, \mu_1, K_1, T_1^{AT}) - \delta_K K_1 & K_1(0) = K_0 \\ \dot{M}_1(t) = \Phi M_1 + \alpha E(t, \mu_1, K_1) e_1 & M_1(0) = M_0 \\ \dot{T}_1(t) = \zeta T_1 + \xi_1 F(t, M_1^{AT}) e_1 & T_1(0) = T_0 \\ \dot{Z}_1(t) = -h\big(T_1(t)\big) Z_1(t) & Z_1(0) = 1 \\ \dot{K}_2(t \,|\, \tau) = s_2 Q_2(t, \mu_2, K_2, T_2^{AT}) - \delta_K K_2 & K_2(\tau \,|\, \tau) = \varepsilon\big(T_1^{AT}(\tau)\big) K_1(\tau) \\ \dot{M}_2(t \,|\, \tau) = \Phi_2 M_2 + \alpha E(t, \mu_2, K_2) e_1 & M_2(\tau \,|\, \tau) = M_1(\tau) \\ \dot{T}_2(t \,|\, \tau) = \zeta_2 T_2 + \xi_1 F(t, M_2^{AT}) e_1 & T_2(\tau \,|\, \tau) = T_1(\tau) \\ \dot{Z}_2(t \,|\, \tau) = 0 & Z_2(\tau \,|\, \tau) = h(T_1(\tau)) Z_1(\tau) \end{cases}$$

Maximum Principle

A Hamiltonian function is defined for each stage:

$$\begin{aligned} H_1 &= H_1(t, s_1, \mu_1, K_1, M_1, T_1, \lambda, \xi), \\ H_2 &= H_2(t, s_2, \mu_2, K_2, M_2, T_2, \xi) \end{aligned}$$

where λ, ξ denote the co-state vectors of Stage 1 and 2 respectively. In particular, we will denote by λ_X the adjoint variable of the state variable X_1, and by ξ_X the adjoint variable of the state variable X_2. Of course, the co-state trajectories depend on the same variables as the corresponding state trajectories:

$$\lambda = \lambda(t), \qquad \xi = \xi(t \mid \tau)$$

From now on we may omit the arguments of H_1, H_2 to lighten the notation.

$$\begin{aligned} H_1 &= Z_1 L u(c_1) + \lambda_K \left[s_1 Q - \delta_K K_1\right] + \lambda_M \cdot \Phi M_1 + \lambda_{M^{AT}} \alpha E + \lambda_T \cdot \zeta T_1 + \lambda_{T^{AT}} \xi_1 F \\ &\qquad + \xi_K \varepsilon K_1 + \xi_M \cdot M_1 + \xi_T \cdot T_1 + (\xi_Z - \lambda_Z) h Z_1 \\ H_2 &= Z_2 L u(c_2) + \xi_K \left[s_2 Q_2 - \delta_K K_2\right] + \xi_M \cdot \Phi_2 M_2 + \xi_{M^{AT}} \alpha E + \xi_T \cdot \zeta_2 T_2 + \xi_{T^{AT}} \xi_1 F \end{aligned}$$

Observe that the Stage 2 adjoint variables enter the Stage 1 Hamiltonian: through this, the anticipating effect of the tipping point influences the Stage 1 behavior.

Let us denote $(s_i^*, \mu_i^*, K_i^*, M_i^*, T_i^*)$, $i = 1, 2$, the optimal process. The maximality condition from the Maximum Principle states that, for almost all t, τ,

$$\begin{aligned} \big(s_1^*(t), \mu_1^*(t)\big) &\in \operatorname*{arg\,max}_{s,\mu} H_1\big(t, s, \mu, K_1^*(t), M_1^*(t), T_1^*(t), \lambda(t), \xi(t \mid t)\big) \\ \big(s_2^*(t \mid \tau), \mu_2^*(t \mid \tau)\big) &\in \operatorname*{arg\,max}_{s,\mu} H_2\big(t, s, \mu, K_2^*(t \mid \tau), M_2^*(t \mid \tau), T_2^*(t \mid \tau), \xi(t \mid \tau)\big) \end{aligned}$$

where the co-state trajectories satisfy

$$\begin{cases} \dot\lambda_X(t) = \rho \lambda_X(t) - \nabla_X H_1\big(t, s_1^*(t), \mu_1^*(t), K_1^*(t), M_1^*(t), T_1^*(t), \lambda(t), \xi(t \mid t)\big) \\ \lambda_X(T) = 0 \end{cases}$$

$$\begin{cases} \dot\xi_X(t \mid \tau) = \rho \xi_X(t \mid \tau) - \nabla_X H_2\big(t, s_2^*(t \mid \tau), \mu_2^*(t \mid \tau), K_2^*(t \mid \tau), M_2^*(t \mid \tau), T_2^*(t \mid \tau), \xi(t \mid \tau)\big) \\ \xi_X(T \mid \tau) = 0 \end{cases}$$

First Order Conditions

The maximum condition yields that, whenever s_1^* or s_2^* are inner controls, i.e., they belong to the inner part $(0,1)$ of the control set $[0,1]$, they are stationary points for the respective Hamiltionians:

$$\begin{aligned} 0 &= \partial_s H_1 = \left[-Z_1 u_c(c_1) + \lambda_K\right] Q \\ 0 &= \partial_s H_2 = \left[-Z_2 u_c(c_2) + \xi_K\right] Q_2 \end{aligned}$$

So, for all t such that the optimal savings rate $s_1^*(t)$ is an inner solution (arguments are omitted to lighten the notation),

$$Z_1 u_c(c_1) = \lambda_K \tag{2}$$

and for all t, τ such that the optimal savings rate $s_2^*(t \mid \tau)$ is an inner solution,

$$Z_2 u_c(c_2) = \xi_K \tag{3}$$

Analogously, whenever μ_1^* or μ_2^* are inner controls,

$$0 = \partial_\mu H_1 = -\left\{[Z_1 u_c(c_1)(1-s_1^*) + \lambda_K s_1^*]\frac{\Lambda_\mu}{1+\Omega} + \lambda_{M^{AT}}\alpha\sigma\right\}Y$$
$$0 = \partial_\mu H_2 = -\left\{[Z_2 u_c(c_2)(1-s_2^*) + \xi_K s_2^*]\frac{\Lambda_\mu}{1+\Omega_2} + \xi_{M^{AT}}\alpha\sigma\right\}Y$$

If, furthermore, s_1^* or s_2^* respectively are inner controls,

$$0 = \partial_\mu H_1 = -\left\{\lambda_K\frac{\Lambda_\mu}{1+\Omega} + \lambda_{M^{AT}}\alpha\sigma\right\}Y$$
$$0 = \partial_\mu H_2 = -\left\{\xi_K\frac{\Lambda_\mu}{1+\Omega_2} + \xi_{M^{AT}}\alpha\sigma\right\}Y$$

yielding

$$\sigma\left(\alpha\frac{-\lambda_{M^{AT}}}{\lambda_K}\right) = \frac{\Lambda_\mu}{1+\Omega} \tag{4}$$
$$\sigma\left(\alpha\frac{-\xi_{M^{AT}}}{\xi_K}\right) = \frac{\Lambda_\mu}{1+\Omega_2} \tag{5}$$

This is quite meaningful because, since s_i^* is inner, the term inside the parentheses coincides with the Social Cost of Carbon (see Appendix A).

Adjoint Equations

Although we can of course compute the adjoint equations for all the variables, due to their relevance in the calculation of the Social Cost of Carbon (see Appendix A) we are more interested in the co-state trajectories of capital and atmospheric carbon, and particularly in their growth rates.

Here we are omitting the dependence on time and switching time. Recall that in the adjoint equation for Stage 1 at time t, the Stage 2 co-states are evaluated on the diagonal, i.e., $\xi = \xi(t\,|\,t)$.

Observe that in both cases the terms of the two equations are symmetric up until the final one: in general, the Stage 1 shadow values also depend on the Stage 2 shadow values and on the 'switching conditions' of the states (i.e., condition $X_2(t\,|\,t) = \varphi(X_1(t))$). This dependence encloses the awareness of the policymaker in regard to the possibility of a tipping point.

We start with the Stage 1 and 2 adjoint equations for capital:

$$\dot{\lambda}_K = (\rho+\delta_K)\lambda_K - \left\{[Z_1 u_c(c_1)(1-s_1^*) + \lambda_K s_1^*]\frac{1-\Lambda}{1+\Omega} + \lambda_{M^{AT}}\alpha(1-\mu_1^*)\sigma\right\}Y_K - \varepsilon\xi_K$$
$$\dot{\xi}_K = (\rho+\delta_K)\xi_K - \left\{[Z_2 u_c(c_2)(1-s_2^*) + \xi_K s_2^*]\frac{1-\Lambda}{1+\Omega_2} + \xi_{M^{AT}}\alpha(1-\mu_2^*)\sigma\right\}Y_K$$

In this case, the final term in the Stage 1 equation is $-\xi_K(t\,|\,t)\varepsilon((T_1^{AT})^*(t))$. Recalling that $\varepsilon(T^{AT})$ is the fraction of capital that remains after the tipping point, and by educated-guessing that both $\lambda_K, \xi_K > 0$ (which is confirmed by the numerical simulations), we can observe that the preservation of capital upon the regime shift has a frontloading effect on λ_K. This finds an intuitive explanation in the fact that if you know that part of the capital is going to be destroyed at the switch, then, before the switch occurs, it will hold less value than it would if it were to be completely preserved.

Suppose that s_1^*, s_2^* are inner controls and that $\mu_1^*, \mu_2^* \neq 0$.[3] Then, by the FOCs (2-3) and (4-5), the growth rates of λ_k, ξ_K are:

$$\frac{\dot{\lambda}_K}{\lambda_K} = \rho + \delta_K - \left[1 - \Lambda - \Lambda_\mu(1-\mu_1^*)\right]\frac{1}{1+\Omega}Y_K - \varepsilon\frac{\xi_K}{\lambda_K}$$
$$\frac{\dot{\xi}_K}{\xi_K} = \rho + \delta_K - \left[1 - \Lambda - \Lambda_\mu(1-\mu_1^*)\right]\frac{1}{1+\Omega_2}Y_K$$

Next, we compute the adjoint equations for the atmospheric carbon:

$$\dot{\lambda}_{M^{AT}} = \rho\lambda_{M^{AT}} - \delta_1(\lambda_{M^{UP}} - \lambda_{M^{AT}}) - \xi_1 F_{M^{AT}}\lambda_{T^{AT}} - \xi_{M^{AT}}$$
$$\dot{\xi}_{M^{AT}} = \rho\xi_{M^{AT}} - \delta_1^{(2)}(\xi_{M^{UP}} - \xi_{M^{AT}}) - \xi_1 F_{M^{AT}}\xi_{T^{AT}}$$

Here the final term in the Stage 1 equation is $-\xi_{M^{AT}}(t\,|\,t)$. By guessing that both $\lambda_{M^{AT}}, \xi_{M^{AT}} < 0$ (confirmed by the numerical simulations), we observe that the continuity of the atmospheric carbon upon the regime shift constitutes a frontloading factor for (the negative value of) $\lambda_{M^{AT}}$: indeed, knowing that carbon will stay around after the shift makes it worse to accumulate it in Stage 1.

Their growth rates are:

$$\frac{\dot{\lambda}_{M^{AT}}}{\lambda_{M^{AT}}} = \rho + \delta_1\left(1 - \frac{\lambda_{M^{UP}}}{\lambda_{M^{AT}}}\right) - \xi_1 F_{M^{AT}}\frac{\lambda_{T^{AT}}}{\lambda_{M^{AT}}} - \frac{\xi_{M^{AT}}}{\lambda_{M^{AT}}}$$
$$\frac{\dot{\xi}_{M^{AT}}}{\xi_{M^{AT}}} = \rho + \delta_1^{(2)}\left(1 - \frac{\xi_{M^{UP}}}{\xi_{M^{AT}}}\right) - \xi_1 F_{M^{AT}}\frac{\xi_{T^{AT}}}{\xi_{M^{AT}}}$$

Another recurrent term that is worth considering is $\xi_Z(t\,|\,t) - \lambda_Z(t)$. To understand its meaning we compute the adjoint equations for λ_Z, ξ_Z and complete them with the suitable transversality conditions:

$$\begin{cases}\dot{\lambda}_Z = \rho\lambda_Z - Lu(c_1) - (\xi_Z - \lambda_Z)h \\ \lambda_Z(T) = 0\end{cases} \qquad \begin{cases}\dot{\xi}_Z = \rho\xi_Z - Lu(c_2) \\ \xi_Z(T\,|\,\tau) = 0\end{cases}$$

Recall that all the functions are evaluated in the optimal controls and the corresponding states.

Define the value function of Stage 2:

$$V_2(t, \mathtt{X}) := \max_{s,\mu}\int_t^T e^{-\rho(\theta-t)}L(\theta)\,u\big(c_2(\theta)\big)\,d\theta$$

[3]This assumption is reasonable, since from the numerical simulations we obtain that s_i^* is inner up to the last 5 years in the planning horizon, and that μ_i^* is always nonzero except at the final time.

subject to:

$$\begin{cases} \dot{X}(\theta) = f_2(\theta, X(\theta), s(\theta), \mu(\theta)) & \theta > t \\ X(t) = \mathtt{X} \end{cases}$$

and the value function of the 2-stage problem (maximizing the expectation of the 2-stage payoff) assuming optimal behavior in Stage 2:

$$V(t, \mathtt{X}, \mathtt{Z}) := \max_{s,\mu} \int_t^T e^{-\rho(\theta-t)} Z(\theta) \left[L(\theta)\, u\big(c_1(\theta)\big) + h(T(\theta)) V_2\big(\theta, \varphi(X(\theta))\big) \right] dt$$

subject to:

$$\begin{cases} \dot{X}(\theta) = f_1(\theta, X(\theta), s(\theta), \mu(\theta)) & \theta > t, \qquad X(t) = \mathtt{X} \\ \dot{Z}(\theta) = -h(T(\theta)) Z(\theta) & \theta > t, \qquad Z(t) = \mathtt{Z} \end{cases}$$

Observe that

$$\xi_Z(t \,|\, \tau) = \int_t^T e^{-\rho(\theta-t)} L(\theta)\, u\big(c_2(\theta \,|\, \tau)\big)\, d\theta = V_2\big(t, X_2^*(t \,|\, \tau)\big)$$

and, in particular,

$$\xi_Z(t \,|\, t) = V_2\big(t, X_2^*(t \,|\, t)\big) = V_2\big(t, \varphi(X_1^*(t))\big)$$

whereas

$$\begin{aligned} \lambda_Z(t) &= \int_t^T e^{-\rho(\theta-t)} \frac{Z_1^*(\theta)}{Z_1^*(t)} \left[L(\theta) u(c_1(\theta)) + h(T_1^*(\theta)) V_2\big(\theta, \varphi(X_1^*(\theta))\big) \right] d\theta \\ &= V\big(t, X_1^*(t), Z_1^*(t)\big) / Z_1^*(t) \end{aligned}$$

which is the value function of the whole problem, actualized to time t by dividing it by the probability of still being in Stage 1 at time t. Therefore

$$\xi_Z(t \,|\, t) - \lambda_Z(t) = V_2\big(t, \varphi(X_1^*(t))\big) - V\big(t, X_1^*(t), Z_1^*(t)\big) / Z_1^*(t)$$

represents the instantaneous 'desirability' of the switch at time t: it measures the expected[4] gain from switching now, if the shift has not occurred yet. The sign of this quantity tells us whether switching now is advantageous (positive) or not (negative) compared to still being in Stage 1. However, comparing its value for different times is not meaningful because it is an undiscounted quantity: if we want to compare the expected gain from switching at different times, we need to multiply this quantity by the discount factor.

To highlight the meaning of the discounted quantity, we add and subtract the payoff from

[4]Recall that V is the maximized *expectation* of the two-stage objective.

0 to s to obtain the following:

$$\begin{aligned}
e^{-\rho s}\left[\xi_Z(s\,|\,s)-\lambda_Z(s)\right] &= \int_0^s e^{-\rho t}\,L(t)\,u\big(c_1(t)\big)\,dt + e^{-\rho s}V_2\big(s,\varphi(X_1^*(s))\big)\\
&\quad-\left[\int_0^s e^{-\rho t}\,L(t)\,u\big(c_1(t)\big)\,dt + e^{-\rho s}V\big(s,X_1^*(s),Z_1^*(t)\big)/Z_1(s)\right]\\
&= J(s)-\mathbb{E}\left[J(\tau)\,|\,\tau\ge s\right]
\end{aligned}$$

where we denoted by $J(s)$ the realization of the payoff for $\tau = s \le T$, i.e.,

$$\begin{aligned}
J(s) &= \int_0^s e^{-\rho t}\,L(t)\,u\big(c_1(t)\big)\,dt + \int_s^T e^{-\rho t}\,L(t)\,u\big(c_2(t\,|\,s)\big)\,dt\\
&= \int_0^s e^{-\rho t}\,L(t)\,u\big(c_1(t)\big)\,dt + e^{-\rho s}V_2\big(s,\varphi(X_1^*(s))\big)
\end{aligned}$$

Social Cost of Carbon

The Social Cost of Carbon (SCC) is defined as the Marginal Rate of Substitution (MRS) of consumption for emissions: it measures how much consumption one is willing to give up to reduce CO_2 emissions by 1 unit. Whenever s_1^* or s_2^* are inner controls, then

$$SCC_1 = \alpha\frac{-\lambda_{M^{AT}}}{\lambda_K}, \qquad SCC_2 = \alpha\frac{-\xi_{M^{AT}}}{\xi_K}$$

respectively (proved in Appendix A). As we will see from the numerical simulations, the savings rate (both in Stage 1 and 2) is inner up to the last ~5 years in the planning horizon, so we can safely use these formulae for the SCC.

Hence

$$\frac{\dot{SCC}_1}{SCC_1} = \frac{\dot\lambda_{M^{AT}}}{\lambda_{M^{AT}}} - \frac{\dot\lambda_K}{\lambda_K}, \qquad \frac{\dot{SCC}_2}{SCC_2} = \frac{\dot\xi_{M^{AT}}}{\xi_{M^{AT}}} - \frac{\dot\xi_K}{\xi_K}$$

Substituting the growth rates of the co-states, which were computed in the previous subsection under the further hypothesis that $\mu_1^*, \mu_2^* \neq 0$ (which is confirmed by the numerical simulations, except in the final time),

$$\begin{aligned}
\frac{\dot{SCC}_1}{SCC_1} &= \left[1-\Lambda-(1-\mu_1^*)\Lambda_\mu\right]\frac{1}{1+\Omega}Y_K - \delta_K \\
&\quad+\delta_1\left(1-\frac{\lambda_{M^{UP}}}{\lambda_{M^{AT}}}\right) - \xi_1 F_{M^{AT}}\frac{\lambda_{T^{AT}}}{\lambda_{M^{AT}}}\\
&\quad+\varepsilon\frac{\xi_K}{\lambda_K} - \frac{\xi_{M^{AT}}}{\lambda_{M^{AT}}}
\end{aligned} \tag{6}$$

$$\begin{aligned}
\frac{\dot{SCC}_2}{SCC_2} &= \left[1-\Lambda-(1-\mu_2^*)\Lambda_\mu\right]\frac{1}{1+\Omega}Y_K - \delta_K \\
&\quad+\delta_1^{(2)}\left(1-\frac{\xi_{M^{UP}}}{\xi_{M^{AT}}}\right) - \xi_1 F_{M^{AT}}\frac{\xi_{T^{AT}}}{\xi_{M^{AT}}}
\end{aligned} \tag{7}$$

where, in both stages, the first line comes from the growth rate of the co-state of capital and the second line comes from the growth rate of the co-state of the atmospheric carbon; in Stage 1 additional terms are present, which reflect the anticipation of the tipping point.

One can prove that:

$$\xi_X(t \mid \tau) = Z_2^*(t \mid \tau) \nabla_X V_2\big(t, X_2^*(t \mid \tau)\big)$$
$$\lambda_X(t) = \nabla_X V\big(t, X_1^*(t), Z_1^*(t)\big)$$

Through further calculations we can attach a meaning to those terms:

$$\begin{aligned}
\varepsilon \frac{\xi_K}{\lambda_K} - \frac{\xi_{M^{AT}}}{\lambda_{M^{AT}}} &= Z_2^*(t \mid t) \left[\frac{\varepsilon \partial_K V_2\big(t, \varphi(X_1^*(t))\big)}{\partial_K V\big(t, X_1^*(t), Z_1^*(t)\big)} - \frac{\partial_{M^{AT}} V_2\big(t, \varphi(X_1^*(t))\big)}{\partial_{M^{AT}} V\big(t, X_1^*(t), Z_1^*(t)\big)} \right] \\
&= h(T^*(t)) \left[\frac{\partial_K \left[V_2\big(t, \varphi(\mathtt{X})\big)\right]}{\partial_K V\big(t, \mathtt{X}, Z_1^*(t)\big)/Z_1^*(t)} - \frac{\partial_{M^{AT}} \left[V_2\big(t, \varphi(\mathtt{X})\big)\right]}{\partial_{M^{AT}} V\big(t, \mathtt{X}, Z_1^*(t)\big)/Z_1^*(t)} \right] \Bigg|_{\mathtt{X}=X^*(t)}
\end{aligned}$$

where the first (resp. second) ratio is the marginal gain in the Stage 2 value from a unit increment in K_1 (resp. M_1^{AT}) if switching now, compared to the marginal gain in the expected value of the 2-stage problem (actualized with the probability of still being in Stage 1 at time t) from a unit increment in K_1 (resp. M_1^{AT}). All that is weighted by the hazard rate of τ at time t.

Guessing that the numerator and denominator are both positive in the first fraction, and both negative in the second fraction, we could say that the more is gained from switching *now* with a unit increment in K_1, the more the SCC is backloaded, and the more is lost from switching *now* with one unit increment in M_1^{AT}, the more the SCC is frontloaded.

3 Numerical results: four tipping point scenarios

In this section we present the numerical results in four different tipping point scenarios: A) Melting of sea ice or ice sheets, B) Destruction of capital due to an environmental catastrophe, C) Transformation of a carbon sink into a carbon source, and D) Increase in the impact of temperature on production.

We set a time horizon of 250 years. For the myopic scenario, where the planner does not expect a tipping point, and for the parametrization of Stage 1 in the problem with the tipping point, we use the continuous-time version of the functional forms and parameters in Freiberger et al. [2022], that were extracted from the official GAMS code DICE-2016R.

As for the Stage 2 parametrization, we purposely exaggerated the effects of the tipping points on the system, leading to exaggerated (and hopefully unrealistic!) outcomes in terms of the Stage 2 temperature and/or carbon. Keep in mind that this was done in order to better highlight the *qualitative* adjustments to the optimal behavior in preparation for and adaptation to the tipping point, and not to accurately quantify the tipping point's effect or the relative optimal strategy. A sensitivity analysis will be performed in future research for this purpose.

The numerical results were obtained through a gradient descent method based on the Maximum Principle for heterogeneous systems: starting from an initial guess for the controls, at each step the corresponding states and co-states are computed through forward and backward integration respectively, then a line search in a descent direction for $-H_1$ and $-H_2$ with respect to the controls is performed by comparing objective values, and finally the controls are updated.

3.1 How to read the figures

We include essentially two types of figures.

In the first kind we compare a variable's behavior in the *myopic* scenario, in gray, with its behavior in *anticipation* of the tipping point, in black (see figure 1).

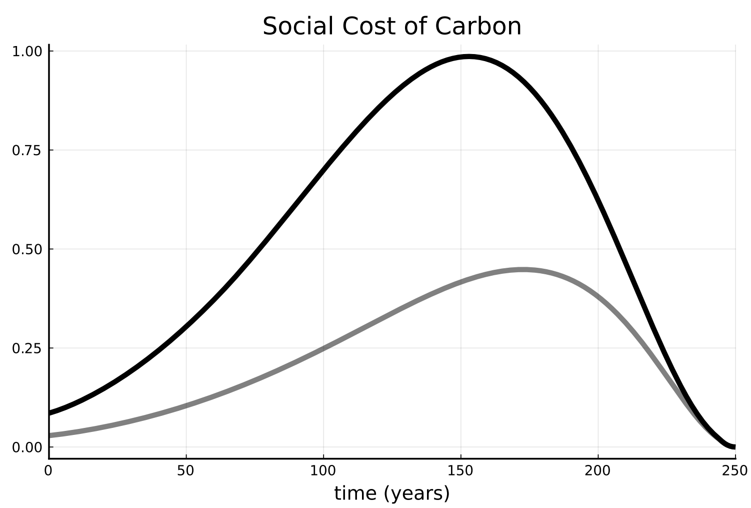


Figure 1: Anticipating (black) vs myopic (gray)

The second kind features the anticipating *Stage 1* behavior (in black) and the *Stage 2* behavior after the tipping point (in color). A Stage 2 line will live in the time interval between the tipping point τ and the final time T, and since τ could occur at any time, in principle there should be a colored line starting from every instant. Of course this would be impossible to represent, so we only display ten Stage 2 lines (for one realization of τ every 25 years). To consult the variable's global behavior for a specific occurrence $\tau = s$, one should follow the black Stage 1 line from $t = 0$ to $t = s$, and then jump to the colored Stage 2 line which starts from $t = s$ (see figure 2).

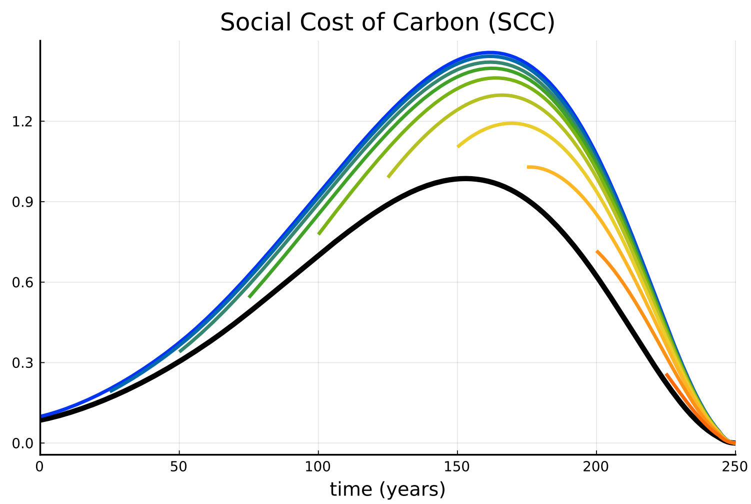


Figure 2: Stage 1 (black) vs Stage 2 (color)

3.2 A) Melting of sea ice or ice sheets

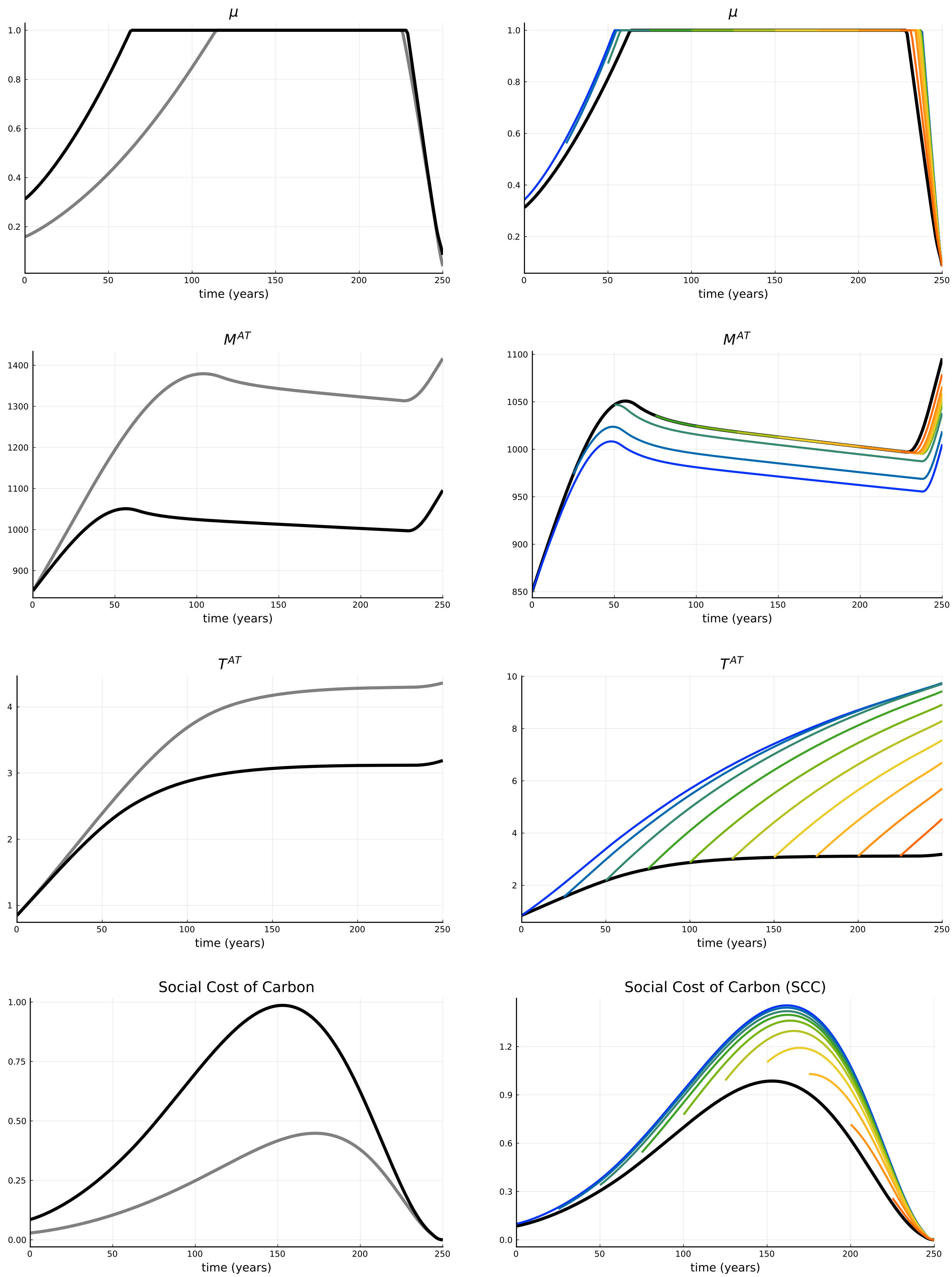
The ice reservoirs on the Earth's surface are important allies in keeping the atmosphere cool: this is due to the ice-albedo effect, which consists in the ice's ability to reflect part of the sun's radiation back to space. In the DICE model, this effect is enclosed in the self-cooling term $-\xi_2 T^{AT}$ in the atmospheric temperature's dynamics (1).

However, global warming is triggering a feedback mechanism involving the ice-albedo effect: higher temperatures reduce the extension of the ice sheets, which in turn reflect less of the radiation away from our atmosphere, allowing for the temperature to rise even more.

Supposing that a tipping point may consist in the complete melting of an important ice reservoir, such as the Arctic sea-ice or the Greenland ice sheet, we model this effect as a drastic lowering of the atmosphere's cooling rate, upon the regime shift:

$$\xi_2^{(2)} < \xi_2$$

In the following simulation, $\xi_2^{(2)} = \xi_2/5$.



The tipping point has a catastrophic effect on atmospheric temperature: in Stage 2 the mean temperature change rises up to an exaggerated 10°C. Compared to the myopic scenario, in anticipation of the tipping point we observe a precautionary increase in the emission abatement μ. The purpose is to postpone the tipping point by keeping the atmospheric temperature (which is the only risk factor) as low as possible: the higher the abatement, the lower the carbon in the atmosphere, the lower the temperature. The SCC peaks about 20 years earlier than in the myopic scenario, and at more than double the value: indeed emissions are way more costly when trying to fend off such a catastrophic event.

3.3 B) Destruction of capital due to an environmental catastrophe

A realization of this scenario may consist in a serious climate event, such as severe wind or precipitation, flooding, wildfire, extreme heatwave, ecosystem disruption, etc. Even though these catastrophes have a regional characterization, there may be an impact on the global economy, which could be due to side-effects of the event such as mass migrations, pandemic outbreaks, disruptions in global supply chains, etc.

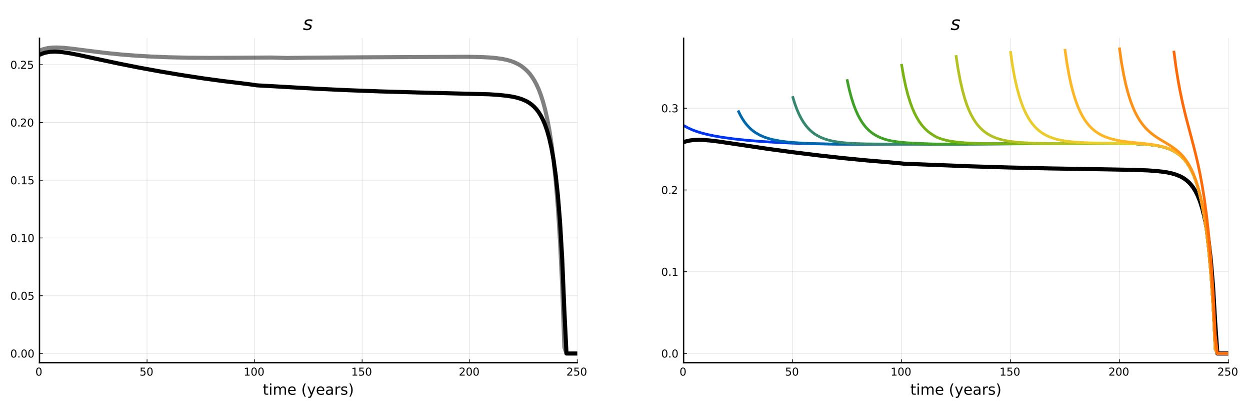
This is not technically a tipping point, because even though there is an instantaneous damage, the system is not permanently changed. This means that, once the only switch has occurred, we are back to the original system where the planner is not expecting any tipping point, i.e., the myopic scenario. This fact will be evident from the simulation figures below.

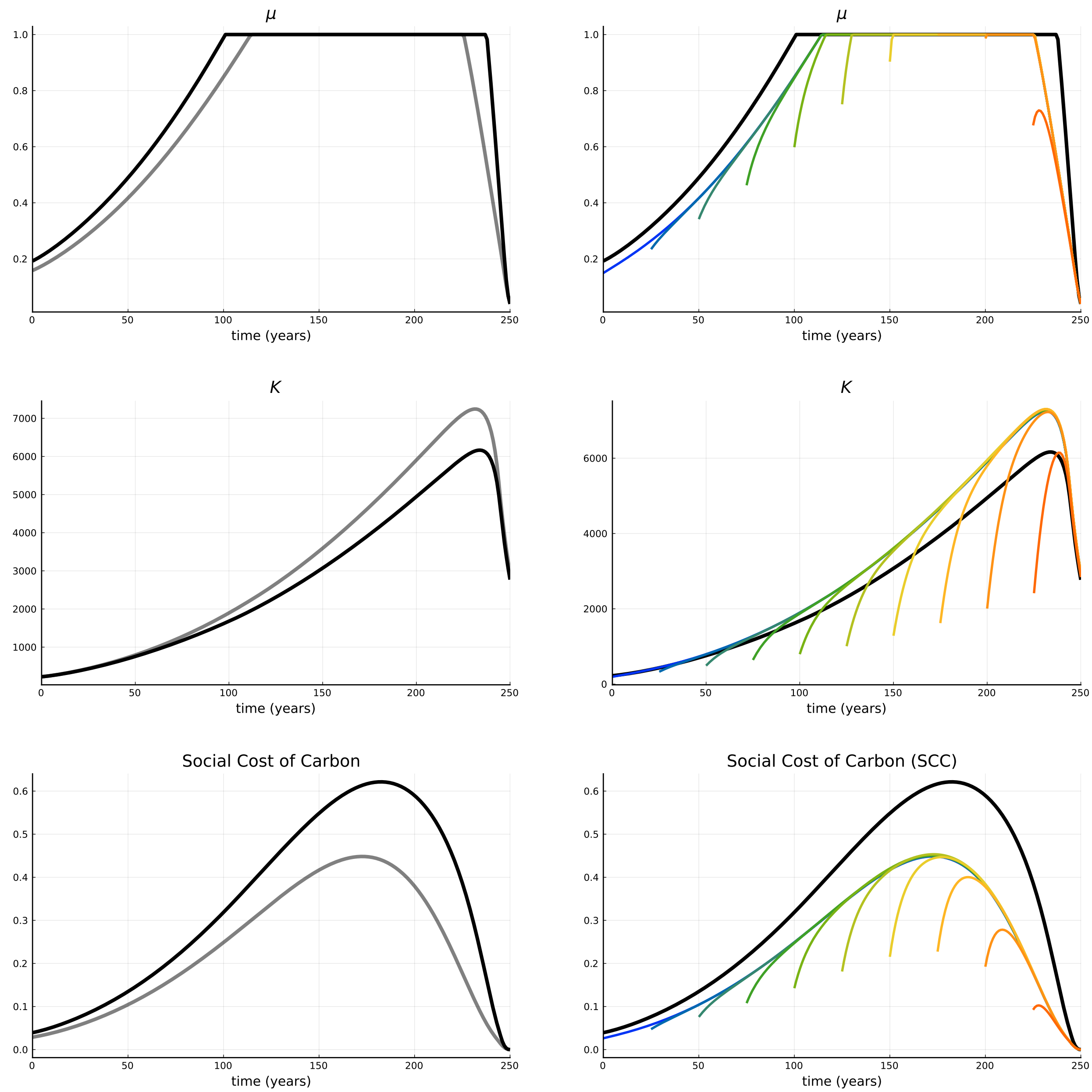
We assume the damage to be proportional to the capital itself, and to the global mean temperature change in the atmosphere:

$$K_2(\tau \,|\, \tau) = K_1(\tau) - \varepsilon_T T_1^{AT}(\tau) K_1(\tau)$$

$$\varepsilon(T^{AT}) = 1 - \varepsilon_T T_1^{AT}$$

In the following simulation, $\varepsilon_T = 0.15$.





Compared to the myopic scenario, in anticipation of the tipping point the planner is saving less, thus accumulating less capital: indeed, since part of the capital is going to be destroyed at the switch, it is less valuable in Stage 1. The abatement is higher in order to postpone the switch and to reduce the damage (both depend on atmospheric temperature). The SCC is higher because of the negative effect of emissions, but peaks slightly later: indeed capital – and therefore, damage – is increasing with time, so a later switch yields greater damage, and therefore emissions are more costly later on.

Observe that, as we anticipated, all the variables' behavior in Stage 2 retrace the myopic case, after an adjustment period of about 25 years, in which we restore capital through higher savings and lower abatement.

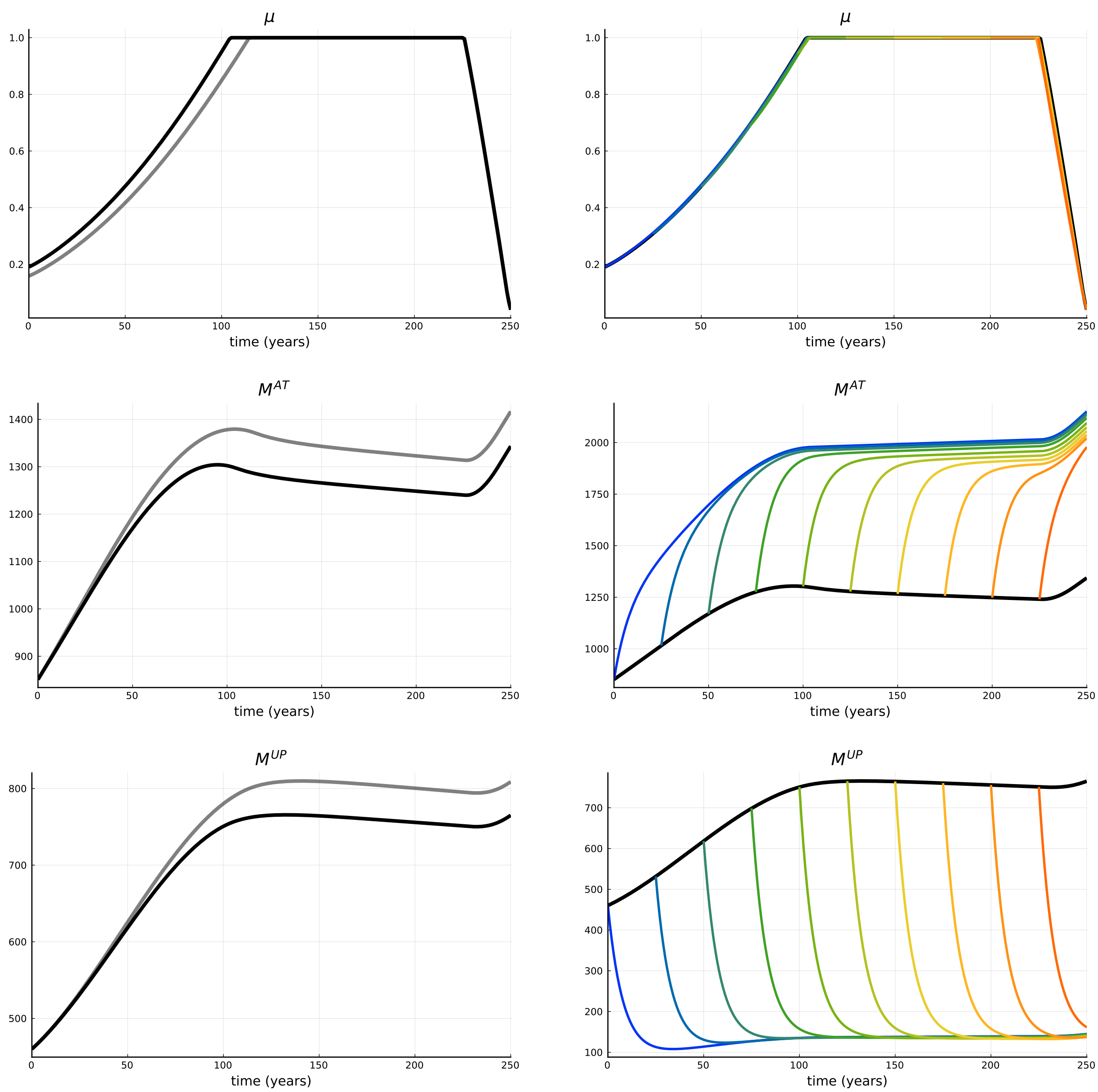
3.4 C) Transformation of a carbon sink into a carbon source

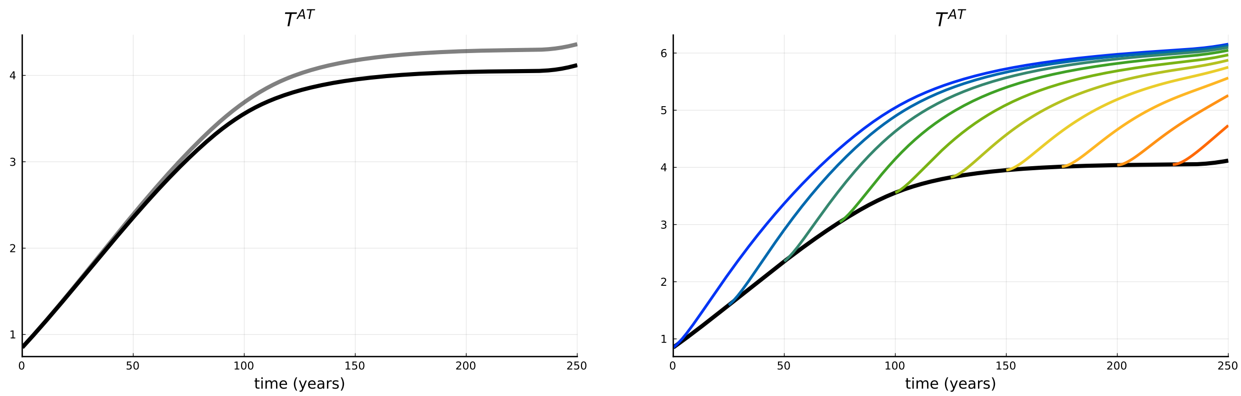
The Earth's biosphere is an important ally in taking up the atmospheric carbon, however, anthropogenic factors are endangering and compromising its activity. Some examples of this phenomenon are: the destruction of the Amazon rainforest and of the Boreal forest, the die-off of coral reefs, and the thawing of the Permafrost. When such an ecosystem is destroyed, the CO_2 that was absorbed and stored is released back into the atmosphere.

This could be modeled in Stage 2 by reducing the carbon uptake rate from the atmosphere to the middle layer, and increasing the release rate:

$$\delta_1^{(2)} < \delta_1, \qquad \delta_2^{(2)} > \delta_2$$

In the following simulation, $\delta_1^{(2)} = \delta_1/3$ and $\delta_2^{(2)} = 3 \cdot \delta_2$.





Very little is done to fend off the tipping point: compared to the myopic scenario, the abatement in the anticipating case is only slightly higher (hence the carbon and temperature are slightly lower).

Interestingly enough, nothing changes in the planner's policy from Stage 1 to Stage 2: indeed, the carbon stocks and temperature reach stable values that are different from the Stage 1 stable values (higher for the atmosphere, lower for the biosphere).

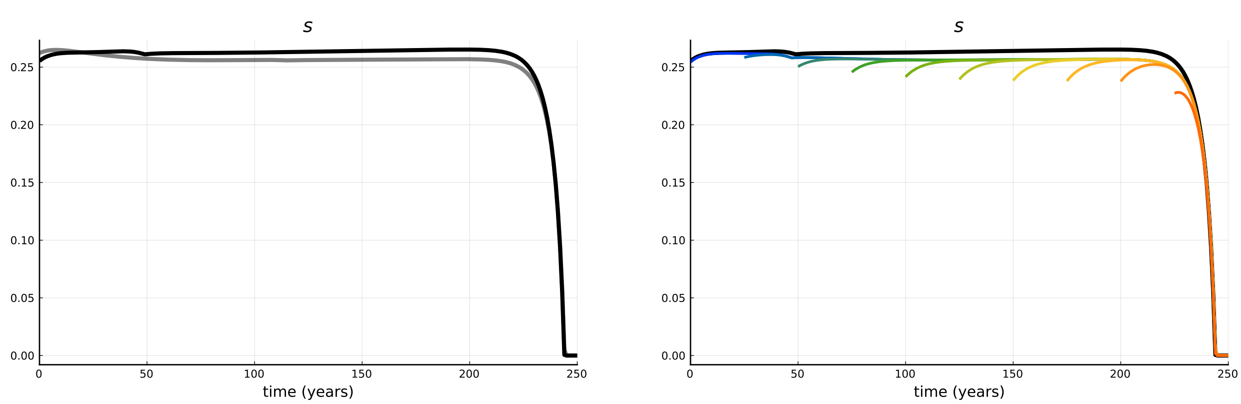
3.5 D) Increased impact of temperature on production

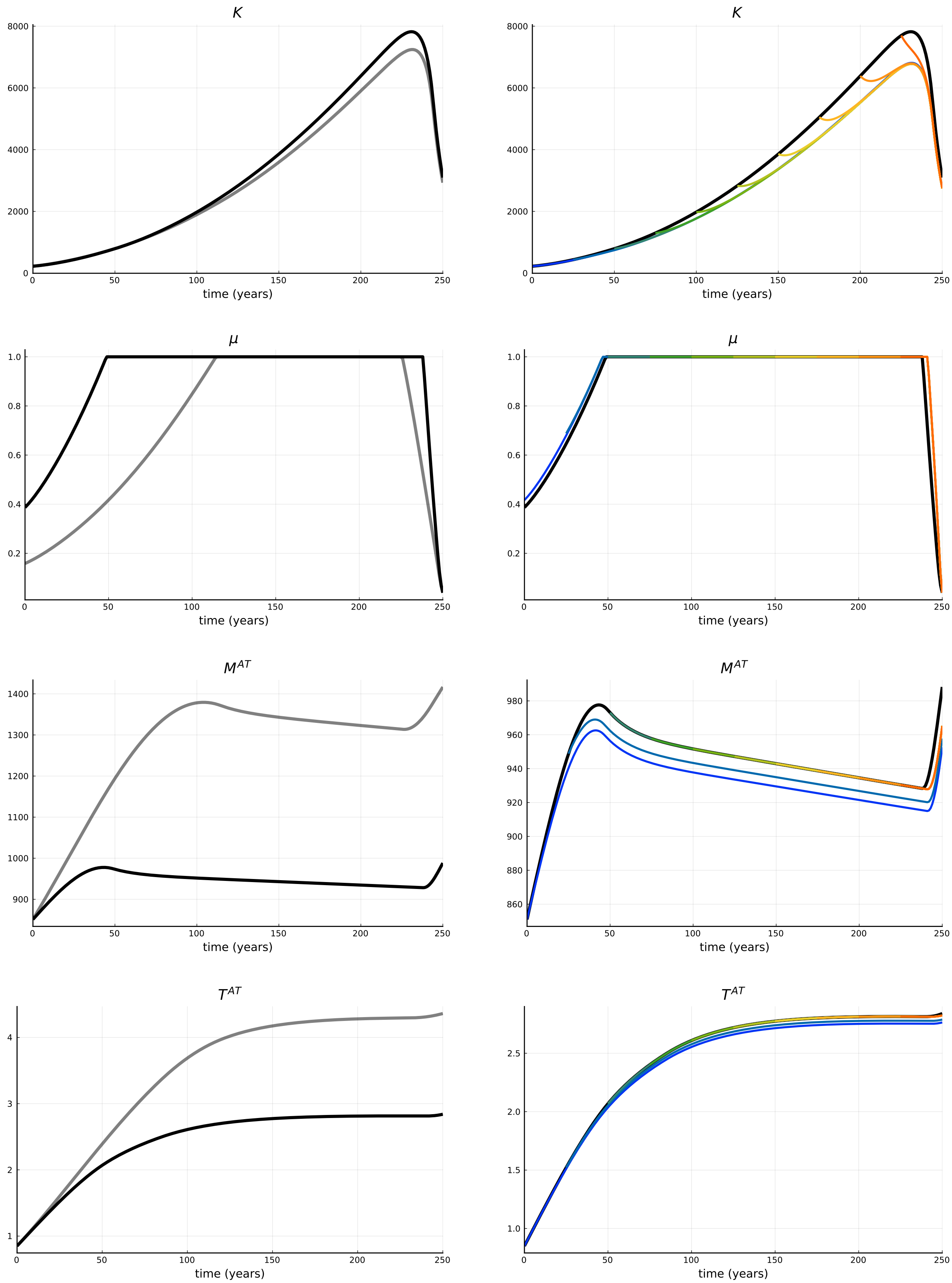
Some production sectors, like agriculture, highly depend on climate conditions that are not represented in this model, such as humidity or rainfall. In this scenario, we suppose that the climate change brought on by the tipping point alters some of these climate variables, thus making the economic production more sensitive to global warming.

This is modeled by increasing the temperature damage function upon the regime shift:

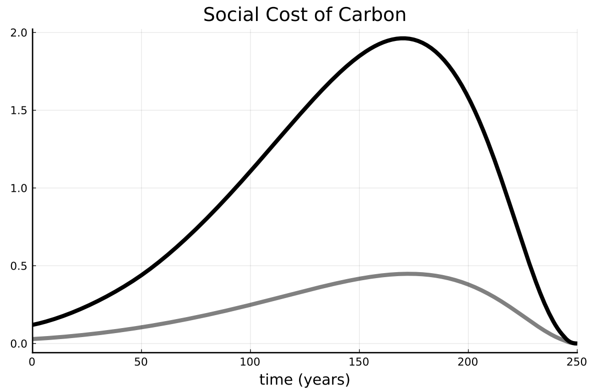
$$Q_2 = \frac{1 - \Lambda(t, \mu)}{1 + \Omega_2(T^{AT})} Y(t, K) \quad \text{with} \quad \Omega_2(T^{AT}) > \Omega(T^{AT})$$

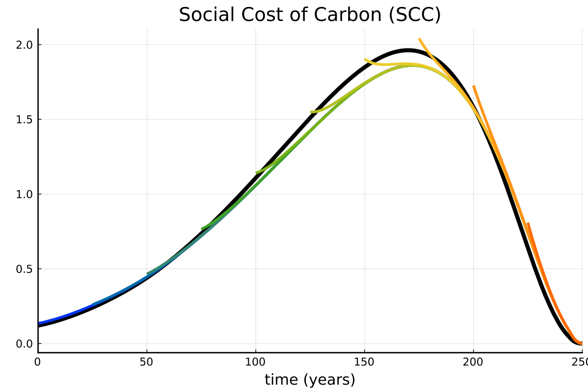
In the following simulation, $\Omega_2 = 5 \cdot \Omega$.





K
time (years)
K
time (years)
μ
time (years)
μ
time (years)
M^AT
time (years)
M^AT
time (years)
T^AT
time (years)
T^AT
time (years)





Compared to the myopic scenario, in the anticipating case much more abatement is employed to fend off the tipping point (high abatement implies low carbon, hence low temperature). The importance of preventing the tipping point rather than adjusting to it in this case is also evident in the anticipating vs myopic SCC: the anticipating peak is more than quadruple the myopic peak, reflecting the extremely costly emissions in anticipation of the event.

As for the Stage 1 to Stage 2 adaptation, there is only a slight change in policy. One can observe that the savings rate in Stage 2 retraces the Stage 1 policy after an adjustment period, but the capital stays lower due to the decreased production output.

3.6 Decomposition of the SCC

In section A, we decomposed the growth rate of the Social Cost of Carbon into the contribution of different terms (see equations 6 and 7). In both stages we can separate a backloading term,

$$[1 - \Lambda - (1 - \mu_1^*)\Lambda_\mu] \frac{1}{1+\Omega} Y_K - \delta_K$$

representing the effect of capital and the production dynamics, and a frontloading term,

$$\delta_1 \left(1 - \frac{\lambda_{M^{UP}}}{\lambda_{M^{AT}}}\right) - \xi_1 F_{M^{AT}} \frac{\lambda_{T^{AT}}}{\lambda_{M^{AT}}}$$

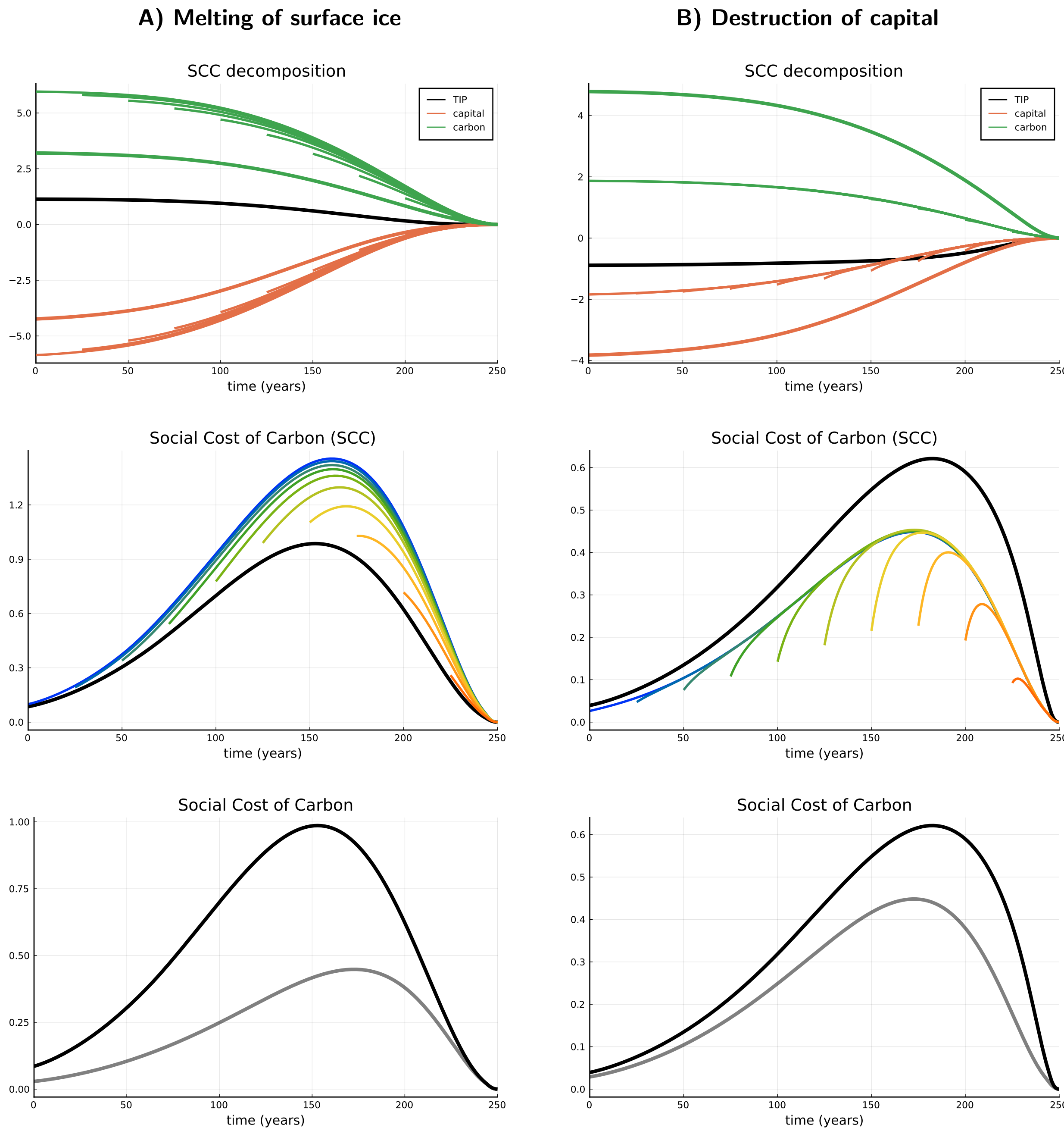
representing the effect of carbon and the geophysical dynamics.

In Stage 1 there is also a term representing the effect of the anticipation of the tipping point:

$$\varepsilon \frac{\xi_K}{\lambda_K} - \frac{\xi_{M^{AT}}}{\lambda_{M^{AT}}}$$

which is in turn made up of a backloading term related to capital and a frontloading term related to carbon. Depending on the scenario, one of the two may prevail, thus making the anticipation of the tipping point either a backloading or a frontloading factor for the SCC.

Here below we show two figures with the SCC decomposition in scenarios A and B, respectively, and the resulting SCCs for reference. The red lines (one for Stage 1 and the other ten for Stage 2) represent the backloading contribution of the capital dynamics, the green lines (again, one for Stage 1 and the other ten for Stage 2) represent the frontloading contribution of the carbon dynamics. The black line (Stage 1 only) represents the anticipating effect of the tipping point: it is frontloading in scenario A, and backloading in scenario B.



We can observe how in scenario A the tipping point's frontloading effect shifts the SCC's Stage 1 peak sooner than it would occur in the myopic scenario and in Stage 2, whereas in scenario B the tipping point's backloading effect shifts the SCC's peak later.

4 Conclusion

We began our analysis by formulating the continuous-time version of the DICE model; we then included the possibility of a climate tipping point, modeled as a random instant whose hazard rate increases with the global temperature, and whose effect is either a permanent or an instantaneous change in the system. The resulting problem is an optimal control problem where the planner controls the savings rate and the abatement rate, aiming to maximize the expectation of a 2-stage payoff. Explicitly computing the expectation, we obtained a heterogeneous system where the Stage 1 variables depend on time alone, whereas the Stage 2 variables depend on both time and switching time.

We wrote the necessary conditions for the optimal controls and the co-states, through a version of the Maximum Principle that was derived as a special case of Veliov [2008]; we then employed such conditions to gain insight about the composition of the Social Cost of Carbon.

We made numerical simulations in four different tipping point scenarios, reporting the relevant figures that highlight the changes to the optimal policies in anticipation of the tipping point (compared to a myopic scenario) and in the new regime in response to the tipping point. Thanks to the analytical decomposition of the SCC we were also able to identify – through backward numerical integration – which factors have a frontloading/backloading effect on the SCC.

The planner's takeaway from our simulations is the importance of prevention. Whereas in some scenarios there is no significant adjustment in the optimal policy upon the regime shift, in all of them we see some degree of prevention effort to fend off the tipping point through higher abatement rates (compared to the myopic scenario). Ideally we want to postpone the regime shift towards the end of the planning horizon or – even better – after the end.

Our results at the moment are purely qualitative. To complete the work we will perform the Stage 2 parameter calibration and the robustness analysis of the effects of the tipping point, in order to make a meaningful comparison of different scenarios.

In future developments we are going to update the current Stage 1 parameters to the most recent DICE values, and also integrate the possibility of negative emissions from a certain time onward.

With the upcoming advancements in the theoretical equipment, we intend to include multiple tipping points and represent the interconnections between them, that is given by feedback mechanisms and domino effects.

A Appendix: SCC derivation

Let $V(C, E)$ be the Value function depending on total consumption and emissions. Suppose that the level set of V given by $\{V = V(C^*, E^*)\}$ can be locally parametrized as the graph of a function $C(E)$

$$\{(C(E), E) : E \in (E^* - \varepsilon, E^* + \varepsilon)\}$$

(which is true in a neighborhood of (C^*, E^*) if $V_C(C^*, E^*) \neq 0$).

The Social Cost of Carbon (SCC) is defined as the Marginal Rate of Substitution (MRS) of consumption for emissions:

$$SCC = \frac{dC}{dE}.$$

Since V is constant on the level set,

$$\begin{aligned}
0 &= \frac{dV}{dE}(C(E), E) \\
&= V_C \frac{dC}{dE} + V_E \\
&= V_C \cdot SCC + V_E
\end{aligned}$$

which yields

$$SCC = -\frac{V_E}{V_C} = -\frac{\Lambda_E}{\Lambda_C}.$$

Since E and C are not state variables and V does not depend on them explicitly, we identify which quantities that are directly impacting V they influence, compute the derivatives of such quantities with respect to E and C, and then apply the chain rule to V_E and V_C.

E enters the dynamics of M^{AT}:

$$\dot{M}^{AT} = -\delta_1 M^{AT} + \delta_2 M^{UP} + \alpha E$$

therefore the derivative $\frac{dM^{AT}}{dE}(t)$ is to be intended in the sense of Fréchet, i.e. the variation of $M^{AT}(t)$ that is produced by $\Delta E = 1$ in the small interval preceding t and 0 otherwise ("blip")

$$\begin{aligned}
\frac{d}{dt}(M^{AT} + \Delta M^{AT}) &= -\delta_1(M^{AT} + \Delta M^{AT}) + \delta_2 M^{UP} + \alpha(E + 1) \\
&= -\delta_1 M^{AT} + \delta_2 M^{UP} + \alpha E - \delta_1 \Delta M^{AT} + \alpha \\
&= \dot{M}^{AT} - \delta_1 \Delta M^{AT} + \alpha
\end{aligned}$$

yielding the Cauchy problem

$$\begin{cases}
\frac{d}{dt}\Delta M^{AT} = -\delta_1 \Delta M^{AT} + \alpha \\
\Delta M^{AT}(t - \varepsilon) = 0
\end{cases}$$

$$\Delta M^{AT}(t) = \frac{\alpha}{\delta_1}\left(1 - e^{-\delta_1 \varepsilon}\right)$$

$$\begin{aligned}\frac{dM^{AT}}{dE}(t) &= \lim_{\varepsilon\to 0^+}\frac{\Delta M^{AT}(t)}{\varepsilon}\\ &= \frac{\alpha}{\delta_1}\lim_{\varepsilon\to 0^+}\frac{1-e^{-\delta_1\varepsilon}}{\varepsilon}\\ &= \alpha\end{aligned}$$

So $V_E = V_{M^{AT}}\frac{dM^{AT}}{dE} = \lambda_{M^{AT}}\alpha$.

C **enters the utility function** $u(c) = u\big(C/L\big)$**:**

$$\frac{du}{dC} = u_c(c)/L$$

The derivative $\frac{dV}{du}$ is again a Fréchet derivative, this time backwards: it is the variation in $V(t)$ that is produced by $\Delta u = 1$ in the small interval following t.

$$\Delta V(t) = \int_t^{t+\varepsilon} e^{-\rho(s-t)} Z_1(s)L(s)\,ds$$

$$\begin{aligned}\frac{dV}{du}(t) &= \lim_{\varepsilon\to 0^+}\frac{\Delta V(t)}{\varepsilon}\\ &= \lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\int_t^{t+\varepsilon} e^{-\rho(s-t)} Z_1(s)L(s)\,ds = Z_1(t)L(t)\end{aligned}$$

by continuity of the integrand function. So $V_C = \frac{dV}{du}\frac{du}{dC} = Z_1 u_c(c)$, which, **if** s **is an inner control,** equals λ_K.

Putting everything together,

$$SCC = -\frac{V_E}{V_C} = \alpha\frac{-\lambda_{M^{AT}}}{Z_1 u_c} = \alpha\frac{-\lambda_{M^{AT}}}{\lambda_K}.$$

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